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RESEARCH ANNOUNCEMENTS ON "UPPER BOUND ESTIMATES OF EIGENVALUES FOR HÖRMANDER OPERATORS ON NON-EQUIREGULAR SUB-RIEMANNIAN MANIFOLDS"

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1 Introduction and Main Results

For $n \geq 2$, let M be a n-dimensional connected smooth manifold. We take a fixed smooth measure on M with strictly positive density, and write dx as the measure when we integrate. Also, we simply denote the measure of a set $E \subset M$ by $|E|$.

Suppose that $X = (X_1, X_2, \cdots, X_m)$ are the real C^∞ vector fields defined on M, which satisfy the so-called Hörmander's condition with Hörmander's index Q: there exists a smallest positive integer Q such that these vector fields $X_1, X_2, \ldots, X_m$ together with their commutators of length at most Q span the tangent space at each point of M.

Starting from the vector fields $X = (X_1, X_2, \cdots, X_m)$ which satisfy the Hörmander's condition, we can construct a sub-Riemannian manifold (M, D, g) endowed with the canonical sub-Riemannian structure (D, g), where the distribution D is a family of linear subspaces $D_x \subset T_x(M)$ such that $D_x = \text{span}\{X_1(x), X_2(x), \cdots, X_m(x)\}$ depending smoothly on $x \in M$, and the sub-Riemannian metric $g : TM \to \mathbb{R} \cup \{+\infty\}$ is a function given by $g(x, v) = \inf \left\{ \sum\limits_{i=1}^{m} u_i^2 | v = \sum\limits_{i=1}^{m} u_i X_i(x) \right\}$ for $x \in M$ and $v \in T_x(M)$. Observe that $g(x, \cdot)$ is a positive definite quadratic form on D_x and $g(x, v) = +\infty$ for $v \notin D_x$.

For each $x \in M$, the sub-Riemannian flag at x is the sequence of nested vector subspaces

$$\{0\} = D_x^0 \subset D_x = D_x^1 \subset D_x^2 \subset \cdots \subset D_x^{r(x)-1} \subsetneq D_x^{r(x)} = T_x(M)$$

defined in terms of successive Lie brackets, and $r(x) \leq Q$ is the degree of nonholonomy at x. Here for each $1 \leq j \leq r(x)$, D_x^j is the subspaces of the tangent space at x spanned by all

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commutators of $X_1,\ldots,X_m$ with length at most j. Setting $\nu_j(x)=\dim D_x^j$ for $1\le j\le r(x)$ with $\nu_0(x):=0$, the integer

$$\nu(x)=\sum_{j=1}^{r(x)} j(\nu_j(x)-\nu_{j-1}(x))$$

is called the pointwise homogeneous dimension at x. Besides, a point $x\in M$ is regular if, for every $1\le j\le r(x)$, the dimension $\nu_j(y)$ is a constant as y varies in a open neighborhood of x. Otherwise, x is said to be singular. A set $S\subset M$ is equiregular if every point of S is regular. The set $S\subset M$ is said to be non-equiregular if it contains some singular points. The equiregular assumption in sub-Riemannian geometry is also known as the Métivier's condition in PDEs. For an equiregular connected set S, we know that the pointwise homogeneous dimension $\nu(x)$ is a constant ν which coincides with the Hausdorff dimension of S related to the vector fields X, and this constant ν is also called the Métivier's index. Furthermore, if the set $S\subset M$ is non-equiregular, we can introduce the so-called generalized Métivier's index by

$$\tilde{\nu}_S:=\max_{x\in\overline{S}}\nu(x).$$

The generalized Métivier's index is also known as the non-isotropic dimension of S related to the vector fields X, which plays a crucial role in the geometry and functional settings associated to the vector fields X. Note that $n+\max\limits_{x\in\overline{S}} r(x)-1\le\tilde{\nu}_S<nQ$ for $Q>1$, and $\tilde{\nu}_S=\nu$ if the closure of S is equiregular and connected.

