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ABSTRACT CAUCHY-KOVALEVSKAYA THEOREM IN GEVREY SPACE: ENERGY METHOD

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Abstract: In this paper, we study the well-posedness of nonlinear Cauchy problem. The main tool is the combination of the classical energy method and abstract Cauchy-Kovalevskaya Theorem. We obtain that the nonlinear Cauchy problem is well-posed in the Gevrey space, which is an extension on the existing literature in the aspect of well-posedness for the nonlinear Cauchy problem.

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1 Introduction and Main Results

As early as 1918, M. Gevrey first proposed the concept of smoothness of Gevrey classes in [1]. Gevrey class is a function space which lies between analytic class and C^∞ function space. Accurately speaking, J. Hadamard put forward a similar concept quasianalytic class earlier in [2], while Gevrey class can be regarded as a special case of quasianalytic class. As we all konw, it is not very convenient to solve general partial differential equations because of the mathematical characterization of C^∞ smoothness. But for the analytic C^ω smoothness, we have the Cauchy-Kovalevskaya theorem on local existence. Cauchy-Kovalevskaya Theorem [3] tells us that as long as the coefficients of the equations are analytic, the solutions of the non-characteristic Cauchy problems of higher order differential equations exist at least locally, although we can't get the continuous dependence of the solutions. In order to extend the abstract Cauchy-Kovalevakaya theorem to the non-analytic function set, M. Gevrey introduced the concept of Gevrey class. Then, in the study of quasianalytic function class, La Vallée Poussin and his collaborators found that Gevrey class functions could be described by exponential decay of their Fourier coefficients in [4], which was later reflected in [5] by J. Kopeé and J. Musielak. Although mathematicians knew the equivalent description of Fourier coefficients of Gevrey class space for a long time, they didn't know that it had other applications at that time. Until 1989, C. Foias and R. Temam creatively applied this Fourier space technique in [6]. At present, it is a standard practice to study the analytic properties of solutions of a large class to dissipative equations in different function spaces, which is due

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to Fourier space method of Gevrey class functions, and their work is much simpler than the earlier method that C. Kahane used the original definition of Gevrey class in [7] to analyze the analytic properties of solutions to the incompressible Navier-Stokes equations.

The most suitable function space for studying hydrodynamic equations is Sobolev space, because the definition of energy in Sobolev space is very simple. However, many basic problems of hydrodynamic equations have not been satisfactorily worked in Sobolev space. For example, Prandtl boundary layer problem is ill-posed in Sobolev space in many cases. On the other hand, according to Cauchy-Kovalevskaya theorem, these equations are locally solvable in the analytic function space. However, analytic function space does not contain compactly supported functions, so it is not the suitable function space for studying hydrodynamic equations. Therefore, Gevrey space, the transition space between Sobolev space and analytic function space, is naturally considered. Now, we recall the definition for the functions in the Gevrey class. Let Ω be an open subset of $\mathbb{R}^d$ and $1 \leq s < +\infty$. f is the real value function defined on Ω. We say that $f \in G^s(\Omega)$ if $f \in C^\infty(\Omega)$ and for any compact subset K of Ω, there exists a constant (say Gevrey constant of f) $C = C_K$, depending only on K and f, such that for all multi-indices $\alpha \in \mathbb{N}^d$,

$$\|\partial^\alpha f\|_{L^\infty(\Omega)} \leq C_K^{|\alpha|+1}(\alpha!)^s. \tag{1.1}$$

If W is a closed subset of $\mathbb{R}^d$, $G^s(W)$ denotes the restriction of $G^s(\widetilde{W})$ on W where $\widetilde{W}$ is an open neighborhood of W. The condition (1.1) is equivalent to the following estimate(see [8, 9]):

$$\|\partial^\alpha f\|_{L^2(K)} \leq C_K^{|\alpha|+1}(\alpha!)^s.$$

Now, let us pay attention to the following abstract Cauchy problem:

$$\begin{cases} u'' + A(t)u = f(t, u(t)), \\ u(0) = u_0,\ u'(0) = u_1, \end{cases} \tag{1.2}$$

