

μ -Bergman 空间到 Zygmund 型空间的加权复合算子

赵艳辉, 廖春艳, 唐亚林
(湖南科技学院理学院, 湖南 永州 425199)

摘要: 本文研究了单位圆盘 D 上的 μ -Bergman 空间到 Zygmund 型空间的加权复合算子的有界性和紧性问题. 利用泛函分析多复变的方法, 获得了单位圆盘上 μ -Bergman 空间到 Zygmund 型空间的加权复合算子为有界算子和紧算子的充要条件.

关键词: μ -Bergman 空间; Zygmund 型空间; 加权复合算子; 有界性; 紧性

MR(2010) 主题分类号: 47B38; 47B33; 30H05; 46E15 中图分类号: O174.56; O177.2

文献标识码: A 文章编号: 0255-7797(2021)02-0159-11

1 引言

设 $D = \{z : |z| < 1\}$ 为复平面上的单位圆盘, $\partial D = \{z : |z| = 1\}$ 是 D 的边界, $H(D)$ 和 $H(D, D)$ 分别表示 D 上的解析函数全体和解析映射全体.

定义 1.1 对 $[0, 1)$ 上的连续函数 $\mu(r) > 0$, 如果存在常数 $0 < s < t$, 使得

(i) $\frac{\mu(r)}{(1-r)^s}$ 在 $[0, 1)$ 上单调递减且 $\lim_{r \rightarrow 1^-} \frac{\mu(r)}{(1-r)^s} = 0$,

(ii) $\frac{\mu(r)}{(1-r)^t}$ 在 $[0, 1)$ 上单调递增且 $\lim_{r \rightarrow 1^-} \frac{\mu(r)}{(1-r)^t} = \infty$,

则称 μ 为 $[0, 1)$ 上的正规函数^[1].

定义 1.2 设 μ 为 $[0, 1)$ 上的正规函数, 对 $0 < p < \infty$, 记

$$H(p, \mu) = \{f \in H(D) : \|f\|_{H(p, \mu)}^p = \int_D |f(z)|^p \frac{\mu^p(|z|)}{1 - |z|} dA(z) < \infty\}$$

表示 D 上的 μ -Bergman 空间. 当 $1 \leq p < \infty$ 时, $H(p, \mu)$ 在范数 $\|\cdot\|_{H(p, \mu)}$ 下是 Banach 空间. 当 $0 < p < 1$ 时, $H(p, \mu)$ 在范数 $\|\cdot\|_{H(p, \mu)}$ 下为度量空间, 也是 Frechet 空间. 特别地, 当 $\mu(r) = (1-r)^q, q > -1$ 时, $H(p, \mu)$ 就是经典的加权 Bergman 空间 \mathcal{A}_q^p ^[2].

定义 1.3 D 上的 Zygmund 空间 Z 是指满足 $f \in H(D) \cap C(\bar{D})$ 且

$$\sup_{0 < \theta \leq 2\pi} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty$$

的函数全体, 其中 $e^{i\theta} \in \partial D, h > 0$. 由文献 [3] 中的定理 5.3 和闭图像定理得 $f \in Z$ 当且仅当 $\sup_{z \in D} (1 - |z|^2)|f''(z)| < \infty$.

定义范数

$$\|f\|_Z = |f(0)| + |f'(0)| + \sup_{z \in D} (1 - |z|^2)|f''(z)|,$$

*收稿日期: 2020-07-15 接收日期: 2020-09-01

基金项目: 湖南省自然科学基金资助 (2019JJ40089); 湖南科技学院应用特色学科建设项目资助.

作者简介: 赵艳辉 (1969-), 女, 湖南益阳, 教授, 主要研究方向: 多复变函数论.

则 Zygmund 空间在此范数下是 Banach 空间. 小 Zygmund 空间是指满足 $f \in Z$ 且 $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)|f''(z)| = 0$ 的函数全体, 记为 Z_0 , 显然 Z_0 是 Z 的闭子空间.

对 $\alpha > 0$, D 上的 Zygmund 型空间是指满足 $f \in H(D)$ 且 $\sup_{z \in D} (1 - |z|^2)^\alpha |f''(z)| < \infty$ 的函数全体, 记为 Z^α . 定义范数

$$\|f\|_{Z^\alpha} = |f(0)| + |f'(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f''(z)|,$$

则 Zygmund 型空间在此范数下是 Banach 空间. 小 Zygmund 型空间是指满足 $f \in Z^\alpha$ 且 $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |f''(z)| = 0$ 的函数全体, 记为 Z_0^α , 显然 Z_0^α 是 Z^α 的闭子空间.

定义 1.4 设 φ 是 D 到 D 上的非常值解析自映射, $\psi \in H(D)$, $H(D)$ 上的加权复合算子 $T_{\varphi, \psi}$ 定义如下:

$$T_{\varphi, \psi} = \psi(z)f \circ \varphi(z), \forall f \in H(D).$$

显然, 加权复合算子 $T_{\varphi, \psi}$ 是线性算子. 特别地, 当 $\psi = 1$ 时即为复合算子 T_φ .

单位圆盘上函数空间中的复合算子和加权复合算子的有界性和紧性问题已有非常广泛的研究, 得到了很好的结论, 如文献 [4-8] 讨论了单位圆盘上与 Zygmund 型空间有关的复合算子问题.

文献 [2, 9-11] 讨论了 Bergman 空间上的复合算子的有界性和紧性问题. 而对于单位球上 Bergman 空间上的复合算子, 文献 [12-16] 有了一些较好的结果. 本文受文献 [2, 7] 工作的启发, 将讨论 D 上的 μ -Bergman 空间到 Zygmund 型空间上的加权复合算子的有界性和紧性. 本文中 $z = (z_1, \dots, z_n)$, $\omega = (\omega_1, \dots, \omega_n)$, $\langle z, \omega \rangle = \sum_{j=1}^n z_j \bar{\omega}_j$, 我们将用记号 C, C_1, C_2, C_3 来表示与变量 z, ω 无关的正数, C, C_1, C_2, C_3 可以与某些范数或有界量有关, 不同的地方可以表示不同的正常数.

