

# INTERPOLATION OF LORENTZ MARTINGALE SPACES WITH VARIABLE EXPONENTS

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**Abstract:** In this paper, we apply function parameters to real interpolation of Lorentz martingale spaces with variable exponents. Some new interpolation theorems concerning variable exponents Lorentz martingale spaces are formulated. The results that we obtain generalize some fundamental interpolation theorems in classical martingale  $H_p$  theory.

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## 1 Introduction

The real interpolation spaces  $\bar{A}_{\theta,q}$  were introduced in [1], and the theory of the spaces  $\bar{A}_{\theta,q}$  has been applied as a powerful tool to many branches of mathematics. These spaces are defined by using the function norm:

$$\phi_{\theta,q} = \left( \int_0^\infty (t^{-\theta}\varphi(t))^q \frac{dt}{t} \right)^{1/q}.$$

For the further applications of interpolation space theory, the idea of replacing  $\bar{A}_{\theta,q}$  by a more general function norm to obtain more general interpolation spaces appeared. The initial work on such spaces was due to Kalugina [2]. He used the function norm

$$\phi_{f,q} = \left( \int_0^\infty (\varphi(t)/f(t))^q \frac{dt}{t} \right)^{1/q}$$

to replace  $\phi_{\theta,q}$ , where  $f$  is a function parameter, which belongs to the function class  $B_k$ . Later on, the theory of interpolation with a function parameter has been developed in an astounding way. For example see [3-7].

Interpolation of martingale Hardy spaces is one of the main topics in martingale  $H_p$  theory, and its theory has been successfully applied to Fourier analysis. In classical martingale  $H_p$  theory, it was proved by Weisz [8] and Long [9] that the interpolation spaces of martingale Hardy spaces were martingale Hardy-Lorentz spaces. Recently, Jiao [10,11] studied real

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interpolation of some weak martingale spaces, and Fan [12] considered real interpolation of some Lorentz martingale spaces. Ren and Guo[13] applied function parameters to consider the interpolation with a function parameter for Lorentz martingale spaces. Motivated by [13], the aim of this paper is to consider the interpolation with a function parameter for Lorentz martingale spaces with variable exponents.

## 2 Preliminaries

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space, and  $\mathcal{F}_n$  be a nondecreasing sequence of sub- $\sigma$ -algebra of  $\mathcal{F}$  such that  $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$ , where  $\mathcal{F}_n$  is generated by countably many atoms. The conditional expectation operators relative to  $\mathcal{F}_n$  are denoted by  $E_n$ .

For  $0 < q \leq \infty$ , we use  $L_q^*$  to denote the Lebesgue space  $L_q^*(0, \infty; dt/t)$ , which is the collection of all measurable functions  $f$  such that

$$\|f\|_{L_q^*} = \begin{cases} \left( \int_0^\infty |f(t)|^q \frac{dt}{t} \right)^{1/q}, & q < \infty, \\ \sup_{t>0} |f(t)|, & q = \infty. \end{cases}$$

Let  $p(\cdot) : \Omega \rightarrow (0, \infty)$  be an  $\mathcal{F}$ -measurable function, we define

$$p^- = \text{ess inf}\{p(x) : x \in \Omega\}, \quad p^+ = \text{ess sup}\{p(x) : x \in \Omega\}.$$

Let  $\mathcal{P}(\Omega)$  denote the collection of all  $\mathcal{F}$ -measurable functions  $p(\cdot) : \Omega \rightarrow (0, \infty)$  such that  $0 < p_- \leq p_+ < \infty$ . The Lebesgue space with variable exponent  $p(\cdot)$  denoted by  $L_{p(\cdot)}$  is defined as the set of all  $\mathcal{F}$ -measurable functions  $f$  satisfying

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho(|f(x)|/\lambda) \leq 1 \right\} < \infty,$$

where

$$\rho(f) = \int_{\Omega} |f(x)|^{p(x)} dP.$$

Let  $p(\cdot) \in \mathcal{P}(\Omega)$  and  $0 < q \leq \infty$ . Then  $L_{p(\cdot), q}(\Omega)$  is the collection of all measurable functions  $f$  such that

$$\|f\|_{L_{p(\cdot), q}} = \begin{cases} \left( \int_0^\infty \lambda^q \|\chi_{\{|f|>\lambda\}}\|_{p(\cdot)}^q \frac{d\lambda}{\lambda} \right)^{1/q}, & q < \infty, \\ \sup_{\lambda} \lambda \|\chi_{\{|f|>\lambda\}}\|_{p(\cdot)}, & q = \infty. \end{cases}$$