in a Hilbert space H, where $A(t)$ is a nonnegative unbound operator. If $A(t)$ satisfies some strict coercivity assumptions, i.e. when the equation in (1.2) is strictly hyperbolic, the local solvability for the Cauchy problem (1.2) is well-known, provided $A(t)$ is Lipschitz continuous in time and f is smooth enough. Kato gave an extensive theory on this problem, including most of the concrete results in the Sobolev space with optimal regularity assumptions. (see[10, 11]) On the other hand, when $A(t) \geq 0$ is allowed to be degenerate, i.e. when the equation of (1.2) is weakly hyperbolic, we need much stronger assumptions in order that the Cauchy problem (1.2) is locally solvable. This is the same to linear equations such as

$$u_{tt} = a(t)u_{xx}, \tag{1.3}$$

which may not be locally solvable in C^∞ for a suitable nonnegative $a(t) \in C^\infty$. It is possible to overcome this difficulty by requiring that the data and the coefficients are more regular in space variables. It was proved in [12, 13] that the equations

$$u_{tt} = \sum_{i,j} a_{ij}(t,x)u_{x_i x_j} + \sum_j b_j(t,x)u_{x_j}, \quad \sum a_{ij}\xi_i\xi_j \geq 0 \tag{1.4}$$

are globally sovable in the Gevrey space. Besides, the case of hyperbolic equations of higher order was proved in [14]. If the coefficients a_{ij} are Hölder continuous in time with exponent λ and Gevrey of order s in x, then (1.4) is uniquely solvable in $G^s(\mathbb{R}^d)$ provided

$$s < 1 + \lambda/2 \tag{1.5}$$

and locally solvable if there is equality in (1.5). A conjecture was inspired by these remarks that an equation like

$$u_{tt} = \sum a_{ij}(t,x)u_{x_ix_j} + f(u,u_x) \tag{1.6}$$

may be locally solvable in Gevrey classes, provided the function f has suitable smoothness properties. In 1967, Leray and Ohya proved that if the system is (weakly) hyperbolic with smooth characteristic roots, the Cauchy problem for the general semilinear system is well-posed in the Gevrey class in [15]. And Kajitani removed the assumption of smoothness. Besides, he further improved the result by showing that it was sufficient to assume Hölder continuity in time of the coefficients, provided (1.5) holds.

In this paper, we discuss the existence and uniqueness of local solution in the Gervey space to the following abstract Cauchy problem:

$$\begin{cases} \partial_t^2 u = F(t, \partial_x u), \ (x;t) \in Q_{\infty,T} \\ u|_{t=0} = u_0, \ x \in \mathbb{R} \\ \partial_t u|_{t=0} = v_0, \ x \in \mathbb{R} \end{cases} \tag{1.7}$$

where $Q_{\infty,T} = \mathbb{R} \times (0,T]$ and $F(t,\partial_x u)$ are analytic with respect to both x and t.

Now, we will introduce the Gevrey function space.

Definition 1.1 Let $0 < \rho < 1$, the Gevrey function space G_ρ consists of all smooth vector-valued functions u such that the Gevrey norm $\|u\|_\rho < +\infty$, where $\|\cdot\|_\rho$ is defined below. We define

$$\|u\|_\rho = \sup_{m\geq 1} \frac{\rho^m}{(m!)^2} \|\partial_x^m u\|_{H^1}.$$

Our result can be stated as follows.

Theorem 1.1 Suppose the initial datum u_0 and v_0 belong to G_{ρ_0} for some $\rho_0 > 0$. Then the system (1.7) admits a unique solution $u \in L^\infty\big([0,T];\ G_\rho\big)$ for some $T > 0$ and some $0 < \rho < \rho_0 \leq 1$.

Remark 1.1 Similar results also hold with different Gevrey index for more general Cauchy problem

$$\begin{cases} \partial_t^k u = F(t, \partial_x u, \cdots, \partial_x^{k-1} u), \ (x;t) \in Q_{\infty,T} \\ \partial_x^j u|_{t=0} = u_j, \ x \in \mathbb{R}, \quad j = 0, \cdots, k-1. \end{cases}$$

The rest of this paper is organized as follows. In Section 2, we will prove a priori estimate. In Section 3, we will prove the Theorem 1.1.