2 有关引理

引理 2.1^[11] 设 $0 < p < \infty, \forall f \in H(p, \mu), z \in D$, 有

$$|f(z)| \leq \frac{C\|f\|_{H(p, \mu)}}{\mu(|z|)(1 - |z|^2)^{\frac{1}{p}}}. \quad (2.1)$$

由文献 [17] 中的定理 2, 令 $m = 1, 2, p = q$, 则分别得以下结论:

引理 2.2 $0 < p < \infty, \mu$ 为 $[0, 1)$ 上的正规函数, $\forall f \in H(D)$, 则有

- (i) $\{\int_D |f(z)|^p \frac{\mu^p(|z|)}{1-|z|} dA(z)\}^{\frac{1}{p}} \approx |f(0)| + \{\int_D |f'(z)|^p (1 - |z|^2)^p \frac{\mu^p(|z|)}{1-|z|} dA(z)\}^{\frac{1}{p}}$,
(ii) $\{\int_D |f(z)|^p \frac{\mu^p(|z|)}{1-|z|} dA(z)\}^{\frac{1}{p}} \approx |f(0)| + |f'(0)| + \{\int_D |f''(z)|^p (1 - |z|^2)^{2p} \frac{\mu^p(|z|)}{1-|z|} dA(z)\}^{\frac{1}{p}}$.

设 $f \in H(p, \mu)$, 则由引理 2.1 和引理 2.2 可知, $\forall z \in D$, 有

$$|f'(z)| \leq \frac{C\|f\|_{H(p, \mu)}}{\mu(|z|)(1 - |z|^2)^{\frac{1}{p}+1}}, |f''(z)| \leq \frac{C\|f\|_{H(p, \mu)}}{\mu(|z|)(1 - |z|^2)^{\frac{1}{p}+2}}. \quad (2.2)$$

引理 2.3 设 $0 < p < \infty, \alpha > 0, f \in H(D)$, 则 $T_{\varphi, \psi}$ 是 $H(p, \mu)$ 空间到 Z^α 型空间上的紧算子的充要条件是对任意满足条件: (1) 在 $H(p, \mu)$ 中有界; (2) 在 D 的任一紧子集上一致收敛于 0 的序列 $\{f_j\}$, 都有

$$\|T_{\varphi, \psi}(f_j)\|_{H(p, \mu)} \rightarrow 0 \quad (j \rightarrow \infty).$$

证 由引理 2.1 和 Montel 定理按紧算子的定义可证.

3 主要结果

定理 3.1 设 $\varphi \in H(D, D)$, $\psi \in H(D)$, 则 $T_{\varphi, \psi} : H(p, \mu) \rightarrow Z^\alpha$ 为有界算子的充要条件是下列条件同时成立:

$$M = \sup_{z \in D} \frac{(1 - |z|^2)^\alpha |\psi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}}} < \infty, \quad (3.1)$$

$$N = \sup_{z \in D} \frac{(1 - |z|^2)^\alpha |2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+1}} < \infty, \quad (3.2)$$

$$K = \sup_{z \in D} \frac{(1 - |z|^2)^\alpha |\psi(z)(\varphi'(z))^2|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+2}} < \infty. \quad (3.3)$$

证 先证充分性 设 (3.1)–(3.3) 式成立, $\forall z \in D, f \in H(p, \mu)$, 由三角不等式和 (2.1)–(2.2) 式有:

$$\begin{aligned} (1 - |z|^2)^\alpha |(T_{\varphi, \psi} f)''(z)| &= (1 - |z|^2)^\alpha |[\psi'(z)f(\varphi(z)) + \psi(z)f'(\varphi(z))\varphi'(z)]'| \quad (3.4) \\ &\leq (1 - |z|^2)^\alpha \{ |\psi''(z)f(\varphi(z))| + |(2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z))f'(\varphi(z))| + |\psi(z)(\varphi'(z))^2 f''(\varphi(z))| \} \\ &\leq C(1 - |z|^2)^\alpha \left\{ \frac{|\psi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}}} + \frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+1}} \right. \\ &\quad \left. + \frac{|\psi(z)(\varphi'(z))^2|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+2}} \right\} \|f\|_{H(p, \mu)} \\ &\leq C(M + N + K) \|f\|_{H(p, \mu)}. \end{aligned}$$

另一方面,

$$|(T_{\varphi, \psi} f)(0)| = |\psi(0)f(\varphi(0))| \leq \frac{C|\psi(0)| \|f\|_{H(p, \mu)}}{\mu(|\varphi(0)|)(1 - |\varphi(0)|^2)^{\frac{1}{p}}}, \quad (3.5)$$

$$\begin{aligned} |(T_{\varphi, \psi} f)'(0)| &= |\psi'(0)f(\varphi(0)) + \psi(0)f'(\varphi(0))\varphi'(0)| \\ &\leq C \left[\frac{|\psi'(0)|}{\mu(|\varphi(0)|)(1 - |\varphi(0)|^2)^{\frac{1}{p}}} + \frac{|\psi(0)\varphi'(0)|}{\mu(|\varphi(0)|)(1 - |\varphi(0)|^2)^{\frac{1}{p}+1}} \right] \|f\|_{H(p, \mu)}. \quad (3.6) \end{aligned}$$

所以由 (3.4)–(3.6) 式知: $T_{\varphi, \psi} : H(p, \mu) \rightarrow Z^\alpha$ 为有界算子.

必要性 设 $T_{\varphi, \psi} : H(p, \mu) \rightarrow Z^\alpha$ 为有界算子, 则存在常数 C 使 $\forall f \in H(p, \mu)$, 有 $\|T_{\varphi, \psi}(f)\|_{Z^\alpha} \leq C \|f\|_{H(p, \mu)}$.