For measurable function  $f, t > 0$ , define

$$h(t) = \|\chi_{\{|f|>t\}}\|_{p(\cdot)}, \quad f_*(t) = \sup\{\lambda > 0 : h(\lambda) \geq t\}.$$

Let  $\varphi$  be a nonnegative and local integrable function on  $[0, \infty)$  ( $\varphi \neq 0$ ), and the Lorentz spaces with variable exponents are defined as

$$\Lambda_q(\varphi) = \left\{ f : \|f\|_{\Lambda_q(\varphi)} = \|f_*(t)\varphi(t)\|_{L_q^*} < \infty \right\}, \quad (0 < q \leq \infty). \quad (2.1)$$

For a complex valued martingale  $f = (f_n)_{n \geq 0}$  relative to  $(\Omega, \mathcal{F}, P; (\mathcal{F}_n)_{n \geq 0})$ , denote its martingale difference by  $df_i = f_i - f_{i-1}$  (with convention  $df_{-1} = 0$ ) and define the maximal function, the square function and the conditional square function of  $f$  respectively as follows:

$$\begin{aligned} M_n(f) &= \sup_{0 \leq k \leq n} |f_k|, & M(f) &= \sup_{n \geq 0} |f_n|, \\ S_n(f) &= \left( \sum_{i=0}^n |df_i|^2 \right)^{1/2}, & S(f) &= \left( \sum_{i=0}^{\infty} |df_i|^2 \right)^{1/2}, \\ \sigma_n(f) &= \left( \sum_{i=0}^n E_{i-1}(|df_i|^2) \right)^{\frac{1}{2}}, & \sigma(f) &= \left( \sum_{i=0}^{\infty} E_{i-1}(|df_i|^2) \right)^{\frac{1}{2}}. \end{aligned}$$

Let  $\Lambda$  be the collection of all sequences  $(\lambda_n)_{n \geq 0}$  of nondecreasing, nonnegative and adapted functions, set  $\lambda_\infty = \lim_{n \rightarrow \infty} \lambda_n$ . Thus the Lorentz martingale spaces with variable exponents are defined as follows.

$$\begin{aligned} \hat{\Lambda}_q(\varphi) &= \{f = (f_n)_{n \geq 0} : \|f\|_{\hat{\Lambda}_q(\varphi)} = \sup_n \|f_n\|_{\Lambda_q(\varphi)} < \infty\}; \\ \Lambda_q^M(\varphi) &= \{f = (f_n)_{n \geq 0} : \|f\|_{\Lambda_q^M(\varphi)} = \|M(f)\|_{\Lambda_q(\varphi)} < \infty\}; \\ \Lambda_q^S(\varphi) &= \{f = (f_n)_{n \geq 0} : \|f\|_{\Lambda_q^S(\varphi)} = \|S(f)\|_{\Lambda_q(\varphi)} < \infty\}; \\ \Lambda_q^\sigma(\varphi) &= \{f = (f_n)_{n \geq 0} : \|f\|_{\Lambda_q^\sigma(\varphi)} = \|\sigma(f)\|_{\Lambda_q(\varphi)} < \infty\}; \\ \mathcal{Q}_{\Lambda_q(\varphi)} &= \{f = (f_n)_{n \geq 0} : \exists (\lambda_n) \in \Lambda, s.t. S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in \Lambda_q(\varphi)\}, \\ &\|f\|_{\mathcal{Q}_{\Lambda_q(\varphi)}} = \inf_\lambda \|\lambda_\infty\|_{\Lambda_q(\varphi)}; \\ \mathcal{D}_{\Lambda_q(\varphi)} &= \{f = (f_n)_{n \geq 0} : \exists (\lambda_n) \in \Lambda, s.t. |f_n| \leq \lambda_{n-1}, \lambda_\infty \in \Lambda_q(\varphi)\}, \\ &\|f\|_{\mathcal{D}_{\Lambda_q(\varphi)}} = \inf_\lambda \|\lambda_\infty\|_{\Lambda_q(\varphi)}. \end{aligned}$$

