Vol. 41 (2021)
No. 6

数 学 杂 志
J. of Math. (PRC)

APPROXIMATIONS OF THE IDENTITY ADAPTED TO CONTINUOUS ELLIPSOID COVER

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Abstract: In this paper we develop some approximation of the identity results adapted to continuous multi-level ellipsoid cover. By using real-variable methods of harmonic analysis, we obtain two approximations of the identity results uniformly in some compact subset of $\mathbb{R}^n$ and in $L^1(\mathbb{R}^n)$ norm, respectively. These results generalize the corresponding classical and anisotropic approximation of the identity results.

Keywords: approximation of the identity; ellipsoid cover; anisotropy

2010 MR Subject Classification: 42B35; 46E30; 40A10

Document code: A **Article ID:** 0255-7797(2021)03-489-06

1 Introduction

As we all know, approximation of identity plays an important role in analysis, see [1-3]. There are numerous approximations of identity results associated with the Euclidian balls in $\mathbb{R}^n$. For example, let φ be an integrable function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, and for $t > 0$ define $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. Then, if $f \in L^1(\mathbb{R}^n)$, $\varphi_t * f \to f$ $(t \to 0)$ in $L^1(\mathbb{R}^n)$.

In 2010, the continuous multi-level ellipsoid cover Θ introduced by Dahmen, Dekel and Petrushev [4] consist of ellipsoids $\theta_{x,t} = M_{x,t}(\mathbb{B}^n)+x$, where $M_{x,t}$ is an invertible matrix and $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$ (see Definition 2.1). The flexible framework of continuous ellipsoid cover Θ introduced in this paper may have the ability to solve the following problems. For example, the formation of shocks results in jump discontinuities of solutions of hyperbolic conservation laws across lower dimensional manifolds. The case such jumps cause a serious obstruction to appropriate regularity theorems, since the available regularity scales are either inherently isotropic or coordinate biased or are subject to an uncontrollable restricted regularity range. For more development of continuous ellipsoid cover, see [5-7].

Inspired by the above work, for any $\theta_{x,t} = M_{x,t}(\mathbb{R}^n) + x \in \Theta$, let φ be an integrable function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi\,dx = 1$, we can define

$$\varphi_{x,\,t}(y) := |\det(M^{-1}_{x,\,t})|\varphi(M^{-1}_{x,\,t}y).$$

* **Received date:** 2020-10-07 **Accepted date:** 2021-03-17

Foundation item: Supported by Xinjiang Training of Innovative Personnel Natural Science Foundation of China (2020D01C048) and National Natural Science Foundation of China (11861062).

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And then a question arises: Is it possible to obtain some approximations of the identity results adapted to ellipsoid cover Θ such as $f * \varphi_{x,t}(x) \to f(x)$ $(t \to \infty)$ in various senses? This article gives some affirmative answers for the question. It is worth pointing out that the approximation of the identity in this paper is done in $C_c(\mathbb{R}^n)$, which is a dense subset of $L^1(\mathbb{R}^n)$, and the approximation of the identity in $L^1(\mathbb{R}^n)$ is difficult for us, which is still open at the moment.

The organization of this article is as follows. In Section 2, we first present some notation and notions used in this article including continuous ellipsoid cover Θ and describe our main theorem. In Section 3, we show the proof details of the main theorem.

2 Preliminaries and Main Results

In this section we recall the properties of ellipsoid cover which was originally introduced by Dahmen, Dekel, and Petruschev [4]. An ellipsoid ξ in $\mathbb{R}^n$ is an image of the Euclidean unit ball $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| < 1\}$ under an affine transform, i.e.,

$$\xi := M_\xi(\mathbb{B}^n) + c_\xi,$$

where M_ξ is an invertible matrix and c_ξ is the center.

Definition 2.1 We say that

$$\Theta := \{\theta_{x,\,t} : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

is a continuous ellipsoid cover of $\mathbb{R}^n$, if there exist constants $\mathbf{p}(\Theta) := \{a_1, \dots, a_6\}$ such that:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, there exists an ellipsoid $\theta_{x,\,t} := M_{x,\,t}(\mathbb{B}^n) + x$, where $M_{x,\,t}$ is a invertible matrix and x is the center, satisfying

$$a_1 2^{-t} \le |\theta_{x,\,t}| \le a_2 2^{-t}. \tag{2.1}$$

(ii) Intersecting ellipsoids from Θ satisfy "shape condition", i.e., for any $x,\ y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \ge 0$, if $\theta_{x,\,t} \cap \theta_{y,\,t+s} \ne \emptyset$, then

$$a_3 2^{-a_4 s} \le 1/\|(M_{y,\,t+s})^{-1} M_{x,\,t}\| \le \|(M_{x,\,t})^{-1} M_{y,\,t+s}\| \le a_5 2^{-a_6 s}. \tag{2.2}$$

Here, $\|\cdot\|$ is the matrix norm given by $\|M\| := \max_{|x|=1} |Mx|$ for invertible matrix M.