固定 $\omega \in D$ 及常数 a, b , 取

$$f_\omega(z) = \frac{a(1 - |\varphi(\omega)|^2)}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)z})^{\frac{1}{p}+1}} - \frac{(1 - |\varphi(\omega)|^2)^2}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)z})^{\frac{1}{p}+2}} + \frac{b(1 - |\varphi(\omega)|^2)^3}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)z})^{\frac{1}{p}+3}}. \quad (3.7)$$

由文献 [18] 中的引理 2.2 和文献 [19] 中的定理 1.12 有:

$$\|f_\omega\|_{H(p, \mu)}^p = \int_D |f_\omega(z)|^p \frac{\mu^p(|z|)}{1 - |z|} dA(z) \quad (3.8)$$

$$\begin{aligned}
&\leq C \int_D \left[\frac{|a|^p(1-|\varphi(\omega)|^2)^p}{\mu^p(|\varphi(\omega)|)|1-\overline{\varphi(\omega)}z|^{p+1}} + \frac{(1-|\varphi(\omega)|^2)^{2p}}{\mu^p(|\varphi(\omega)|)|1-\overline{\varphi(\omega)}z|^{1+2p}} + \frac{|b|^p(1-|\varphi(\omega)|^2)^{3p}}{\mu^p(|\varphi(\omega)|)|1-\overline{\varphi(\omega)}z|^{1+3p}} \right] \frac{\mu^p(|z|)}{1-|z|} dA(z) \\
&\leq C \int_D \left\{ \frac{(1-|z|^2)^{sp}}{(1-|\varphi(\omega)|^2)^{sp}} + \frac{(1-|z|^2)^{tp}}{(1-|\varphi(\omega)|^2)^{tp}} \right\} \left[\frac{|a|^p(1-|\varphi(\omega)|^2)^p}{|1-\overline{\varphi(\omega)}z|^{p+1}(1-|z|)} \right. \\
&\quad \left. + \frac{(1-|\varphi(\omega)|^2)^{2p}}{|1-\overline{\varphi(\omega)}z|^{1+2p}(1-|z|)} + \frac{|b|^p(1-|\varphi(\omega)|^2)^{3p}}{|1-\overline{\varphi(\omega)}z|^{1+3p}(1-|z|)} \right] dA(z) \\
&\leq 2C \{ |a|^p [(1-|\varphi(\omega)|^2)^{p-sp} \int_D \frac{(1-|z|^2)^{sp-1}}{|1-\overline{\varphi(\omega)}z|^{p+1}} dA(z) + (1-|\varphi(\omega)|^2)^{p-tp} \int_D \frac{(1-|z|^2)^{tp-1}}{|1-\overline{\varphi(\omega)}z|^{p+1}} dA(z)] \\
&\quad + [(1-|\varphi(\omega)|^2)^{2p-sp} \int_D \frac{(1-|z|^2)^{sp-1}}{|1-\overline{\varphi(\omega)}z|^{2p+1}} dA(z) + (1-|\varphi(\omega)|^2)^{2p-tp} \int_D \frac{(1-|z|^2)^{tp-1}}{|1-\overline{\varphi(\omega)}z|^{2p+1}} dA(z)] \\
&\quad + |b|^p [(1-|\varphi(\omega)|^2)^{3p-sp} \int_D \frac{(1-|z|^2)^{sp-1}}{|1-\overline{\varphi(\omega)}z|^{3p+1}} dA(z) + (1-|\varphi(\omega)|^2)^{3p-tp} \int_D \frac{(1-|z|^2)^{tp-1}}{|1-\overline{\varphi(\omega)}z|^{3p+1}} dA(z)] \} \\
&\approx 4C_1(|a|^p + 1 + |b|^p) \leq C.
\end{aligned}$$

所以由 (3.8) 式知 $f_\omega \in H(p, \mu)$, 且 $\|f_\omega\|_{H(p, \mu)} \leq C$. 直接计算有:

$$f'_\omega(z) = \overline{\varphi(\omega)} \left\{ \frac{a(\frac{1}{p} + 1)(1-|\varphi(\omega)|^2)}{\mu(|\varphi(\omega)|)(1-\overline{\varphi(\omega)}z)^{\frac{1}{p}+2}} - \frac{(\frac{1}{p} + 2)(1-|\varphi(\omega)|^2)^2}{\mu(|\varphi(\omega)|)(1-\overline{\varphi(\omega)}z)^{\frac{1}{p}+3}} + \frac{b(\frac{1}{p} + 3)(1-|\varphi(\omega)|^2)^3}{\mu(|\varphi(\omega)|)(1-\overline{\varphi(\omega)}z)^{\frac{1}{p}+4}} \right\}. \quad (3.9)$$

$$\begin{aligned}
f''_\omega(z) &= \overline{(\varphi(\omega))^2} \left\{ \frac{a(\frac{1}{p} + 1)(\frac{1}{p} + 2)(1-|\varphi(\omega)|^2)}{\mu(|\varphi(\omega)|)(1-\overline{\varphi(\omega)}z)^{\frac{1}{p}+3}} - \frac{(\frac{1}{p} + 2)(\frac{1}{p} + 3)(1-|\varphi(\omega)|^2)^2}{\mu(|\varphi(\omega)|)(1-\overline{\varphi(\omega)}z)^{\frac{1}{p}+4}} \right. \\
&\quad \left. + \frac{b(\frac{1}{p} + 3)(\frac{1}{p} + 4)(1-|\varphi(\omega)|^2)^3}{\mu(|\varphi(\omega)|)(1-\overline{\varphi(\omega)}z)^{\frac{1}{p}+5}} \right\}. \quad (3.10)
\end{aligned}$$

由 (3.7) 式和 (3.9)–(3.10) 式得:

$$f_\omega(\varphi(\omega)) = \frac{a-1+b}{\mu(|\varphi(\omega)|)(1-|\varphi(\omega)|^2)^{\frac{1}{p}}}, \quad (3.11)$$

$$f'_\omega(\varphi(\omega)) = \frac{\overline{\varphi(\omega)}[a(\frac{1}{p} + 1) - (\frac{1}{p} + 2) + b(\frac{1}{p} + 3)]}{\mu(|\varphi(\omega)|)(1-|\varphi(\omega)|^2)^{\frac{1}{p}+1}}, \quad (3.12)$$

$$f''_\omega(\varphi(\omega)) = \frac{\overline{(\varphi(\omega))^2}[a(\frac{1}{p} + 1)(\frac{1}{p} + 2) - (\frac{1}{p} + 2)(\frac{1}{p} + 3) + b(\frac{1}{p} + 3)(\frac{1}{p} + 4)]}{\mu(|\varphi(\omega)|)(1-|\varphi(\omega)|^2)^{\frac{1}{p}+2}}. \quad (3.13)$$