**Remark 2.1** It is clear that if  $\varphi(t) = t$ , then

$$\begin{aligned} \hat{\Lambda}_q &= L_{p(\cdot), q}, & \Lambda_q^M(\varphi) &= H_{p(\cdot), q}^M, & \Lambda_q^S(\varphi) &= H_{p(\cdot), q}^S, \\ \Lambda_q^\sigma(\varphi) &= H_{p(\cdot), q}^\sigma, & \mathcal{Q}_{\Lambda_q(\varphi)} &= \mathcal{Q}_{p(\cdot), q}, & \mathcal{D}_{\Lambda_q(\varphi)} &= \mathcal{D}_{p(\cdot), q}. \end{aligned}$$

Let  $a_0$  and  $a_1$  be real numbers such that  $a_0 < a_1$ . The class  $Q[a_0, a_1]$  consists of all functions  $\varphi(t)$  on  $(0, \infty)$  such that  $\varphi(t)t^{-a_0}$  is nondecreasing and  $\varphi(t)t^{-a_1}$  is non-increasing. A function is said to belong to the class  $Q(a_0, a_1)$ , if  $\varphi(t) \in Q[a_0 + \varepsilon, a_1 - \varepsilon]$  for some  $\varepsilon > 0$ . The notation  $\varphi(t) \in Q(a_0, -)$  (or  $\varphi(t) \in Q(-, a_1)$ ) means that  $\varphi(t) \in Q(a_0, b)$  (or  $\varphi(t) \in Q(b, a_1)$ ) for some real number  $b$ . It was shown in [5] that the function class  $Q(0, 1)$  is larger than the function class  $B_\Psi$  introduced by Kalugina [2]. It is easy to see that  $\varphi(t) = t^\theta$  ( $0 < \theta < 1$ ) belongs to the function class  $Q(0, 1)$ .

Let  $X_0$  and  $X_1$  be two quasi-Banach spaces, which are both embedded into a topological vector space  $X$ . We call  $\bar{X} = (X_0, X_1)$  an interpolation couple in this case. Then the space  $X_0 + X_1$  is defined as the set of all  $x \in X$ , which may be written as  $x = x_0 + x_1$  for which

$x_0 \in X_0$  and  $x_1 \in X_1$ . For any  $x \in X_0 + X_1$  and  $0 < t < \infty$ , the so-called  $K$ -functional is defined as

$$K(x, t; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\},$$

where the infimum is taken over all  $x = x_0 + x_1$  for which  $x_0 \in X_0$  and  $x_1 \in X_1$ . For  $\varphi$  a function parameter,  $0 < q \leq \infty$ , the interpolation spaces  $(X_0, X_1)_{\varphi, q}$  between  $X_0$  and  $X_1$  are defined as the spaces of all functions  $x \in X_0 + X_1$ , such that

$$\|x\|_{(X_0, X_1)_{\varphi, q}} = \left\| \frac{K(x, t; X_0, X_1)}{\varphi(t)} \right\|_{L^*_q} < \infty.$$

Throughout this paper, we use  $c$  to denote some positive constant and may be different at each occurrence.  $a \preceq b$  means that  $a \leq cb$  for some positive constants  $c$  and the equivalence  $a \approx b$  means that  $c_1a \leq b \leq c_2a$  for some positive constants  $c_1$  and  $c_2$ .

### 3 Some Lemmas

In order to prove our main results, we collect some lemmas in this section.

**Lemma 3.1** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty$ ,  $0 < q < \infty$  then for all martingales  $f = (f_n)_{n \geq 0} \in H^{\sigma}_{p(\cdot), q} + H^{\sigma}_{p(\cdot), \infty}$ ,

$$K(f, t; H^{\sigma}_{p(\cdot), q}, H^{\sigma}_{p(\cdot), \infty}) \approx \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q}.$$