Proposition 2.2 For any $x \in \mathbb{R}^n$ and $\{\theta_{x,t}\}_{t\in\mathbb{R}} \subset \Theta$, it holds true that

$$\theta_{x,t} = M_{x,\,t}(\mathbb{B}^n) + x \to \mathbb{R}^n \quad \text{as} \quad t \to -\infty. \tag{2.3}$$

Proof For any $y \in \mathbb{B}^n$ and $t < 0$, by (2.2), we obtain

$$|M_{x,\,t}^{-1} y| \le \|M_{x,\,t}^{-1} M_{x,\,0}\| \|M_{x,\,0}^{-1}\| |y| \le a_5 2^{a_6 t} \|M_{x,\,0}^{-1}\| |y| < a_5 2^{a_6 t} \|M_{x,\,0}^{-1}\|,$$

which implies that

$$a_5^{-1}\|M_{x,0}^{-1}\|^{-1}2^{-a_6 t}\mathbb{B}^n \subset M_{x,t}(\mathbb{B}^n)$$

and hence $\theta_{x,t} = M_{x,t}(\mathbb{B}^n) + x \to \mathbb{R}^n$ as $t \to -\infty$.

Definition 2.3 For each $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\theta_{x,t} = M_{x,t}(\mathbb{B}^n) + x \in \Theta$, denote

$$\varphi_{x,t}(y) := |\det(M_{x,t}^{-1})|\varphi(M_{x,t}^{-1}y)$$

and, for each measurable function f,

$$f * \varphi_{x,t}(x) := \int_{\mathbb{R}^n} f(y)\varphi_{x,t}(x-y)dy.$$

Theorem 2.4

(i) Let $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$ and $f \in C_c(\mathbb{R}^n)$, which collects all continuous functions of compact support. Then

$$\lim_{t\to+\infty}\sup_{x\in\operatorname{supp}f}|f*\varphi_{x,t}(x) - f(x)| = 0.$$

(ii) Let $f \in C_c(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$ and $\operatorname{supp}\varphi \subset \mathbb{B}^n$. Moreover, if the cover Θ is zero-level-bounded, i.e., there exists a positive constant C such that, for any $x \in \mathbb{R}^n$ and matrix $M_{x,0}$ of ellipsoid $\theta_{x,0} = M_{x,0}(\mathbb{B}^n) + x$,

$$\|M_{x,0}\| \le C. \tag{2.4}$$

Then

$$\lim_{t\to+\infty}\int_{\mathbb{R}^n}|f*\varphi_{x,t}(x) - f(x)|\,dx = 0.$$

Remark 2.5 The assumption (2.4) is a mild condition. When the ellipsoid cover Θ is reduced to the following typical ellipsoid covers, the assumption (2.4) holds true automatically.

(i) The regular cover of $\mathbb{R}^n$ consisting of all balls in $\mathbb{R}^n$:

$$\Theta := \{\theta(x,t) = 2^{-\frac{t}{n}}\mathrm{I}_n(\mathbb{B}^n) + x : t \in \mathbb{R}, x \in \mathbb{R}^n\}.$$

Obviously, $\|M_{x,0}\| = 1$.

(ii) The one-parameter family of diagonal dilation matrices

$$D_t := \operatorname{diag}(2^{-tb_1}, 2^{-tb_2}, \cdots, 2^{-tb_n}), \quad b_j > 0,\ j = 1, \cdots, n,\ \sum_{j=1}^n b_j = 1,$$

induces a continuous ellipsoid cover of $\mathbb{R}^n$ with $M_{x,t} = D_t$. Obviously, $\|M_{x,0}\| = 1$.