设 $v = \frac{1}{p} + 1$, 则由 $av - (v+1) + b(v+2) = 0$, $av(v+1) - (v+1)(v+2) + b(v+2)(v+3) = 0$ 得: $a = \frac{v+1}{2v}$, $b = \frac{v+1}{2v+4}$, 此时 $a-1+b = \frac{1}{v(v+2)} = \frac{p^2}{(1+p)(1+3p)}$. 由 (3.11)–(3.13) 式有

$$f_\omega(\varphi(\omega)) = \frac{\frac{p^2}{(1+p)(1+3p)}}{\mu(|\varphi(\omega)|)(1-|\varphi(\omega)|^2)^{\frac{1}{p}}}, f'_\omega(\varphi(\omega)) = f''_\omega(\varphi(\omega)) = 0. \quad (3.14)$$

由 $T_{\varphi, \psi}$ 是 $H(p, \mu)$ 到 Z^α 的有界算子及 (3.14) 式知

$$C \geq \|(T_{\varphi, \psi} f_\omega)(z)\|_{Z^\alpha} \geq (1-|\omega|^2)^\alpha |(T_{\varphi, \psi} f_\omega)''(\omega)| = (1-|\omega|^2)^\alpha \frac{|\psi''(\omega)| \frac{p^2}{(1+p)(1+3p)}}{\mu(|\varphi(\omega)|)(1-|\varphi(\omega)|^2)^{\frac{1}{p}}}.$$

即

$$\frac{(1 - |\omega|^2)^\alpha |\psi''(\omega)|}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}}} \leq C. \quad (3.15)$$

由 ω 的任意性及 (3.15) 式知 (3.1) 式成立.

下面证明 (3.2) 式成立. 固定 $\omega \in D$ 及常数 c, d , 取

$$g_\omega(z) = -\frac{1 - |\varphi(\omega)|^2}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+1}} + \frac{c(1 - |\varphi(\omega)|^2)^2}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+2}} + \frac{d(1 - |\varphi(\omega)|^2)^3}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+3}}. \quad (3.16)$$

类似 $f_\omega(z)$ 的证明知 $g_\omega \in H(p, \mu)$, 且 $\|g_\omega\|_{H(p, \mu)} \leq C$. 直接计算得

$$g'_\omega(z) = \frac{\overline{\varphi(\omega)}}{\mu(|\varphi(\omega)|)} \left[-\frac{(\frac{1}{p} + 1)(1 - |\varphi(\omega)|^2)}{(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+2}} + \frac{c(\frac{1}{p} + 2)(1 - |\varphi(\omega)|^2)^2}{(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+3}} + \frac{d(\frac{1}{p} + 3)(1 - |\varphi(\omega)|^2)^3}{(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+4}} \right], \quad (3.17)$$

$$g''_\omega(z) = \frac{(\overline{\varphi(\omega)})^2}{\mu(|\varphi(\omega)|)} \left[-\frac{(\frac{1}{p} + 1)(\frac{1}{p} + 2)(1 - |\varphi(\omega)|^2)}{(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+3}} + \frac{c(\frac{1}{p} + 2)(\frac{1}{p} + 3)(1 - |\varphi(\omega)|^2)^2}{(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+4}} + \frac{d(\frac{1}{p} + 3)(\frac{1}{p} + 4)(1 - |\varphi(\omega)|^2)^3}{(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+5}} \right]. \quad (3.18)$$

且由 (3.16)–(3.18) 式有

$$g_\omega(\varphi(\omega)) = \frac{c + d - 1}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}}}, \quad (3.19)$$

$$g'_\omega(\varphi(\omega)) = \frac{\overline{\varphi(\omega)} \left[-(\frac{1}{p} + 1) + c(\frac{1}{p} + 2) + d(\frac{1}{p} + 3) \right]}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}}, \quad (3.20)$$

$$g''_\omega(\varphi(\omega)) = \frac{(\overline{\varphi(\omega)})^2 \left[-(\frac{1}{p} + 1)(\frac{1}{p} + 2) + c(\frac{1}{p} + 2)(\frac{1}{p} + 3) + d(\frac{1}{p} + 3)(\frac{1}{p} + 4) \right]}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+2}}. \quad (3.21)$$

设 $v = \frac{1}{p} + 1$, 则由 $c + d - 1 = 0$, $-v(v + 1) + c(v + 1)(v + 2) + d(v + 2)(v + 3) = 0$ 得 $c = \frac{2v+3}{v+2}$, $d = -\frac{v+1}{v+2}$, 此时 $-v + c(v + 1) + d(v + 2) = \frac{1}{v+2} = \frac{p}{1+3p}$. 由 (3.19)–(3.21) 式有

$$g'_\omega(\varphi(\omega)) = \frac{\frac{p}{(1+3p)} \overline{\varphi(\omega)}}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}}}, \quad g_\omega(\varphi(\omega)) = g''_\omega(\varphi(\omega)) = 0. \quad (3.22)$$

由 $T_{\varphi, \psi}$ 是 $H(p, \mu)$ 到 Z^α 的有界算子及 (3.22) 式知

$$\begin{aligned} C &\geq \|(T_{\varphi, \psi} g_\omega)(z)\|_{Z^\alpha} \geq (1 - |\omega|^2)^\alpha |(T_{\varphi, \psi} g_\omega)''(\omega)| \\ &= (1 - |\omega|^2)^\alpha |2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega)| |g'_\omega(\varphi(\omega))| \\ &= \frac{(1 - |\omega|^2)^\alpha |2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega)| |\varphi(\omega)|^{\frac{p}{(1+3p)}}}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}}. \end{aligned} \quad (3.23)$$

当 $|\varphi(\omega)| > \frac{1}{2}$ 时, 由 (3.23) 式有

$$\frac{(1 - |\omega|^2)^\alpha |2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega)|}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}} \leq C. \quad (3.24)$$