**Proof** Let  $f \in H^{\sigma}_{p(\cdot), q} + H^{\sigma}_{p(\cdot), \infty}$ ,  $f = g + h$ , with  $g \in H^{\sigma}_{p(\cdot), q}$ ,  $h \in H^{\sigma}_{p(\cdot), \infty}$ . Then for any  $u > 0$ , we have

$$\sigma(f)_*(u) \preceq (\sigma(g) + \sigma(h))_*(u) \preceq \sigma(g)_*(u) + \sigma(h)_*(u) \preceq \sigma(g)_*(u) + \|h\|_{H^{\sigma}_{p(\cdot), \infty}}. \tag{3.1}$$

So

$$\begin{aligned} \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q} &\preceq \left( \int_0^{t^q} (\sigma(g)_*(u) + \|h\|_{H^{\sigma}_{p(\cdot), \infty}})^q du \right)^{1/q} \\ &\preceq \left( \int_0^{t^q} (\sigma(g)_*(u))^q du \right)^{1/q} + t\|h\|_{H^{\sigma}_{p(\cdot), \infty}} \\ &\preceq \left( \int_0^{\infty} (\sigma(g)_*(u))^q du \right)^{1/q} + t\|h\|_{H^{\sigma}_{p(\cdot), \infty}} \\ &\preceq \|g\|_{H^{\sigma}_{p(\cdot), q}} + t\|h\|_{H^{\sigma}_{p(\cdot), \infty}}. \end{aligned} \tag{3.2}$$

Taking the infimum over all decompositions  $f = g + h \in H^{\sigma}_{p(\cdot), q} + H^{\sigma}_{p(\cdot), \infty}$ , we obtain

$$\left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q} \preceq K(f, t; H^{\sigma}_{p(\cdot), q}, H^{\sigma}_{p(\cdot), \infty}). \tag{3.3}$$

To prove the converse, for any  $t > 0$ , let  $\alpha = \sigma(f)_*(t^q)$ , and define the stopping time,

$$\tau = \inf\{n \in N : \sigma_{n+1}(f) > \alpha\},$$

then  $\{\tau < \infty\} = \{\sigma(f) > \alpha\}$ . Let  $h = f^\tau = (f_{n \wedge \tau})_{n \geq 0}$ ,  $g = f - h$ , then we have

$$\|h\|_{H_{p(\cdot), \infty}^\sigma} \preceq \|\sigma(h)\|_\infty = \|\sigma_\tau(f)\|_\infty \leq \alpha = \sigma(f)_*(t^q), \quad (3.4)$$

and

$$\begin{aligned} \|g\|_{H_{p(\cdot), q}^\sigma} &= \left( \int_0^\infty \lambda^q \|\chi_{\{|\sigma(g)| > \lambda\}}\|_{p(\cdot)}^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &= \left( \int_0^\infty \lambda^q \|\chi_{\{|\sigma(f-f^\tau)| > \lambda\}} \chi_{\{\tau < \infty\}}\|_{p(\cdot)}^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &\preceq \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q}. \end{aligned} \quad (3.5)$$

Hence

$$\|g\|_{H_{p(\cdot), q}^\sigma} + t\|h\|_{H_{p(\cdot), \infty}^\sigma} \preceq \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q} + t\sigma(f)_*(t^q) \preceq \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q} \quad (3.6)$$

It follows that

$$K(f, t; H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma) \preceq \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q}. \quad (3.7)$$

The proof is completed.

**Lemma 3.2** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty$ ,  $0 < q < \infty$  then for all martingales  $f = (f_n)_{n \geq 0} \in \mathcal{Q}_{p(\cdot), q} + \mathcal{Q}_{p(\cdot), \infty}$  and for any admissible majorant  $\lambda = \{\lambda_n\}_{n \geq 0}$  of  $S_n(f)$ , we have

$$K(f, t; \mathcal{Q}_{p(\cdot), q}, \mathcal{Q}_{p(\cdot), \infty}) \approx \inf_\lambda \left( \int_0^{t^q} ((\lambda_\infty)_*(u))^q du \right)^{1/q}.$$