In this paper, we concerned with the eigenvalue problems of self-adjoint Hörmander operator $\triangle_X:=-\sum\limits_{i=1}^{m}X_i^*X_i$ on non-equiregular sub-Riemannian manifolds, where X_i^* denotes the formal adjoint of X_i. Precisely, we first study the closed eigenvalue problem of $-\triangle_X$, i.e.

$$-\triangle_X u=\mu u \qquad \text{in } M, \tag{1.1}$$

where M is a n-dimensional connected compact smooth manifold without boundary. Secondly, we study the Dirichlet eigenvalue problem of $-\triangle_X$. For simplicity we assume that M is an open connected domain in $\mathbb{R}^n$ endowed with Lebesgue measure, and $\Omega\subset\subset M$ is a bounded connected open subset with smooth boundary $\partial\Omega$ which is non-characteristic for $X=(X_1,X_2,\cdots,X_m)$. The Dirichlet eigenvalue problem of $-\triangle_X$ will be considered as follows

$$\begin{cases} -\triangle_X u=\lambda u, & \text{in } \Omega;\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In both cases above, the Hörmander's condition ensures that the positive self-adjoint operator $-\triangle_X$ possesses discrete eigenvalues, which will be denoted by $\{\mu_k\}_{k\ge0}$ and $\{\lambda_k\}_{k\ge1}$ respectively. Thus we have

$$\begin{aligned} 0 &= \mu_0<\mu_1\le\mu_2\le\cdots\le\mu_{k-1}\le\mu_k\le\cdots,\\ 0 &< \lambda_1<\lambda_2\le\cdots\le\lambda_{k-1}\le\lambda_k\le\cdots, \end{aligned}$$

and $\mu_k \to +\infty$, $\lambda_k \to +\infty$ as $k \to +\infty$.

Combining the Rayleigh-Ritz formula with Jerison and Sánchez-Calle's subelliptic heat kernel estimates in [6, 8], we can obtain the following estimate for closed eigenvalue problem (1.1).

Theorem 1.1 Let $X = (X_1, X_2, \cdots, X_m)$ be the C^∞ real vector fields defined on the compact manifold M, which satisfy the Hörmander's condition in M. Denote by μ_k the k^{th} eigenvalue of the problem (1.1). Then for any $0 < t < 1$ and any $k \geq 0$, we have

$$\mu_{k+1}\left[A_1\int_M \frac{dx}{|B_{d_X}(x,\sqrt{t})|} - (k+1)\right] + \sum_{j=0}^{k}\mu_j \leq \frac{C_{1,1}}{t}\int_M \frac{dx}{|B_{d_X}(x,\sqrt{t})|}, \tag{1.3}$$

where $B_{d_X}(x, r)$ denotes the subunit ball induced by the subunit metric, A_1 and $C_{1,1}$ are some positive constants which depend only on the sub-Riemannian structure of M.

Remark 1.1 From Theorem 1.1, we can recover the lower bound estimate of the counting function given by Fefferman and Phong in [4], namely

$$N(\lambda) \geq c_1 \int_M \frac{dx}{|B_{d_X}(x, \lambda^{-\frac{1}{2}})|}$$

holds for sufficient large $\lambda > 0$, where $N(\lambda) := \#\{k \mid \mu_k \leq \lambda\}$ is the spectral counting function and $c_1 > 0$ is a constant depending on the sub-Riemannian structure. This means (1.3) possesses the optimal growth order.

Next, if we denote $H := \{x \in M \mid \nu(x) = \tilde{\nu} = \max_{x\in M} \nu(x)\}$ as a subset of M. Then the following result gives the explicit upper bound of μ_k.

Theorem 1.2 Suppose that $X = (X_1, X_2, \cdots, X_m)$ and M satisfy the conditions in Theorem 1.1. Denote by $\tilde{\nu} = \max_{x\in M} \nu(x)$ the non-isotropic dimension of M related to the vector fields X. If the subset $H := \{x \in M \mid \nu(x) = \tilde{\nu} = \max_{x\in M} \nu(x)\}$ possesses a positive measure (i.e. $|H| > 0$), then for any $k \geq 1$ we have