当 $|\varphi(\omega)| \leq \frac{1}{2}$ 时, 取 $g_\omega(z) = \frac{z}{\mu(|\varphi(\omega)|)}$, 由文献 [18] 中的引理 2.2, 直接计算得

$$\begin{aligned} \|g_\omega\|_{H(p,\mu)}^p &= \int_D \frac{|z|^p \mu^p(|z|)}{\mu^p(|\varphi(\omega)|)(1 - |z|)} dA(z) \\ &\leq 2C \int_D \left[\frac{(1 - |z|^2)^{sp-1}}{(1 - |\varphi(\omega)|^2)^{sp}} + \frac{(1 - |z|^2)^{tp-1}}{(1 - |\varphi(\omega)|^2)^{tp}} \right] dA(z) \\ &\leq \frac{2C \cdot 2\pi}{(1 - |\varphi(\omega)|^2)^{sp}} \int_0^1 (1 - r^2)^{sp-1} r dr + \frac{2C \cdot 2\pi}{(1 - |\varphi(\omega)|^2)^{tp}} \int_0^1 (1 - r^2)^{tp-1} r dr \\ &= \frac{2\pi C}{sp(1 - |\varphi(\omega)|^2)^{sp}} + \frac{2\pi C}{tp(1 - |\varphi(\omega)|^2)^{tp}} \leq C, \end{aligned}$$

所以 $g_\omega \in H(p, \mu)$, 且 $\|g_\omega\|_{H(p,\mu)} \leq C$.

因为 $T_{\varphi,\psi}$ 是 $H(p, \mu)$ 到 Z^α 的有界算子, 且 $g_\omega(\varphi(\omega)) = \frac{\varphi(\omega)}{\mu(|\varphi(\omega)|)}$, $g'_\omega(\varphi(\omega)) = \frac{1}{\mu(|\varphi(\omega)|)}$, $g''_\omega(\varphi(\omega)) = 0$, 所以

$$C \geq \|(T_{\varphi,\psi} g_\omega)\|_{Z^\alpha} \geq (1 - |\omega|^2)^\alpha |\psi''(\omega)g_\omega(\varphi(\omega)) + (2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega))g'_\omega(\varphi(\omega))|.$$

从而由三角不等式有

$$\frac{(1 - |\omega|^2)^\alpha |(2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega))|}{\mu(|\varphi(\omega)|)} \leq C + \frac{(1 - |\omega|^2)^\alpha |\psi''(\omega)\varphi(\omega)|}{\mu(|\varphi(\omega)|)}.$$

上式两边同乘以 $\frac{1}{(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}}$, 结合 (3.1) 式及 $\frac{1}{1 - |\varphi(\omega)|^2} \leq \frac{4}{3}$ 有

$$\begin{aligned} &\frac{(1 - |\omega|^2)^\alpha |(2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega))|}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}} \\ &\leq \frac{C}{(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}} + \frac{(1 - |\omega|^2)^\alpha |\psi''(\omega)|}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}}} \frac{|\varphi(\omega)|}{(1 - |\varphi(\omega)|^2)} \leq C\left(\frac{4}{3}\right)^{\frac{1}{p}+1} + \frac{2}{3}M \leq C. \end{aligned} \quad (3.25)$$

结合 (3.24)–(3.25) 式知 (3.2) 式成立.

最后证明 (3.3) 式成立. 当 $|\varphi(\omega)| > \frac{1}{2}$ 时, 取

$$h_\omega(z) = \frac{e(1 - |\varphi(\omega)|^2)}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+1}} + \frac{f(1 - |\varphi(\omega)|^2)^2}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+2}} - \frac{(1 - |\varphi(\omega)|^2)^3}{\mu(|\varphi(\omega)|)(1 - \overline{\varphi(\omega)}z)^{\frac{1}{p}+3}}.$$

易知 $h_\omega \in H(p, \mu)$, 且 $\|h_\omega\|_{H(p,\mu)} \leq C$. 直接计算得

$$\begin{aligned} h_\omega(\varphi(\omega)) &= \frac{e + f - 1}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}}}, \quad h'_\omega(\varphi(\omega)) = \frac{\overline{\varphi(\omega)}[e(\frac{1}{p} + 1) + f(\frac{1}{p} + 2) - (\frac{1}{p} + 3)]}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+1}}, \\ h''_\omega(\varphi(\omega)) &= \frac{(\overline{\varphi(\omega)})^2[e(\frac{1}{p} + 1)(\frac{1}{p} + 2) + f(\frac{1}{p} + 2)(\frac{1}{p} + 3) - (\frac{1}{p} + 3)(\frac{1}{p} + 4)]}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p}+2}}. \end{aligned}$$

设 $v = \frac{1}{p} + 1$, 则由 $e + f - 1 = 0, ev + f(v + 1) - (v + 2) = 0$ 得 $e = -1, f = 2$, 此时 $e(\frac{1}{p} + 1)(\frac{1}{p} + 2) + f(\frac{1}{p} + 2)(\frac{1}{p} + 3) - (\frac{1}{p} + 3)(\frac{1}{p} + 4) = -2, h_\omega(\varphi(\omega)) = h'_\omega(\varphi(\omega)) = 0, h''_\omega(\varphi(\omega)) = \frac{-2(\varphi(\omega))^2}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p} + 2}}$. 因为 $T_{\varphi, \psi}$ 是 $H(p, \mu)$ 到 Z^α 的有界算子, 所以

$$C \geq \|(T_{\varphi, \psi} h_\omega)\|_{Z^\alpha} \geq \frac{(1 - |\omega|^2)^\alpha |\psi(\omega)(\varphi'(\omega))^2| |\varphi(\omega)|^2}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p} + 2}},$$

即

$$\frac{(1 - |\omega|^2)^\alpha |\psi(\omega)(\varphi'(\omega))^2|}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p} + 2}} \leq C. \quad (3.26)$$

当 $|\varphi(\omega)| \leq \frac{1}{2}$ 时, 取 $h_\omega(z) = \frac{z^2}{\mu(|\varphi(\omega)|)}$, 易知 $h_\omega \in H(p, \mu)$, 且 $\|h_\omega\|_{H(p, \mu)} \leq C$.