2 A Priori Estimate

Suppose $u \in L^\infty\big([0,T];\, G_\rho\big)$ is a solution to the system (1.7) with initial datum $u_0 \in G_{\rho_0}$ and $v_0 \in G_{\rho_0}$. Now, we will rewrite the system (1.7) as follows:

$$\begin{cases} \partial_t u = v, \ (x;t) \in Q_{\infty,T} \\ \partial_t v = F(t, \partial_x u), \ (x;t) \in Q_{\infty,T} \\ u|_{t=0} = u_0, \ x \in \mathbb{R} \\ v|_{t=0} = v_0, \ x \in \mathbb{R} \end{cases} \tag{2.1}$$

where $Q_{\infty,T} = \mathbb{R} \times (0,T]$ and $F(t, \partial_x u)$ are analytic with respect to both x and t.

In this section, we will derive a priori estimate for (u,v). We denote $\vec{a} = (u,v)$ and define $|\cdot|_\rho$ for each $0 < \rho < 1$ by

$$|\vec{a}|_\rho = \sup_{m\geq 1} \frac{\rho^m}{(m!)^2}\|\partial_x^m u\|_{H^1} + \sup_{m\geq 1} \frac{\rho^{m+1}}{[(m+1)!]^2} m \|\partial_x^m v\|_{H^1}. \tag{2.2}$$

From the definition of $|\cdot|_\rho$ given in (2.2), it follows that

$$\forall\ 0<\rho<1,\ \forall\, m \geq 1, \quad \|\partial_x^m u\|_{H^1} \leq \frac{(m!)^2}{\rho^m}|\vec{a}|_\rho \text{ and } \|\partial_x^m v\|_{H^1} \leq \frac{1}{m}\frac{[(m+1)!]^2}{\rho^{m+1}}|\vec{a}|_\rho.$$

We calculate by using the above estimates, for any $\tilde\rho$ with $0<\rho<\tilde\rho\leq 1$,

$$\begin{aligned} \frac{\rho^{2m}}{(m!)^4}\int_0^t \|\partial_x^m v\|_{H^1}\|\partial_x^m u\|_{H^1} ds &\leq \frac{\rho^{2m}}{(m!)^4}\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho}}{m}\frac{[(m+1)!]^2}{\tilde\rho^{m+1}}\frac{(m!)^2}{\tilde\rho^m}\,|\vec{a}(s)|_{\tilde\rho}\, ds \\ &\leq 4\int_0^t \frac{m}{\tilde\rho}\frac{\rho^{2m}}{\tilde\rho^{2m}}\,|\vec{a}(s)|_{\tilde\rho}^2\, ds \leq 4\int_0^t \frac{m}{\tilde\rho}\frac{\rho^m}{\tilde\rho^m}\,|\vec{a}(s)|_{\tilde\rho}^2\, ds \\ &\leq 4\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho}^2}{\tilde\rho-\rho} ds. \end{aligned} \tag{2.3}$$

The penultimate inequality is used the fact that for any $0<\rho<\tilde\rho\leq 1$,

$$\frac{\rho^2}{\tilde\rho^2} \leq \frac{\rho}{\tilde\rho}.$$

And the last inequality is used the fact that for any integer $m \geq 1$ and for any pair $(\rho, \tilde\rho)$ with $0<\rho<\tilde\rho\leq 1$,

$$m\left(\frac{\rho}{\tilde\rho}\right)^m \leq \frac{\tilde\rho}{\tilde\rho-\rho}. \tag{2.4}$$

We note that the inequality (2.4) follows from the fact that

$$\frac{1}{1-r} = \sum_{j=0}^{\infty} r^j \geq \sum_{j=1}^{m} r^j \geq \sum_{j=1}^{m} r^m = mr^m.$$

Applying the similar argument as above, we can get that

$$\frac{\rho^{2(m+1)}m^2}{[(m+1)!]^4}\int_0^t \|\partial_x^m v\|_{H^1}\|\partial_x^{m+1} u\|_{H^1} ds \leq \frac{\rho^{2(m+1)}m^2}{[(m+1)!]^4}\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho}^2}{m}\frac{[(m+1)!]^2}{\tilde\rho^{m+1}}\frac{[(m+1)!]^2}{\tilde\rho^{m+1}} ds$$

$$\leq \int_0^t m\frac{\rho^{2(m+1)}}{\tilde{\rho}^{2(m+1)}}\left|\vec{a}(s)\right|_{\tilde{\rho}}^2 ds \leq \int_0^t m\frac{\rho^{m+1}}{\tilde{\rho}^{m+1}}\left|\vec{a}(s)\right|_{\tilde{\rho}}^2 ds \leq \int_0^t m\frac{\rho^{m}}{\tilde{\rho}^{m}}\left|\vec{a}(s)\right|_{\tilde{\rho}}^2 ds \leq \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho}-\rho}ds. \tag{2.5}$$

Now we will state the main a priori estimate.