$$\sum_{j=1}^{k}\mu_j \leq \frac{C_{1,1}}{\widehat{C_1}}\cdot\left(\frac{\widehat{C_2}}{A_1}\right)^{1+\frac{2}{\tilde{\nu}}}\cdot\frac{|M|}{|H|^{1+\frac{2}{\tilde{\nu}}}}\cdot(k+1)^{1+\frac{2}{\tilde{\nu}}}, \tag{1.4}$$

and

$$\mu_k \leq \frac{C_{1,1}}{\widehat{C_1}}\cdot\left(\frac{2\widehat{C_2}}{A_1}\right)^{1+\frac{2}{\tilde{\nu}}}\cdot\frac{|M|}{|H|^{1+\frac{2}{\tilde{\nu}}}}\cdot k^{\frac{2}{\tilde{\nu}}}, \tag{1.5}$$

where $\widehat{C_1}$ and $\widehat{C_2}$ are some positive constants depending only on the sub-Riemannian structure of M.

Remark 1.2 From the asymptotic results in [1], we know the upper bounds (1.4) and (1.5) for μ_k in Theorem 1.2 are optimal in terms of the order on k. In particular, if M is equiregular, then $\tilde{\nu} = \nu$ and $H = M$. In this case, the upper bound estimate (1.5) above gives the similar results by Kokarev [7] and Hassannezhad-Kokarev [5]. However, our results will be suitable for general equiregular sub-Riemannian manifolds and non-equiregular sub-Riemannian manifolds.

Furthermore, we can also obtain the following inequality for the Dirichlet eigenvalues of the problem (1.2).

Theorem 1.3 Let $X = (X_1, X_2, \cdots, X_m)$ be C^∞ real vector fields defined on a connected open domain M in $\mathbb{R}^n$, which satisfy the Hörmander's condition. Assume that $\Omega \subset\subset M$ is a bounded connected open subset with smooth boundary such that $\partial\Omega$ is non-characteristic for X. Denote by λ_k the k^{th} eigenvalue of the problem (1.2). Then for any compact subset $K \subset \Omega$, there exists a positive constant $\delta(K)$, such that for any $0 < t \le \delta(K)$ and any $k \ge 1$, we have

$$\lambda_{k+1}\left[\frac{A_2}{2}\cdot\int_K \frac{dx}{|B_{d_X}(x,\sqrt{t})|} - k\right] + \sum_{j=1}^{k}\lambda_j \le \frac{2A_3}{t}\int_\Omega \frac{1}{|B_{d_X}(x,\sqrt{t})|}dx, \tag{1.6}$$

where A_2 and A_3 are some positive constants which depend only on the sub-Riemannian structure.

Similarly, when the subset H has a positive measure, Theorem 1.3 also indicates the following explicit upper bounds of λ_k which are optimal in terms of the order on k, and also compatible with the asymptotic results in [2].

Theorem 1.4 Suppose that $X = (X_1, X_2, \cdots, X_m)$ and Ω satisfy the conditions in Theorem 1.3. Let $\tilde{\nu} = \max_{x\in\overline{\Omega}} \nu(x)$ be the non-isotropic dimension of Ω related to the vector fields X. If the subset $H := \{x \in \Omega \mid \nu(x) = \tilde{\nu} = \max_{x\in\overline{\Omega}} \nu(x)\}$ admits a positive measure, then for any $k \ge 1$ we have

$$\sum_{j=1}^{k}\lambda_j \le \frac{2A_3}{\widehat{C_1}}\cdot\left(\frac{4\widehat{C_2}}{A_2}\right)^{1+\frac{2}{\tilde{\nu}}}\cdot\frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}}\cdot k^{1+\frac{2}{\tilde{\nu}}} \tag{1.7}$$

and

$$\lambda_{k+1} \le \frac{2A_3}{\widehat{C_1}}\cdot\left(\frac{8\widehat{C_2}}{A_2}\right)^{1+\frac{2}{\tilde{\nu}}}\cdot\frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}}\cdot k^{\frac{2}{\tilde{\nu}}}, \tag{1.8}$$

where $\widehat{C_1}$ and $\widehat{C_2}$ are some positive constants depending only on the sub-Riemannian structure of M.

The details of proofs for Theorem 1.1–Theorem 1.4 have been given in [3].

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