Vol. 41 ( 2021 )
No. 2

数 学 杂 志
J. of Math. (PRC)

# LOWER BOUND FOR THE BLOW-UP TIME OF THE SOLUTION TO A QUASI-LINEAR PARABOLIC PROBLEM

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**Abstract:** In this paper, using a delicate application of general Sobolev inequality, we establish the lower bound for the blow-up time of the solution to a quasi-linear parabolic problem,which improves the result of Theorem 2.1, Theorem 3.1 in [1], and the model (4.1) in [2].

**Keywords:** Quasi-linear parabolic equation; Initial-boundary value problem; lower bound for blow-up time

**2010 MR Subject Classification:** 35K59

**Document code:** A **Article ID:** 0255-7797(2021)02-0109-06

## 1 Introduction

In this paper, we will establish the lower bound for the blow-up time of the solution to the following problems:

$$\begin{cases} u_t = \triangle u^m + au^p \int_\Omega u^q \mathrm{d}x, & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $a > 0, m > 1$, $p \geq 0$ and $q \geq 0$, $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a smooth bounded domain, $\nu$ is the outward norm vector. The initial data $u_0(x)$ is a continuous nonnegative function and satisfies the compatible conditions. In [3], LI and XIE proved that the solution to (1.1) exists globally if $p+q < m$ or $p+q = m$ and $a$ is sufficiently small, while the solution will blow-up in finite time if $p+q > m$ and the initial data $u_0(x)$ is sufficiently large.

The direct motivation of this paper comes from the papers [1] and [2]. In [1], the authors estimated the lower bounds for the blow-up time of solution to (1.1) subject to Dirichlet boundary condition and Neumann boundary condition in 3-dimension space. In [2], the authors only established the lower bounds for the blow-up time of the solution to

* **Received date:** 2020-06-17 **Accepted date:** 2020-10-06

**Foundation item:** Supported by The Basic Scientific Research Foundation of Inner Mongolia Agricultural University(JC2016002).

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(1.1) subject to Dirichlet boundary condition with smooth bounded $\Omega \subset \mathbb{R}^n$ and $n \geq 3$. Naturally, we hope to obtain the lower bound for the blow-up time of the solution to (1.1) subject to Dirichlet boundary condition and Neumann boundary condition with smooth bounded $\Omega \subset \mathbb{R}^n$ and $n \geq 3$ . Inspired by Payne-Schaefer's idea and following the AN and SONG's methods in [4], we will use a delicate application of general Sobolev inequality to deal with both (1.1) subject to Neumann boundary condition and (1.1) subject to Dirichlet boundary condition. There are many results about the estimates of the lower bounds for blow-up time of the solution to parabolic equation. We can refer to [5-13] and the references therein to get more information.

Our main result in this paper can be stated as follows:

**Theorem 1.1** Assume that $u$ is the blow-up solution of (1.1), which will blow-up at time $t = t^*$. Then the lower bound for the blow-up time of the solution is

$$t^* \geq \int_{\phi(0)}^{\infty} \frac{\mathrm{d}\eta}{C_1 + C_2\eta^M}, \tag{1.2}$$

where $\phi(t) = \int_\Omega u^{nk}\mathrm{d}x$, $\phi(0) = \int_\Omega u_0^{nk}\mathrm{d}x$, with $n \geq 3$, $k > \max\{\frac{p+q-m}{2}, \frac{1}{n}\}$, and $M$, $C_1$, $C_2$ are given by the following section.

We will give the details to proof of Theorem 1.1 in the next section.

## 2 Lower Bound for the Blow-Up Time

In this section, using a delicate application of general Sobolev inequality, we will establish the lower bound for the blow-up time of the solution to (1.1).

Proof of Theorem 1.1. Define

$$\phi(t) = \int_\Omega u^{nk}\mathrm{d}x, \tag{2.1}$$

with $n \geq 3$, $k > \max\{\frac{p+q-m}{2}, \frac{1}{n}\}$.