**Proof** Let  $f \in \mathcal{Q}_{p(\cdot), q} + \mathcal{Q}_{p(\cdot), \infty}$ . Since  $\|f\|_{H_{p(\cdot), q}^\sigma} \leq c\|f\|_{\mathcal{Q}_{p(\cdot), q}}$  for  $0 < q \leq \infty$ , it is easy to prove that  $(\int_0^{t^q} (S(f)_*(u))^q du)^{1/q} \leq cK(f, t; \mathcal{Q}_{p(\cdot), q}, \mathcal{Q}_{p(\cdot), \infty})$  for all  $t > 0$ . Thus for each  $t > 0$ , there is  $f$ 's decomposition,  $f = g + h$  with  $g \in \mathcal{Q}_{p(\cdot), q} + \mathcal{Q}_{p(\cdot), \infty}$ , such that

$$\|g\|_{\mathcal{Q}_{p(\cdot), q}} + \|h\|_{\mathcal{Q}_{p(\cdot), \infty}} \leq 2K(f, t; \mathcal{Q}_{p(\cdot), q}, \mathcal{Q}_{p(\cdot), \infty}). \quad (3.8)$$

Let  $\lambda_g^{(t)}$  and  $\lambda_h^{(t)}$  be admissible majorants of  $S_n(g)$  and  $S_n(h)$  such that  $\|\lambda_g^{(t)}\|_{p(\cdot), q} = \|g\|_{\mathcal{Q}_{p(\cdot), q}}$  and  $\|\lambda_h^{(t)}\|_{p(\cdot), \infty} = \|h\|_{\mathcal{Q}_{p(\cdot), \infty}}$ . Define  $\lambda(t) = \lambda_g^{(t)} + \lambda_h^{(t)}$  and  $\lambda^{(1)} = \{\lambda_n^{(1)}\}_{n \geq 0}$ , where  $\lambda_n^{(1)} = \inf_{t > 0} \lambda_n^{(t)}$ . Then we get  $\lambda^{(1)} = \{\lambda_n^{(1)}\}_{n \geq 0}$  an admissible majorant of  $\{S_n(f)\}_{n \geq 0}$  which satisfies

$$\inf_\lambda \left( \int_0^{t^q} (\lambda_\infty)_*(u))^q du \right)^{1/q} \leq \left( \int_0^{t^q} (\lambda_*^{(1)}(u))^q du \right)^{1/q} \leq cK(f, t; \mathcal{Q}_{p(\cdot), q}, \mathcal{Q}_{p(\cdot), \infty}). \quad (3.9)$$

To prove the converse, let  $\lambda = \{\lambda_n\}_{n \geq 0}$  be any admissible majorant of  $S_n(f)$ , for any  $t > 0$ , let  $\alpha = (\lambda_\infty)_*(t^q)$ , and define the stopping time,

$$\tau = \inf\{n \in N : \lambda_n > \alpha\},$$

then  $\{\tau < \infty\} = \{\lambda_\infty > \alpha\}$ . Let  $h = f^\tau = (f_{n \wedge \tau})_{n \geq 0}, g = f - h$ , then we have

$$S(h) = S_\tau(f) \leq \alpha = (\lambda_\infty)_*(t^q),$$

$$S_n(g) = S_n(f - f^\tau) \leq 2S_n(f)\chi_{\{\tau < n\}} \leq 2\lambda_{n-1}\chi_{\{\tau < n\}}.$$

Then we have

$$\|g\|_{\mathcal{Q}_{p(\cdot),q}} \leq c \left( \int_0^{t^q} ((\lambda_\infty)_*(u))^q du \right)^{1/q}. \tag{3.10}$$

Hence

$$\|g\|_{\mathcal{Q}_{p(\cdot),q}} + t\|h\|_{\mathcal{Q}_{p(\cdot),q}} \preceq \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q} + t(\lambda_\infty)_*(t^q) \preceq \left( \int_0^{t^q} ((\lambda_\infty)_*(u))^q du \right)^{1/q}. \tag{3.11}$$

It follows that

$$K(f, t; \mathcal{Q}_{p(\cdot),q}, \mathcal{Q}_{p(\cdot),\infty}) \preceq \inf_\lambda \left( \int_0^{t^q} ((\lambda_\infty)_*(u))^q du \right)^{1/q}. \tag{3.12}$$

The proof is completed.