Theorem 2.1 (A priori estimate in Gevrey space) Let G_ρ be the Gevrey function space. Suppose (u, v) is the solution to the system (2.1) with initial datum $u_0, v_0 \in G_{\rho_0}$. We can find two constant $C_1, C_2 \geq 1$, such that the estimate

$$|\vec{a}(t)|_\rho^2 \leq C_1(\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2) + C_2\int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho}-\rho}ds$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$ and any $t \in [0, T]$, where the constant C_1 can be computed explicitly and the constant C_2 depends only on the Sobolev embedding constants.

Proof First, we multiply both sides of the equation $\partial_t u = v$ by $\rho^m/(m!)^2$ and apply ∂_x^m to both sides of the equation $\partial_t u = v$. We can get that

$$\partial_t \frac{\rho^m}{(m!)^2}\partial_x^m u = \frac{\rho^m}{(m!)^2}\partial_x^m v. \tag{2.6}$$

Then, we take the scalar product with $\frac{\rho^m}{(m!)^2}\partial_x^m u$ and Hölder inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m u\|_{L^2}^2 \leq \frac{\rho^{2m}}{(m!)^4}\|\partial_x^m v\|_{L^2}\|\partial_x^m u\|_{L^2}. \tag{2.7}$$

On the other hand, using the similar method but applying ∂_x^{m+1} to the equation $\partial_t u = v$ we obtain

$$\frac{1}{2}\frac{d}{dt}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^{m+1} u\|_{L^2}^2 \leq \frac{\rho^{2m}}{(m!)^4}\|\partial_x^{m+1} v\|_{L^2}\|\partial_x^{m+1} u\|_{L^2}. \tag{2.8}$$

Adding the equation (2.7) and (2.8), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m u\|_{H^1}^2 &\leq \frac{\rho^{2m}}{(m!)^4}(\|\partial_x^m v\|_{L^2}\|\partial_x^m u\|_{L^2} + \|\partial_x^{m+1} v\|_{L^2}\|\partial_x^{m+1} u\|_{L^2})\\ &\leq C_2\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m v\|_{H^1}\|\partial_x^m u\|_{H^1}.\end{aligned}$$

It follows from the integration that

$$\begin{aligned}\frac{1}{2}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m u(t)\|_{H^1}^2 &\leq \frac{1}{2}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m u_0\|_{H^1}^2 + C_2\int_0^t \frac{\rho^{2m}}{(m!)^4}\|\partial_x^m v\|_{H^1}\|\partial_x^m u\|_{H^1}ds\\ &\leq \frac{1}{2}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m u_0\|_{H^1}^2 + C_2\int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho}-\rho}ds.\end{aligned}$$

The last inequality is used in the result of (2.3). Now taking the upper bound on both sides of the above inequality we obtain

$$\sup_{m\geq 1}\frac{\rho^{2m}}{(m!)^4}\|\partial_x^m u(t)\|_{H^1}^2 \leq C_1\|u_0\|_{\rho_0}^2 + C_2\int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho}-\rho}ds. \tag{2.9}$$

Now, we will deal with the equation $\partial_t v = F(t, \partial_x u)$. Similarly, we multiply both sides of the equation $\partial_t v = F(t, \partial_x u)$ by $(\rho^{m+1} m)/[(m+1)!]^2$ and apply ∂_x^m to both sides of the equation $\partial_t v = F(t, \partial_x u)$. We can get that

$$\partial_t \frac{\rho^{m+1}}{[(m+1)!]^2} m \partial_x^m v = \frac{\rho^{m+1}}{[(m+1)!]^2} m \partial_x^m F.$$

Next, we take the scalar product with $\frac{\rho^{m+1} m}{[(m+1)!]^2} \partial_x^m v$ and Hölder inequality to obtain

$$\frac{1}{2} \frac{d}{dt} \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^m v\|_{L^2}^2 \le \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^m v\|_{L^2} \|\partial_x^m F\|_{L^2}. \tag{2.10}$$