因为 $T_{\varphi, \psi}$ 是 $H(p, \mu)$ 到 Z^α 的有界算子, 所以

$$C \geq \|(T_{\varphi, \psi} h_\omega)\|_{Z^\alpha} \quad (3.27)$$

$$\geq (1 - |\omega|^2)^\alpha |\psi''(\omega)h_\omega(\varphi(\omega)) + (2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega))h'_\omega(\varphi(\omega)) + \psi(\omega)(\varphi'(\omega))^2 h''_\omega(\varphi(\omega))|.$$

又由 $h_\omega(\varphi(\omega)) = \frac{(\varphi(\omega))^2}{\mu(|\varphi(\omega)|)}, g'_\omega(\varphi(\omega)) = \frac{2\varphi(\omega)}{\mu(|\varphi(\omega)|)}, g''_\omega(\varphi(\omega)) = \frac{2}{\mu(|\varphi(\omega)|)}$ 及三角不等式和 (3.27) 式有

$$\begin{aligned} & \frac{2(1 - |\omega|^2)^\alpha |\psi(\omega)(\varphi'(\omega))^2|}{\mu(|\varphi(\omega)|)} \\ & \leq C + \frac{(1 - |\omega|^2)^\alpha |\psi''(\omega)||\varphi(\omega)|^2}{\mu(|\varphi(\omega)|)} + \frac{2(1 - |\omega|^2)^\alpha |(2\psi'(\omega)\varphi'(\omega) + \psi(\omega)\varphi''(\omega))||\varphi(\omega)|}{\mu(|\varphi(\omega)|)}. \end{aligned}$$

上式两边同乘以 $\frac{1}{(1 - |\varphi(\omega)|^2)^{\frac{1}{p} + 2}}$, 结合 (3.1)–(3.2) 式及 $\frac{1}{1 - |\varphi(\omega)|^2} \leq \frac{4}{3}$ 有

$$\frac{(1 - |\omega|^2)^\alpha |\psi(\omega)||\varphi'(\omega)|^2}{\mu(|\varphi(\omega)|)(1 - |\varphi(\omega)|^2)^{\frac{1}{p} + 2}} \leq C. \quad (3.28)$$

结合 (3.26) 式和 (3.28) 式知 (3.3) 式成立. 定理 3.1 证毕.

定理 3.2 设 $\varphi \in H(D, D), \psi \in H(D)$, 则 $T_{\varphi, \psi} : H(p, \mu) \rightarrow Z^\alpha$ 为紧算子的充要条件是 $T_{\varphi, \psi}$ 为有界算子且下列条件同时成立:

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |\psi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}}} = 0, \quad (3.29)$$

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p} + 1}} = 0, \quad (3.30)$$

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |\psi(z)(\varphi'(z))^2|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p} + 2}} = 0. \quad (3.31)$$

证 先证充分性 假设 (3.29)–(3.31) 式同时成立, 则 $\forall \varepsilon > 0$, 存在 $0 < \delta < 1$, 当 $\delta < |\varphi(z)| < 1$ 时, 有

$$\frac{(1 - |z|^2)^\alpha}{\mu(|\varphi(z)|)} \left[\frac{|\psi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}}} + \frac{|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + 1}} + \frac{|\psi(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + 2}} \right] < \varepsilon. \quad (3.32)$$

由 $T_{\varphi, \psi}$ 为有界算子及 (3.1) 式有

$$\begin{aligned} (1 - |z|^2)^\alpha |\psi''(z)| &\leq \frac{(1 - |z|^2)^\alpha |\psi''(z)| \mu(|\varphi(z)|)}{(1 - |\varphi(z)|)^s \mu(|\varphi(z)|)} \\ &\leq \frac{(1 - |z|^2)^\alpha |\psi''(z)|}{\mu(|\varphi(z)|)} \frac{\mu(\delta)}{(1 - \delta)^s} \leq \frac{(1 - |z|^2)^\alpha |\psi''(z)|}{\mu(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}} (1 - \delta)^s} \leq C. \end{aligned} \quad (3.33)$$

同理由 (3.2) 式和 (3.3) 式有

$$(1 - |z|^2)^\alpha |2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)| \leq C, (1 - |z|^2)^\alpha |\psi(z)(\varphi'(z))^2| \leq C. \quad (3.34)$$

设 $\{f_j\}$ 是在 D 的任一紧子集上一致趋于 0 且满足 $\|f_j\|_{H(p, \mu)} \leq 1$ 的任一解析函数序列, 则 $\{f'_j\}$ 和 $\{f''_j\}$ 在 $E = \{\omega : |\omega| \leq \delta\}$ 上一致收敛于 0. 由 (3.33) 式和 (3.34) 式知, 存在正整数 N , 当 $j > N$ 时, 有

$$\begin{aligned} & |(T_{\varphi, \psi} f_j)(0)| + |(T_{\varphi, \psi} f_j)'(0)| + \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^\alpha |(T_{\varphi, \psi} f_j)''(z)| \\ & \leq |\psi(0)| \|f_j(\varphi(0))\| + |\psi'(0)| \|f_j(\varphi(0))\| + |\psi(0)| \|f'_j(\varphi(0))\| |\varphi'(0)| \\ & \quad + \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^\alpha [|\psi''(z) f_j(\varphi(z))| + |2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)| \\ & \quad \times |f'_j(\varphi(z))| + |\psi(z)(\varphi'(z))^2| \|f''_j(\varphi(z))\|] \\ & \leq |\psi(0)| \|f_j(\varphi(0))\| + |\psi'(0)| \|f_j(\varphi(0))\| + |\psi(0)| \|f'_j(\varphi(0))\| |\varphi'(0)| \\ & \quad + C \left[\sup_{|\varphi(z)| \leq \delta} |f_j(\varphi(z))| + \sup_{|\varphi(z)| \leq \delta} |f'_j(\varphi(z))| + \sup_{|\varphi(z)| \leq \delta} |f''_j(\varphi(z))| \right] \\ & \leq C\varepsilon. \end{aligned} \quad (3.35)$$