Similarly to Lemma 3.2, we have

**Lemma 3.3** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty, 0 < q < \infty$  then for all martingales  $f = (f_n)_{n \geq 0} \in \mathcal{D}_{p(\cdot),q} + \mathcal{D}_{p(\cdot),\infty}$  and for any admissible majorant  $\lambda = \{\lambda_n\}_{n \geq 0}$  of  $|f_n|$ , we have

$$K(f, t; \mathcal{D}_{p(\cdot),q}, \mathcal{D}_{p(\cdot),\infty}) \approx \inf_\lambda \left( \int_0^{t^q} ((\lambda_\infty)_*(u))^q du \right)^{1/q}.$$

**Lemma 3.4** ([14]) Let  $\varphi(t) \in Q[a_0, a_1]$ . Then

- (1)  $\varphi(t^\alpha) \in Q[a_0\alpha, a_1\alpha], \alpha > 0;$
- (2)  $t^\alpha(\varphi(t))^\beta \in Q[\alpha + a_1\beta, \alpha + a_0\beta], \alpha \in R, \beta < 0;$
- (3)  $\varphi(\alpha t) \in Q[a_0, a_1], \alpha > 0.$

**Lemma 3.5** ([14]) Let  $0 < q_1 \leq \infty, 0 < q < \infty, \psi(t) \in Q(-, -)$ , and  $h(t)$  a positive and non-increasing function on  $(0, \infty)$ . If  $\varphi(t) \in Q(-, 0)$  Then

$$\left( \int_0^\infty (\varphi(t))^{q_1} \left( \int_0^t (h(u)\psi(u))^q \frac{du}{u} \right)^{q_1/q} \frac{dt}{t} \right)^{1/q_1} \leq c \left( \int_0^\infty (\varphi(t)h(t)\psi(t))^{q_1} \frac{dt}{t} \right)^{1/q_1}.$$

**Lemma 3.6** ([14]) Let  $\varphi_0(t), \varphi_1(t)$  and  $\varphi(t)$  be in the class  $Q(0, 1), 0 < q_0, q_1 < \infty, 0 < q \leq \infty$ . If we put  $\varphi_2(t) = \varphi_0(t)\varphi(\varphi_1(t)/\varphi_0(t)), \varphi_3(t) = \varphi_0(t)\varphi(t/\varphi_0(t)), \varphi_4(t) = \varphi(\varphi_1(t))$ . Then

- (1)  $(\overline{X}_{\varphi_0, q_0}, X_1)_{\varphi, q} = \overline{X}_{\varphi_3, q};$
- (2)  $(X_0, \overline{X}_{\varphi_1, q_1})_{\varphi, q} = \overline{X}_{\varphi_4, q};$
- (3) If, in addition  $\frac{\varphi_1(t)}{\varphi_0(t)} \in Q(0, -)$  or  $\frac{\varphi_0(t)}{\varphi_1(t)} \in Q(0, -)$ , then  $(\overline{X}_{\varphi_0, q_0}, \overline{X}_{\varphi_1, q_1})_{\varphi, q} = \overline{X}_{\varphi_2, q}.$

**Lemma 3.7** ([14]) Let  $\varphi_0(t), \varphi_1(t)$  and  $\varphi(t)$  be in the class  $Q(0, 1)$  and put  $\tau(t) = \varphi_1(t)/\varphi_0(t)$ . If  $\tau(t) \in Q(0, -)$  or  $\tau(t) \in Q(-, 0)$ , then  $\varphi_2(t) = \varphi_0(t)\varphi(\tau(t)) \in Q(0, 1)$ .

## 4 Main Results

**Theorem 4.1** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty$ ,  $0 < q < \infty$ ,  $0 < q_1 \leq \infty$  and  $\varphi \in Q(0, 1)$ . Then

$$(H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, q_1} = \Lambda_{q_1}^\sigma(t^{1/q}/\varphi(t^{1/q})). \quad (4.1)$$

**Proof** If  $0 < q_1 < \infty$ , by Lemma 3.1, it is easy to see that

$$\|f\|_{(H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, q_1}} \approx \left( \int_0^\infty \left( \frac{1}{\varphi(t^{1/q})} \right)^{q_1} \left( \int_0^t (\sigma(f)_*(u))^q du \right)^{q_1/q} \frac{dt}{t} \right)^{1/q_1}. \quad (4.2)$$