On the other hand, we will use the similar method but applying ∂_x^{m+1} to the equation $\partial_t v = F(t, \partial_x u)$ and we obtain

$$\frac{1}{2} \frac{d}{dt} \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^{m+1} v\|_{L^2}^2 \le \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^{m+1} v\|_{L^2} \|\partial_x^{m+1} F\|_{L^2}. \tag{2.11}$$

Adding the equation (2.10) and (2.11), we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^m v\|_{H^1}^2 &\le \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 (\|\partial_x^m v\|_{L^2} \|\partial_x^m F\|_{L^2} + \|\partial_x^{m+1} v\|_{L^2} \|\partial_x^{m+1} F\|_{L^2}) \\
&\le C_2 \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^m v\|_{H^1} \|\partial_x^m F\|_{H^1} \\
&\le C_2 \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^m v\|_{H^1} \|\partial_x^{m+1} u\|_{H^1}.
\end{aligned}$$

Then, through the integration we can get that

$$\begin{aligned}
\frac{1}{2} \frac{\rho^{2(m+1)} m^2}{[(m+1)!]^4} \|\partial_x^m v(t)\|_{H^1}^2 &\le \frac{1}{2} \frac{\rho^{2(m+1)} m^2}{[(m+1)!]^4} \|\partial_x^m v_0\|_{H^1}^2 + C_2 \int_0^t \frac{\rho^{2(m+1)} m^2}{[(m+1)!]^4} \|\partial_x^m v\|_{H^1} \|\partial_x^{m+1} u\|_{H^1} ds \\
&\le \frac{1}{2} \frac{\rho^{2m}}{(m!)^4} \|\partial_x^m v_0\|_{H^1}^2 + C_2 \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho} - \rho} ds.
\end{aligned}$$

The last inequality is used in the result of (2.5) and note the fact that

$$\frac{\rho^{2(m+1)} m^2}{[(m+1)!]^4} \le \frac{\rho^{2m}}{(m!)^4} \frac{m^2}{(m+1)^4} \le \frac{\rho^{2m}}{(m!)^4}.$$

Now taking the upper bound on both sides of the above inequality we obtain

$$\sup_{m \ge 1} \frac{\rho^{2(m+1)}}{[(m+1)!]^4} m^2 \|\partial_x^m v(t)\|_{H^1}^2 \le C_1 \|v_0\|_{\rho_0}^2 + C_2 \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho} - \rho} ds. \tag{2.12}$$

At last, adding the equation (2.9) and (2.12) we obtain

$$|\vec{a}(t)|_\rho^2 \le C_1 (\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2) + C_2 \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}}^2}{\tilde{\rho} - \rho} ds. \tag{2.13}$$

3 Proof of the Main Result

In this section, we will prove the main result on the existence and uniqueness for the system (2.1). We adopt the similar idea in [16].

Proof of Theorem 1.1 The proof relies on the a priori estimate given in Theorem 2.1.

Step 1) First, we will proof the existence. On one hand, we should investigate the existence of approximate solution to the system (2.1)

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon = v_\varepsilon, \ (x;t) \in Q_{\infty,T} \\ \partial_t v_\varepsilon - \varepsilon \partial_x^2 v_\varepsilon = F(t, \partial_x u_\varepsilon), \ (x;t) \in Q_{\infty,T} \\ u_\varepsilon|_{t=0} = u_0, \ x \in \mathbb{R} \\ v_\varepsilon|_{t=0} = v_0, \ x \in \mathbb{R} \end{cases} \tag{3.1}$$

where $Q_{\infty,T} = \mathbb{R} \times (0,T]$ and $F(t, \partial_x u_\varepsilon)$ is analytic with respect to both x and t. On the other hand, we will derive a uniform estimate with respect to ε for the approximate solutions $(u_\varepsilon, v_\varepsilon)$.

The existence for the system (3.1) is standard. Indeed, this is a parabolic system of equations. Suppose that $u_0, v_0 \in G_{\rho_0}$, then we can construct a solution $(u_\varepsilon, v_\varepsilon) \in L^\infty\big([0, \widetilde{T}_\varepsilon]; G_{\rho_0}\big)$ to the system (3.1) for some $\widetilde{T}_\varepsilon > 0$ that may depend on ε.

