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RESEARCH ANNOUNCEMENTS ON "MODERATE DEVIATIONS FOR GRENANDER ESTIMATOR NEAR BOUNDARIES OF THE SUPPORT"

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1 Introduction and Main Results

Let f be a decreasing density with support $[0,\infty)$. Denote by F_n the empirical distribution function of a sample $X_1,...,X_n$ from f. Let $\hat{F}_n$ be the concave majorant of F_n on $[0,\infty)$, i.e. the smallest concave function such that

$$\hat{F}_n(t) \geq F_n(t), \quad \hat{F}_n(0)=0, \quad t\in[0,\infty).$$

The Grenander estimator $\hat{f}_n$ is defined as the left derivative of $\hat{F}_n$.

Prakasa Rao [9] obtained the earliest result on the asymptotic pointwise behavior of $\hat{f}_n(t)$ with $t\in(0,\infty)$

$$|4f(t)f'(t)|^{-1/3}\, n^{1/3}\left(\hat{f}_n(t)-f(t)\right) \xrightarrow{d} \mathrm{argmax}_{t\in\mathbb{R}}\left\{W(t)-t^2\right\},$$

where W denotes a two-sided standard Wiener process originating from zero, $\xrightarrow{d}$ means convergence in distribution. Then Groeneboom [6] provided an elegant proof of the same result based on inverse process of $\hat{f}_n$, which has become a cornerstone in this field. For the associated moderate deviations, one can see Gao et al. [4].

It should be noted that the Grenander estimator $\hat{f}_n$ is not consistent at boundaries ([1], [10]). This phenomenon has great influences on the global measures of deviation, such as the L_k-distances with $k>1$ ([2], [7]) and L_∞-distance ([3]), because the inconsistency at the boundaries will dominate the convergence.

To make the properties of $\hat{f}_n$ near boundaries more clear, Kulikov and Lopuhaä [8] considered asymptotic distribution of

$$n^{\alpha}\left(\hat{f}_n(cn^{-\alpha})-f(cn^{-\alpha})\right),$$

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where $c, \alpha > 0, 0 < \alpha < 1$. To be explicit, for the left boundary zero, suppose the following conditions hold:
(C1) $0 < f(0) = \lim_{t\downarrow 0} f(t) < \infty$;
(C2) there exists some positive constant ε_0 such that f has k-th order continuous derivative in $(0, \varepsilon_0]$ and $f(\varepsilon_0) \neq 0$. Moreover $0 < |f^{(k)}(0)| \leq \sup_{t\geq 0} |f^{(k)}(t)| < \infty$, with $f^{(k)}(0) = \lim_{t\downarrow 0} f^{(k)}(t)$ and $f^{(i)}(0) = 0$ for $1 \leq i \leq k-1$.
Kulikov and Lopuhaä ([7]) formulated that, as $0 < \alpha < 1/(2k+1)$,

$$c^{(1-k)/3} 2^{-2/3} ((k-1)!)^{1/3} n^{1/3+\alpha(k-1)/3} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})}{\left|f(0)f^{(k)}(0)\right|^{1/3}} \xrightarrow{d} \operatorname{argmax}_{t\in\mathbb{R}} \left\{W(t) - t^2\right\}.$$

Moreover, in the case of $1/(2k+1) < \alpha < 1$,

$$n^{(1-\alpha)/2} c^{1/2} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})}{f^{1/2}(0)} \xrightarrow{d} \sqrt{\operatorname{argmax}_{t\in[0,\infty)} \{W(t) - t\}}.$$

In this paper, the moderate deviations of $\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})$ in the above two cases ($0 < \alpha < 1/(2k+1)$ and $1/(2k+1) < \alpha < 1$) will be considered. By using strong approximation technique and comparison method, we obtain the following main results.

Theorem 1.1 When $0 < \alpha < 1/(2k+1)$, let ℓ_n satisfy as $n \to \infty$

$$\ell_n \to \infty, \quad \frac{n^\alpha}{\ell_n^{15}(\log n)^2} \to \infty.$$

Then, under conditions (C1) and (C2), the sequence

$$\left\{ \frac{n^{1/3+\alpha(k-1)/3}}{\ell_n} c^{(1-k)/3} 2^{-2/3} ((k-1)!)^{1/3} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})}{\left|f(0)f^{(k)}(0)\right|^{1/3}}, n \geq 1 \right\}$$

satisfies the moderate deviations in $\mathbb{R}$ with speed ℓ_n^3 and rate function $I(x) = \frac{2}{3}|x|^3$, that is, for any open subset G of $\mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\frac{n^{1/3+\alpha(k-1)/3}}{\ell_n} c^{(1-k)/3} 2^{-2/3} ((k-1)!)^{1/3} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})}{\left|f(0)f^{(k)}(0)\right|^{1/3}} \in G \right) \geq -\inf_{x\in G} I(x),$$

and for any closed subset F of $\mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\frac{n^{1/3+\alpha(k-1)/3}}{\ell_n} c^{(1-k)/3} 2^{-2/3} ((k-1)!)^{1/3} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})}{\left|f(0)f^{(k)}(0)\right|^{1/3}} \in F \right) \leq -\inf_{x\in F} I(x).$$

Remark 1 For any $x > 0$, by Theorem 1.1, we have

$$\lim_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\pm \frac{n^{1/3+\alpha(k-1)/3}}{\ell_n} c^{(1-k)/3} 2^{-2/3} ((k-1)!)^{1/3} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha})}{\left|f(0)f^{(k)}(0)\right|^{1/3}} \geq x \right) = -\frac{2}{3}|x|^3.$$

Theorem 1.2 When $1/(2k+1) < \alpha < 1$, let $\{\lambda_n\}$ satisfy that

$$\lambda_n \to \infty, \quad \frac{n^{(1-\alpha)}}{\lambda_n^{30}(\log n)^6} \to \infty, \quad \frac{n^{(2k+1)\alpha-1}}{\lambda_n^{4k+10}} \to \infty.$$

Then, under conditions (C1) and (C2), the sequence

$$\left\{ \frac{n^{(1-\alpha)/2}}{\lambda_n} c^{1/2} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha}t)}{f^{1/2}(0)}, n \geq 1 \right\}$$

satisfies the moderate deviations in $\mathbb{R}^+$ with speed λ_n^2 and rate function $J(x) = \frac{x^2}{2}$.

Remark 2 For any $x > 0$, by Theorem 1.2, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \log P\left(\frac{n^{(1-\alpha)/2}}{\lambda_n} c^{1/2} \frac{\hat{f}_n(cn^{-\alpha}) - f(cn^{-\alpha}t)}{f^{1/2}(0)} \geq x \right) = -\frac{x^2}{2}.$$

Remark 3 If f has compact support, without loss of generality, assume it is the interval $[0, 1]$. The moderate deviations of $\hat{f}_n$ near the right boundary 1 (similar to Theorem 1.1 and Theorem 1.2) can also be obtained, and the details are omitted here.

For a detail study of the moderate deviations for Grenander estimator near boundaries, please refer to [5].

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