By Lemma 3.4 we see that  $1/\varphi(t^{1/q}) \in Q(-1/q, 0)$ . Therefore, by Lemma 3.5 we have

$$\left( \int_0^\infty \left( \frac{1}{\varphi(t^{1/q})} \right)^{q_1} \left( \int_0^t (\sigma(f)_*(u))^q du \right)^{q_1/q} \frac{dt}{t} \right)^{1/q_1} \leq c \left( \int_0^\infty \left( \frac{t\sigma(f)_*(t)}{\varphi(t^{1/q})} \right)^{q_1} \frac{dt}{t} \right)^{1/q_1}. \quad (4.3)$$

Hence,  $\Lambda_{q_1}^\sigma(t^{1/q}/\varphi(t^{1/q})) \subseteq (H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, q_1}$ . Since

$$t(\sigma(f)_*(t))^q \leq \int_0^t (\sigma(f)_*(u))^q du,$$

we get

$$\left( \int_0^\infty \left( \frac{t\sigma(f)_*(t)}{\varphi(t^{1/q})} \right)^{q_1} \frac{dt}{t} \right)^{1/q_1} \leq \left( \int_0^\infty \left( \frac{1}{\varphi(t^{1/q})} \right)^{q_1} \left( \int_0^t (\sigma(f)_*(u))^q du \right)^{q_1/q} \frac{dt}{t} \right)^{1/q_1}. \quad (4.4)$$

So we have  $(H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, q_1} \subseteq \Lambda_{q_1}^\sigma(t^{1/q}/\varphi(t^{1/q}))$ .

If  $q_1 = \infty$ , since

$$t\sigma(f)_*(t^q) \leq \left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q},$$

we get

$$\begin{aligned} \|f\|_{\Lambda_\infty^\sigma(t^{1/q}/\varphi(t^{1/q}))} &= \sup_{t>0} t^{1/q} \varphi(t^{1/q}) \sigma_*(f)(t) = \sup_{t>0} t \varphi(t) \sigma_*(f)(t^q) \\ &\leq c \sup_{t>0} \frac{K(f, t; H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)}{\varphi(t)} \\ &= c \|f\|_{(H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, \infty}}. \end{aligned} \quad (4.5)$$

Hence,  $\Lambda_\infty^\sigma(t^{1/q}/\varphi(t^{1/q})) \subseteq (H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, \infty}$ . For the converse, since  $\varphi(t) \in Q(0, 1)$ , then there exists a constant  $a \in (0, 1)$  such that  $\varphi(t)t^{-a}$  is non-increasing on  $(0, \infty)$ . So we have

$$\begin{aligned} \|f\|_{(H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)_{\varphi, \infty}} &= \sup_{t>0} \frac{K(f, t; H_{p(\cdot), q}^\sigma, H_{p(\cdot), \infty}^\sigma)}{\varphi(t)} \\ &\leq c \sup_{t>0} \frac{\left( \int_0^{t^q} (\sigma(f)_*(u))^q du \right)^{1/q}}{\varphi(t)} \leq c \sup_{t>0} \frac{\left( \int_0^t (\sigma(f)_*(u^q))^q u^{q-1} du \right)^{1/q}}{\varphi(t)} \\ &\leq c \sup_{u>0} \frac{u\sigma(f)_*(u^q)}{\varphi(u)} \cdot \sup_{t>0} \frac{\varphi(u)u^{-a} \left( \int_0^t u^{qa-1} du \right)^{1/q}}{\varphi(t)} \\ &\leq c \|f\|_{\Lambda_\infty^\sigma(t^{1/q}/\varphi(t^{1/q}))}. \end{aligned} \quad (4.6)$$

So we have  $(H_{p(\cdot),q}^\sigma, H_{p(\cdot),\infty}^\sigma)_{\varphi,\infty} \subseteq \Lambda_\infty^\sigma(t^{1/q}/\varphi(t^{1/q}))$ . Thus we complete the proof.

Similar to Theorem 4.1, we have the following two theorems, we omit the proofs.

**Theorem 4.2** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty$ ,  $0 < q < \infty$ ,  $0 < q_1 \leq \infty$  and  $\varphi \in Q(0, 1)$ .

Then

$$(\mathcal{Q}_{p(\cdot),q}, \mathcal{Q}_{p(\cdot),\infty})_{\varphi,q_1} = \mathcal{Q}_{\Lambda_{q_1}^\sigma}(t^{1/q}/\varphi(t^{1/q})). \quad (4.7)$$

**Theorem 4.3** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty$ ,  $0 < q < \infty$ ,  $0 < q_1 \leq \infty$  and  $\varphi \in Q(0, 1)$ .