(iii) For a one parameter continuous subgroup of $GL(\mathbb{R}^n, n)$ of the form $\{A_t : 0 < t < \infty\}$ satisfying $A_t A_s = A_{ts}$ and, there exist constants $1 \leq \alpha \leq \beta < \infty$ such that, $t^\alpha |x| < |A_t x| < t^\beta |x|$ for all $x \in \mathbb{R}^n$ and $t \geq 1$. The infinitesimal generator P of $A_t := \exp(P \ln t)$ satisfies $(Px, x) \geq (x, x)$ and $\det A_t = t^a$, where $(\cdot, \cdot)$ is the standard scalar product in $\mathbb{R}^n$ and a is trace of P. Then we can define a continuous ellipsoid cover Θ of Calderón and Torchinsky [8], that is,

$$\Theta := \{x + A_t(\mathbb{B}^n) : x \in \mathbb{R}^n, t \in (0, \infty)\} := \{x + A_{2^{-t/a}}(\mathbb{B}^n) : x \in \mathbb{R}^n, t \in \mathbb{R}\}.$$

Obviously, $\|M_{x,0}\| = \|A_1\| = \|\exp(P \ln 1)\| = 1$.

(iv) Consider a $n \times n$ real matrix A with eigenvalues λ satisfying $|\lambda| > 1$. By [9, Lemma 2.2], there exists an ellipsoid $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where P is some invertible $n \times n$ matrix, such that $B_k \subset B_{k+1}$, where $B_k := A^k \Delta$ for $k \in \mathbb{Z}$. Then we can define a semi-continuous ellipsoid cover in the sense of [4, Definition 2.5] by

$$\Theta := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\} = \{x + A^k P^{-1}(\mathbb{B}^n) : x \in \mathbb{R}^n, k \in \mathbb{Z}\}.$$

Obviously, $\|M_{x,0}\| = \|P^{-1}\|$.

(v) Lighted by [4, Theorem 7.3], we give a concrete example for the ellipsoid cover Θ:

$$\Theta_0 := \{\theta_{x,t} = M_{x,t}(\mathbb{B}^n) + x : x \in \mathbb{R}^2, t \in \mathbb{R}\}$$

with

$$\theta_{x,t} := \{z = (z_1, z_2) \in \mathbb{R}^2 : \frac{(z_1 - x_1)^2}{\sigma_1^2} + \frac{(z_2 - x_2)^2}{\sigma_2^2} \leq 1\},$$

where

t	σ_1	σ_2
$t \leq 0$	$2^{-\frac{t}{2}}$	$2^{-\frac{t}{2}}$
$0 < t \leq -2\log_2 \lvert x_2 \rvert$	$2^{-\frac{t}{3}}$	$2^{-\frac{2t}{3}}$
$-2\log_2 \lvert x_2 \rvert < t \leq -3\log_2 \lvert x_2 \rvert$	$2^{-\frac{5t}{6}} \frac{1}{\lvert x_2 \rvert}$	$2^{-\frac{t}{6}} \lvert x_2 \rvert$
$t > -3\log_2 \lvert x_2 \rvert$	$2^{-\frac{t}{2}}$	$2^{-\frac{t}{2}}$

Here, $M_{x,t} = \mathrm{diag}(\sigma_1, \sigma_2)$. Obviously, $\|M_{x,0}\| = \|\mathrm{I}_{n\times n}\| = 1$,

3 Proof of Theorem 2.4

Proof (i) By $f \in C_c(\mathbb{R}^n)$, we know that there exists a positive constant N and M such that $\mathrm{supp} f \subset N\mathbb{B}^n$ and $|f(x)| \leq M$ for any $x \in \mathbb{R}^n$, and f is a uniformly continuous function on $\mathbb{R}^n$. By this, we obtain that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ with $|y| < \delta$,

$$|f(x - y) - f(x)| < \varepsilon / (2\|\varphi\|_1).$$

From this, we deduce that, for any $x \in \operatorname{supp} f$,

$$\begin{aligned}|f * \varphi_{x,t}(x) - f(x)| &\le \int_{|y|<\delta} |f(x-y) - f(x)||\varphi_{x,t}(y)|\,dy + \int_{|y|\ge\delta} |f(x-y) - f(x)||\varphi_{x,t}(y)|\,dy \\ &< \frac{\varepsilon}{2} + 2M \int_{|M_{x,t}y|\ge\delta} |\varphi(y)|\,dy. \qquad (3.1)\end{aligned}$$

From Proposition 2.2, it follows that there exists a positive constant t_N large enough such that $N\mathbb{B}^n \subset \theta_{0,-t_N}$ and hence, for any $x \in N\mathbb{B}^n$, $x \in \theta_{0,-t_N}$. By this and (2.2), we obtain

$$\|M_{x,0}\| \le \|M_{0,-t_N}\| \|M_{0,-t_N}^{-1} M_{x,0}\| \le \|M_{0,-t_N}\| a_5 2^{-a_6 t_N} =: C_N.$$