It remains to derive a uniform estimate for the approximate solutions $(u_\varepsilon, v_\varepsilon)$, so that we can remove the ε-dependence of the lifespan $\widetilde{T}_\varepsilon$. To do so we denote

$$\vec{a}_\varepsilon = (u_\varepsilon, v_\varepsilon)$$

and define $|\vec{a}_\varepsilon|$ similarly as that of $|\vec{a}|$ (see (2.2)). We note that

$$\vec{a}_\varepsilon|_{t=0} = (u_0, v_0)\,.$$

Then we can verify directly that

$$\forall\ \rho < \rho_0 \le 1, \quad \left|\vec{a}_\varepsilon(0)\right|_\rho \le \left|\vec{a}_\varepsilon(0)\right|_{\rho_0} \le (\|u_0\|_{\rho_0} + \|v_0\|_{\rho_0}). \tag{3.2}$$

Let $\tau > 1$ be a fixed number to be determined later. We denote

$$C_0 = 2(\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2). \tag{3.3}$$

And we define

$$|\!|\!|\vec{a}_\varepsilon|\!|\!|_{(\tau)} \overset{\text{def}}{=} \sup_{\rho,t} \left(\frac{\rho_0 - \rho - \tau t}{\rho_0 - \rho} \right)^{1/2} \left|\vec{a}_\varepsilon(t)\right|_\rho, \tag{3.4}$$

where the supremum is taken over all pairs (ρ, t) such that $0 < \rho < 1, 0 \le t \le \rho_0/(4\tau)$ and $\rho + \tau t < \rho_0$. For any $t \in [0, \rho_0/(4\tau)]$,

$$\frac{\sqrt{2}}{2} \|u_\varepsilon(t)\|_{\frac{\rho_0}{2}} \le \frac{\sqrt{2}}{2} \left|\vec{a}_\varepsilon(t)\right|_{\frac{\rho_0}{2}} \le \left(\frac{\rho_0 - \frac{\rho_0}{2} - \tau t}{\rho_0 - \frac{\rho_0}{2}} \right)^{1/2} \left|\vec{a}_\varepsilon(t)\right|_{\frac{\rho_0}{2}} \le |\!|\!|\vec{a}_\varepsilon|\!|\!|_{(\tau)}. \tag{3.5}$$

Then similar to Theorem 2.1 we can repeat the argument in Section 2 with minor modification to obtain the following assertion: for any $t \in [0,\ \rho_0/(4\tau)]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \le 1$,

$$|\vec{a}_\varepsilon(t)|_\rho^2 \le C_1(\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2) + C_2 \int_0^t \frac{|\vec{a}_\varepsilon(s)|_{\tilde{\rho}}^2}{\tilde{\rho} - \rho} ds. \tag{3.6}$$

where $C_2 > 0$ is a constant depending only on the numbers ρ_0 and the Sobolev embedding constants but independent of ε, and the constant $C_1 \ge 1$ is just the one given in Theorem 2.1.

We let (ρ, t) be an arbitrary pair which is fixed at moment and satisfies that $0 < \rho < 1$, $t \in [0, \rho_0/(4\tau)]$ and $\rho + \tau t < \rho_0$. Then it follows from the definition (3.4) of $|||\vec{a}_\varepsilon|||_{(\tau)}$ that

$$\forall\, 0 \le s \le t, \quad |\vec{a}_\varepsilon(s)|_\rho \le |||\vec{a}_\varepsilon|||_{(\tau)} \left(\frac{\rho_0 - \rho}{\rho_0 - \rho - \tau s} \right)^{1/2}. \tag{3.7}$$

In addition, we take in particular such a $\tilde{\rho}(s)$ that

$$\tilde{\rho}(s) = \frac{\rho_0 + \rho - \tau s}{2}.$$

Then by direct calculation we can get that

$$\forall\, 0 \le s \le t, \qquad \rho < \tilde{\rho}(s) \quad \text{and} \quad \tilde{\rho}(s) + \tau s < \rho_0, \tag{3.8}$$

and

$$\forall\, 0 \le s \le t, \qquad \tilde{\rho}(s) - \rho = \frac{\rho_0 - \rho - \tau s}{2} = \rho_0 - \tilde{\rho}(s) - \tau s. \tag{3.9}$$

By the inequalities in (3.8) and the second equality in (3.9) it follows that, for any $0 \le s \le t$,

$$|\vec{a}_\varepsilon(s)|_{\tilde{\rho}(s)} \le |||\vec{a}_\varepsilon|||_{(\tau)} \left(\frac{\rho_0 - \tilde{\rho}(s)}{\rho_0 - \tilde{\rho}(s) - \tau s} \right)^{\frac{1}{2}} \le |||\vec{a}_\varepsilon|||_{(\tau)} \left(\frac{2\,(\rho_0 - \rho)}{\rho_0 - \rho - \tau s} \right)^{\frac{1}{2}}. \tag{3.10}$$