Then

$$(\mathcal{D}_{p(\cdot),q}, \mathcal{D}_{p(\cdot),\infty})_{\varphi,q_1} = \mathcal{D}_{\Lambda_{q_1}^\sigma}(t^{1/q}/\varphi(t^{1/q})). \quad (4.8)$$

By interpolation theorem, we can easily have

**Theorem 4.4** Let  $p(\cdot) \in P(\Omega)$  with  $p^+ < \infty$ ,  $0 < q_i < \infty$ ,  $0 < q \leq \infty$ ,  $\varphi_i(t) \in Q(0, -)$ ,  $i = 0, 1$ , and  $\varrho \in Q(0, 1)$ . Then

(1)

$$(\Lambda_{q_0}^\sigma(\varphi_0), H_{p(\cdot),\infty}^\sigma)_{\varrho,q} = \Lambda_q^\sigma(\varphi), \quad (4.9)$$

where  $\varphi(t) = \frac{\varphi_0(t)}{\varrho(\varphi_0(t))}$ .

(2) If, in addition  $\varphi_1 \in Q(0, \frac{1}{q})$ , then

$$(H_{p(\cdot),q_1}^\sigma, \Lambda_{q_0}^\sigma(\varphi_0))_{\varrho,q} = \Lambda_q^\sigma(\varphi), \quad (4.10)$$

where  $\varphi(t) = \frac{t^{1/q}}{\varrho(t^{1/q}/\varphi_1(t))}$ .

(3) If, in addition  $\frac{\varphi_1(t)}{\varphi_0(t)} \in Q(0, -)$  or  $\frac{\varphi_0(t)}{\varphi_1(t)} \in Q(0, -)$ , then

$$(\Lambda_{q_0}^\sigma(\varphi_0), \Lambda_{q_1}^\sigma(\varphi_1))_{\varrho,q} = \Lambda_q^\sigma(\varphi), \quad (4.11)$$

where  $\varphi(t) = \frac{\varphi_0(t)}{\varrho(\varphi_0(t)/\varphi_1(t))}$ .

**Proof** First we prove (3). Put  $\varrho_i(t) = t/\varphi_i(t^q)$ , by (2) in Lemma 3.4, we can choose  $q$  so small that  $\varrho_i(t) \in Q(0, 1)$ ,  $i = 0, 1$ . By the Lemma 3.6, 3.7 and Theorem 4.1 we obtain

$$\begin{aligned} (\Lambda_{q_0}^\sigma(\varphi_0), \Lambda_{q_1}^\sigma(\varphi_1))_{\varrho,q} &= ((H_{p(\cdot),q}^\sigma, H_{p(\cdot),\infty}^\sigma)_{\varrho_0,q_0}, (H_{p(\cdot),q}^\sigma, H_{p(\cdot),\infty}^\sigma)_{\varrho_1,q_1})_{\varrho,q} \\ &= (H_{p(\cdot),q}^\sigma, H_{p(\cdot),\infty}^\sigma)_{\varrho_0\varrho(\varrho_1/\varrho_0),q} = \Lambda_q^\sigma(\varphi), \end{aligned} \quad (4.12)$$

where  $\varphi(t) = \frac{\varphi_0(t)}{\varrho(\varphi_0(t)/\varphi_1(t))}$ .

In order to prove (2), we first note that, by Lemma 3.4, the condition  $\varphi_1 \in Q(0, \frac{1}{q})$  implies that  $\varrho_1(t) = t/\varphi_1(t^q) \in Q(0, 1)$ . By using of Theorem 4.1 and (2) in Lemma 3.6, similar to that of (3), we can prove (2). It is obvious that (1) is an easy consequence of Theorem 4.1 and (1) in Lemma 3.6. The proof is completed.

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## 变指数Lorentz鞅空间的内插

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**摘要:** 本文研究了变指数Lorentz鞅空间的插值问题. 利用函数参数方法, 获得了几类变指数Lorentz鞅空间的插值定理, 推广了常指数 $H_p$ 鞅空间下的结果.

**关键词:** Lorentz鞅空间; 变指数; 函数参数

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