由 (3.32) 式和 (3.35) 式可得, 当 $j > N$ 时, 有

$$\begin{aligned} \|T_{\varphi, \psi} f_j\|_{Z^\alpha} &= |(T_{\varphi, \psi} f_j)(0)| + |(T_{\varphi, \psi} f_j)'(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |(T_{\varphi, \psi} f_j)''(z)| \\ &\leq [|\psi(0)| + |\psi'(0)|] \|f_j(\varphi(0))\| + |\psi(0)| |\varphi'(0)| \|f'_j(\varphi(0))\| \\ &\quad + \sup_{\delta < |\varphi(z)| < 1} (1 - |z|^2)^\alpha |(T_{\varphi, \psi} f_j)''(z)| + \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^\alpha |(T_{\varphi, \psi} f_j)''(z)| \\ &\leq C\varepsilon + C \|f_j\|_{H(p, \mu)} \varepsilon \leq 2C\varepsilon. \end{aligned}$$

由 ε 的任意性知 $\lim_{j \rightarrow \infty} \|T_{\varphi, \psi} f_j\|_{Z^\alpha} = 0$, 由引理 2.3 知 $T_{\varphi, \psi}$ 为 $H(p, \mu)$ 到 Z^α 的紧算子.

必要性 设 $T_{\varphi, \psi}$ 为 $H(p, \mu)$ 到 Z^α 的紧算子, 则 $T_{\varphi, \psi}$ 为 $H(p, \mu)$ 到 Z^α 的有界算子.

先证 (3.29) 成立. 对于 D 中满足 $\lim_{j \rightarrow \infty} |\varphi(z_j)| = 1$ 的任意序列 $\{z_j\}$, ($j = 1, 2, \dots$), 取

$$f_j(z) = \frac{a(1 - |\varphi(z_j)|^2)}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+1}} - \frac{(1 - |\varphi(z_j)|^2)^2}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+2}} + \frac{b(1 - |\varphi(z_j)|^2)^3}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+3}}.$$

类似 (3.7) 式中的讨论知 $\|f_j\|_{H(p, \mu)} \leq C$, $f_j \in H(p, \mu)$, 且当 $a = \frac{1+2p}{2+2p}$, $b = \frac{1+2p}{2+6p}$ 时,

$$f_j(\varphi(z_j)) = \frac{p^2}{\overline{(1+p)(1+3p)}} \frac{1}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}}}, f'_j(\varphi(z_j)) = f''_j(\varphi(z_j)) = 0.$$

下面证明 $\{f_j\}$ 在 D 的任一紧子集上一致收敛于 0. 对任意 $0 < r < 1$, 有

$$\sup_{|z| \leq r} |f_j(z)| \leq \sup_{|z| \leq r} \frac{C(1 - |\varphi(z_j)|^2)}{\mu(|\varphi(z_j)|)(1 - |z|)^{\frac{1}{p}+1}} \leq \frac{C_1(1 - |\varphi(z_j)|^2)}{(1 - r)^{\frac{1}{p}+1}} \rightarrow 0, \quad (j \rightarrow \infty).$$

所以 $\{f_j\}$ 在 D 的任一紧子集上一致收敛于 0. 由引理 2.3 有 $\lim_{j \rightarrow \infty} \|(T_{\varphi, \psi} f_j)\|_{Z^\alpha} = 0$, 即 $\forall \varepsilon > 0$, 存在正整数 N , 当 $j > N$ 时, 有

$$C\varepsilon \geq \|(T_{\varphi, \psi} f_j)\|_{Z^\alpha} \geq (1 - |z_j|^2)^\alpha |\psi''(z_j) f_j(\varphi(z_j))| = \frac{\frac{p^2}{(1+p)(1+3p)} (1 - |z_j|^2)^\alpha |\psi''(z_j)|}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}}}.$$

由此及 $\lim_{j \rightarrow \infty} |\varphi(z_j)| = 1$ 得

$$\lim_{|\varphi(z_j)| \rightarrow 1^-} \frac{(1 - |z_j|^2)^\alpha |\psi''(z_j)|}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}}} = 0.$$

所以 (3.29) 式成立.

下面证明 (3.30) 式成立. 取

$$g_j(z) = -\frac{1 - |\varphi(z_j)|^2}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+1}} + \frac{\frac{2+5p}{1+3p}(1 - |\varphi(z_j)|^2)^2}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+2}} + \frac{\frac{-(1+2p)}{1+3p}(1 - |\varphi(z_j)|^2)^3}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+3}}.$$

类似 (3.16) 式中的讨论知 $\|g_j\|_{H(p, \mu)} \leq C$, $g_j \in H(p, \mu)$, 且 $g_j(\varphi(z_j)) = g_j''(\varphi(z_j)) = 0$, $g_j'(\varphi(z_j)) = \frac{\frac{p}{1+3p}\overline{\varphi(z_j)}}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}+1}}$.

显然, $\{g_j\}$ 在 D 的任一紧子集上一致收敛于 0. 由引理 2.3 有 $\lim_{j \rightarrow \infty} \|(T_{\varphi, \psi} g_j)\|_{Z^\alpha} = 0$, 即 $\forall \varepsilon > 0$, 存在正整数 N , 当 $j > N$ 时, 有

$$\begin{aligned} C\varepsilon \geq \|(T_{\varphi, \psi} g_j)\|_{Z^\alpha} &\geq (1 - |z_j|^2)^\alpha |2\psi'(z_j)\varphi'(z_j) + \psi(z_j)\varphi''(z_j)g_j'(\varphi(z_j))| \\ &\geq \frac{C_1|\overline{\varphi(z_j)}|(1 - |z_j|^2)^\alpha |2\psi'(z_j)\varphi'(z_j) + \psi(z_j)\varphi''(z_j)|}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}+1}}. \end{aligned}$$

由此及 $\lim_{j \rightarrow \infty} |\varphi(z_j)| = 1, \frac{1}{2} < |\varphi(z_j)| \rightarrow 1$ 得

$$\lim_{|\varphi(z_j)| \rightarrow 1^-} \frac{(1 - |z_j|^2)^\alpha |2\psi'(z_j)\varphi'(z_j) + \psi(z_j)\varphi''(z_j)|}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}+1}} = 0.$$

所以 (3.30) 式成立.