Putting (3.10) into the estimate (3.6) and using the first equality in (3.9), we have

$$\begin{aligned} |\vec{a}_\varepsilon(t)|_\rho^2 &\le C_1(\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2) + C_2 |||\vec{a}_\varepsilon|||_{(\tau)}^2 \int_0^t \frac{4\,(\rho_0 - \rho)}{(\rho_0 - \rho - \tau s)^2} ds \\ &\le C_1(\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2) + \frac{C_2}{\tau} |||\vec{a}_\varepsilon|||_{(\tau)}^2 \frac{\rho_0 - \rho}{\rho_0 - \rho - \tau t}. \end{aligned}$$

Thus we multiply both sides by the fact $(\rho_0 - \rho - \tau t)\,/\,(\rho_0 - \rho)$ and observe (ρ, t) is an arbitrary pair with $0 < \rho < 1$, $t \in [0, \rho_0/(4\tau)]$ and $\rho + \tau t < \rho_0$. And we can get that

$$|||\vec{a}_\varepsilon|||_{(\tau)}^2 \le C_1(\|u_0\|_{\rho_0}^2 + \|v_0\|_{\rho_0}^2) + \frac{C_2}{\tau} |||\vec{a}_\varepsilon|||_{(\tau)}^2. \tag{3.11}$$

Now we choose such a τ that

$$1 - \frac{C_2}{\tau} = C_1. \tag{3.12}$$

Then it follows from (3.11) that

$$|||\vec{a}_\varepsilon|||^2_{(\tau)} \leq (\|u_0\|^2_{\rho_0} + \|v_0\|^2_{\rho_0}),$$

which with (3.5) yields

$$\forall\ t \in [0, \rho_0/(4\tau)], \quad \|u_\varepsilon(t)\|^2_{\frac{\rho_0}{2}} \leq 2(\|u_0\|^2_{\rho_0} + \|v_0\|^2_{\rho_0}).$$

Now letting $\varepsilon \to 0$ we have, by compactness arguments, the limit u of u_ε solves the system (2.1). So we obtain

$$\|u(t)\|^2_\rho \leq \|u(t)\|^2_{\frac{\rho_0}{2}} \leq C_0, \tag{3.13}$$

recalling C_0 is given by (3.3).We complete the existence part of Theorem 1.1.

Step 2) Now,we will discuss the uniqueness. Suppose $(u_1, v_1), (u_2, v_2)$ are two solutions of the system (2.1). We define

$$(u, v) \overset{def}{=} (u_1, v_1) - (u_2, v_2) = (u_1 - u_2, v_1 - v_2).$$

So (u, v) satisfy the following equation

$$\begin{cases} \partial_t u = v. \\ \partial_t v = F(t, \partial_x u). \\ u|_{t=0} = 0. \\ v|_{t=0} = 0. \end{cases} \tag{3.14}$$

We replace u_0, v_0 in the system (2.1) with 0. Then we can follow the argument used in the existence part to get

$$|||\vec{a}_\varepsilon|||_{(\tau)} = \sup_{\rho,t} \left(\frac{\rho_0 - \rho - \tau t}{\rho_0 - \rho} \right)^{1/2} |\vec{a}_\varepsilon(t)|_\rho = 0 \tag{3.15}$$

where the supremun is taken over all pairs (ρ, t) such that $0 < \rho < 1$ and $\rho + \tau t < \rho_0$. This means $u = v = 0, i.e.$ for all $0 \leq t \leq T$, $u_1 = u_2$. Thus the proof of Theorem 1.1 is completed.

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Gevrey空间中抽象柯西-柯瓦列夫斯卡娅定理:能量方法

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摘要: 本文研究了非线性柯西问题的适定性问题. 利用经典的能量法和抽象柯西-柯瓦列夫斯卡娅定理, 得到非线性柯西问题在Gevrey空间中是适定的. 推广了已有文献在非线性柯西问题适定性方面的研究.

关键词: 柯西-柯瓦列夫斯卡娅定理; Gevrey空间; 能量法; 柯西问题

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