From this and (2.2), for any $x \in N\mathbb{B}^n$, $y \in \mathbb{R}^n$ and $t > 0$, we deduce that

$$|M_{x,t}y| = |M_{x,0} M_{x,0}^{-1} M_{x,t} y| \le C_N a_5 2^{-a_6 t}|y|,$$

which implies that

$$\{y : |M_{x,t}y| \ge \delta\} \subset \{y : |y| \ge (C_N a_5)^{-1} 2^{a_6 t}\delta\}.$$

By this and $\varphi \in L^1(\mathbb{R}^n)$, we know that, for any $\varepsilon > 0$, there exists a positive constant T large enough such that, for any $t > T$,

$$\int_{|M_{x,t}y|\ge\delta} |\varphi(y)|dy \le \int_{|y|\ge(C_N a_5)^{-1}2^{a_6 t}\delta} |\varphi(y)|dy < \frac{\varepsilon}{4M},$$

which, together with (3.1), implies that

$$\lim_{t\to+\infty} \sup_{x\in\operatorname{supp} f} |f * \varphi_{x,t}(x) - f(x)| = 0.$$

(ii) By $f \in C_c(\mathbb{R}^n)$, we can assume that there exists $N > 0$ such that $\operatorname{supp} f \subset N\mathbb{B}^n$. Notice that

$$\int_{\mathbb{R}^n} |f * \varphi_{x,t}(x) - f(x)|\,dx \le \int_{\mathbb{B}^n} |\varphi(y)| \int_{\mathbb{R}^n} |f(x - M_{x,t}y) - f(x)|\,dx\,dy. \qquad (3.2)$$

Since $\operatorname{supp} f \subset N\mathbb{B}^n$, we only need to estimate the above integral under the following condition:

$$y \in \mathbb{B}^n \text{ and } x \in \{x : x - M_{x,t}y \in N\mathbb{B}^n\} \cup N\mathbb{B}^n.$$

For any $t > 0$, by this, (2.2) and (2.4), we have

$$|x| \le |M_{x,t}y| + N \le \|M_{x,0}\| a_5 2^{-a_6 t} + N \le C a_5 + N$$

and hence $x \in (Ca_5 + N)\mathbb{B}^n$. From this, we further deduce that

$$\int_{\mathbb{R}^n} |f(x - M_{x,t}y) - f(x)|\,dx \le \int_{(Ca_5+N)\mathbb{B}^n} |f(x - M_{x,t}y) - f(x)|\,dx.$$

By this, (3.2), Fatou's lemma, the uniformly continuity of f on $(Ca_5+N)\mathbb{B}^n$ and the fact that
$$|M_{x,t}y| \le \|M_{x,0}\|a_5 2^{-a_6 t} \le Ca_5 2^{-a_6 t} \to 0 \text{ as } t\to+\infty,$$
we have
$$\begin{aligned}\limsup_{t\to+\infty}\int_{\mathbb{B}^n}|f*\varphi_{x,t}(x)-f(x)|\,dx &\le \limsup_{t\to+\infty}\int_{\mathbb{B}^n}|\varphi(y)|\int_{(Ca_5+N)\mathbb{B}^n}|f(x-M_{x,t}y)-f(x)|\,dx\,dy\\ &\le \int_{\mathbb{B}^n}|\varphi(y)|\int_{(Ca_5+N)\mathbb{B}^n}\limsup_{t\to+\infty}|f(x-M_{x,t}y)-f(x)|\,dx\,dy\\ &=0.\end{aligned}$$

This finishes the proof of Theorem 2.4(ii).

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相适应于连续椭球覆盖的恒等逼近

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摘要: 本文研究了相适应于多尺度连续椭球覆盖的恒等逼近问题. 通过使用调和分析中的实变方法, 得到了如下两个恒等逼近结果: 在欧氏空间 $\mathbb{R}^n$上紧子集的一致恒等逼近和 $L^1(\mathbb{R}^n)$范数下的恒等逼近. 该结果推广了经典情况下和各向异性情形下相应的恒等逼近结果.

关键词: 恒等逼近; 椭球覆盖; 各向异性

MR(2010)**主题分类号**: 42B35; 46E30; 40A10 **中图分类号**: O174.1