最后证 (3.31) 式成立. 取

$$h_j(z) = -\frac{1 - |\varphi(z_j)|^2}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+1}} + \frac{2(1 - |\varphi(z_j)|^2)^2}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+2}} + \frac{-(1 - |\varphi(z_j)|^2)^3}{\mu(|\varphi(z_j)|)(1 - \overline{\varphi(z_j)}z)^{\frac{1}{p}+3}}.$$

类似对定理 3.1 中的 h_ω 的讨论知 $\|h_j\|_{H(p, \mu)} \leq C$, $h_j \in H(p, \mu)$, 且 $h_j(\varphi(z_j)) = h_j'(\varphi(z_j)) = 0$, $h_j''(\varphi(z_j)) = \frac{-2\overline{\varphi(z_j)}^2}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}+2}}$.

易知, $\{h_j\}$ 在 D 的任一紧子集上一致收敛于 0. 由引理 2.3 有 $\lim_{j \rightarrow \infty} \|(T_{\varphi, \psi} h_j)\|_{Z^\alpha} = 0$, 即 $\forall \varepsilon > 0$, 存在正整数 N , 当 $j > N$ 时, 有

$$\begin{aligned} C\varepsilon &\geq \|(T_{\varphi, \psi} h_j)\|_{Z^\alpha} \geq (1 - |z_j|^2)^\alpha |\psi(z_j)(\varphi'(z_j))^2| |h_j''(\varphi(z_j))| \\ &= \frac{2|\overline{\varphi(z_j)}|^2 (1 - |z_j|^2)^\alpha |\psi(z_j)(\varphi'(z_j))^2|}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}+2}}. \end{aligned}$$

由此及 $\lim_{j \rightarrow \infty} |\varphi(z_j)| = 1, \frac{1}{2} < |\varphi(z_j)| \rightarrow 1$ 得

$$\lim_{|\varphi(z_j)| \rightarrow 1^-} \frac{(1 - |z_j|^2)^\alpha |\psi(z_j)(\varphi'(z_j))^2|}{\mu(|\varphi(z_j)|)(1 - |\varphi(z_j)|^2)^{\frac{1}{p}+2}} = 0.$$

所以 (3.31) 式成立. 定理 3.2 证毕.

参 考 文 献

- [1] Shields A, Williams D. Bounded projections, duality, and multipliers in spaces of analytic functions[J]. Trans. Amer. Math. Soc., 1971, 162: 287–302.
- [2] 江治杰. Bergman 型空间到 Bers 型空间之加权复合算子 [J]. 数学学报中文版, 2010(1), 53(1): 67–74.
- [3] Duren P. Theory of H^p space[M]. New York: Academic Press, 1970.
- [4] Li S, Stević S. Weighted composition operator from Zygmund spaces into Bloch spaces[J]. Appl. Math. Comput., 2008, 206(2): 825–831.
- [5] Li S, Stević S. Generalized composition operator on Zygmund spaces and Bloch type spaces[J]. J. Math. Anal. Appl., 2008, 338: 1282–1295.
- [6] Liu Y, Yu Y. Weighted differentiation composition operators from mixed-norm to Zygmund spaces[J]. Numer. Funct. Anal. Optim., 2010, 31(8): 936–954.
- [7] 刘永民, 于燕燕. 从 Hardy 空间到 Zygmund-型空间的加权微分复合算子 [J]. 数学年刊, 2014, 35A(4): 399–412.
- [8] 韩秀, 徐辉明. Besov 空间和 Zygmund 空间上的复合算子 [J]. 数学研究, 2009(9), 42(3): 310–319.
- [9] Li S X. Weighted composition operators between different weighted Bergman spaces[J]. J. of jia Ying University, 2004, 22(6): 5–8.
- [10] 唐笑敏, 胡璋剑. Bergman 空间和 q -Bloch 空间之间的复合算子 [J]. 数学年刊, 2006, 27A(1): 109–118.
- [11] Li S X, Stević S. Weighted composition operators from Bergman-type spaces into Bloch spaces[J]. Proc. Indian Acad. Sci. (Math. Sci.), 2007, 117(3): 371–385.
- [12] Zhang X J. Composition type operator from Bergman space to μ -Bloch type space in C^n [J]. J. Math. Anal. Appl., 2004, 298(): 710–721.
- [13] 张学军, 刘竞成. 加权 Bergman 空间到 μ -Bloch 空间的复合算子 [J]. 数学年刊, 2007, 28A(2): 255–266.
- [14] 张学军, 肖建斌, 李敏, 关莹. 单位球上加权 Bergman 空间的复合型算子 [J]. 中国科学: 数学, 2015, 45(2): 141–150.
- [15] 高进寿, 贾厚玉. Bergman 空间上复合算子的紧性刻划 [J]. 高校应用数学学报 A 辑, 2003, 18(4): 446–454.
- [16] Li S L, Zhang X J, Xu S. The compact composition operator on the μ -Bergman space in the unit ball[J]. Acta. Math. Sci., 2017, 37B(2): 425–438.

- [17] Hu Z J. Extended Cesàro operators on mixed norm spaces[J]. Proc. Amer. Math. Soc., 2002, 131(7): 2171–2179.
- [18] Zhang X J, Xi L H, Fan H X, Li J F. Atomic decomposition of μ -Bergman space in C^n [J]. Acta. Math. Scientia, 2014, 34B(3): 779–789.
- [19] Zhu K H. Spaces of Holomorphic functions in the unit ball[M]. New York: Springer-verlag, 2004.

WEIGHTED COMPOSITION OPERATOR FROM μ -BERGMAN SPACES TO ZYGMUND TYPE SPACES

ZHAO Yan-hui, LIAO Chun-yan, TANG Ya-lin

(College of Science, Hunan University of Science and Engineering, Yongzhou 425199)

Abstract: Boundedness and compactness of weighted composition operators from μ -Bergman space to Zygmund type spaces were studied in this paper. By the methods of functional analysis and several complex variables, the necessary and sufficient conditions are given for weighted composition operator to be bounded and compact from μ -Bergman space to Zygmund type spaces in the disk.

Keywords: μ -Bergman space; Zygmund type spaces; Weighted composition operator; Bounded; Compact

2010 MR Subject Classification: 47B38, 47B33; 30H05; 46E15