Using Green formula, we have

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \int_\Omega nku^{nk-1}u_t\mathrm{d}x \\
&= \int_\Omega nku^{nk-1}(\Delta u^m + au^p\int_\Omega u^q\mathrm{d}x)\mathrm{d}x \\
&= \int_\Omega nku^{nk-1}\Delta u^m\mathrm{d}x + nka\int_\Omega u^q\mathrm{d}x\int_\Omega u^{nk+p-1}\mathrm{d}x \\
&= -\int_\Omega nkm(nk-1)u^{nk+m-3}|\nabla u|^2\mathrm{d}x + nka\int_\Omega u^q\mathrm{d}x\int_\Omega u^{nk+p-1}\mathrm{d}x \\
&= -\frac{4nkm(nk-1)}{(nk+m-1)^2}\int_\Omega |\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x + nka\int_\Omega u^q\mathrm{d}x\int_\Omega u^{nk+p-1}\mathrm{d}x.
\end{aligned} \tag{2.2}$$

Using Hölder inequality to the last term in the right of (2.2), we get

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \leq -\frac{4nkm(nk-1)}{(nk+m-1)^2}\int_\Omega |\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x + nka|\Omega|\int_\Omega u^{nk+p+q-1}\mathrm{d}x, \tag{2.3}$$

where $|\Omega|$ denotes the measure of $\Omega$.We denote that $C(\Omega)$ is the best constant in general Sobolev's inequality

$$(\int_\Omega w^{\frac{2n}{n-2}}\mathrm{d}x)^{\frac{n-2}{2n}} \le C(\Omega)(\int_\Omega |w|^2+|\nabla w|^2\mathrm{d}x)^{\frac{1}{2}}$$

for any $w\in H^1(\Omega)$.

In convenience, we denote

$$m_1=\frac{n(2k+m-1)}{n(2k+m-1)-(n-2)(p+q-1)},\quad m_2=\frac{n(2k+m-1)}{(n-2)(p+q-1)},$$
$$m_3=\frac{2k+m}{2k+m-(p+q)},\qquad m_4=\frac{2k+m}{p+q}.$$

Then using Hölder inequality and Young's inequality, we have

$$\begin{aligned}\int_\Omega u^{nk+p+q-1}\mathrm{d}x &\le \left(\int_\Omega u^{nk}\mathrm{d}x\right)^{\frac{1}{m_1}}\left(\int_\Omega u^{\frac{nk+m-1}{2}\cdot\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{1}{m_2}}\\ &\le C(\varepsilon)\left(\int_\Omega u^{nk}\mathrm{d}x\right)^{\frac{m_3}{m_1}}+\varepsilon\left(\int_\Omega u^{\frac{nk+m-1}{2}\cdot\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{m_4}{m_2}}.\end{aligned}\tag{2.4}$$

Here $\varepsilon$ will be chosen later, while

$$C(\varepsilon)=\frac{(\varepsilon m_4)^{-\frac{m_3}{m_4}}}{m_3}=\frac{2k+m-(p+q)}{2k+m}\left(\frac{\varepsilon(2k+m)}{p+q}\right)^{-\frac{p+q}{2k+m-(p+q)}}.$$

Using general Sobolev's inequality and Young's inequality to the last term in the right of (2.4), we get

$$\begin{aligned}&\varepsilon\left(\int_\Omega u^{\frac{nk+m-1}{2}\cdot\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{m_4}{m_2}}\\ \le&\varepsilon\left[(C(\Omega))^{\frac{2n}{n-2}}\left(\int_\Omega |u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x+\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x\right)^{\frac{n}{n-2}}\right]^{\frac{m_4}{m_2}}\\ \le&\varepsilon\left[1-\frac{nm_4}{(n-2)m_2}\right](C(\Omega))^{\frac{2nm_4}{(n-2)m_2}\cdot\frac{1}{1-\frac{nm_4}{(n-2)m_2}}}\\ &+\frac{nm_4\varepsilon}{(n-2)m_2}\left(\int_\Omega |u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x+\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x\right)\\ =&\varepsilon\left[1-\frac{nm_4}{(n-2)m_2}\right](C(\Omega))^{\frac{2(2k+m)(p+q-1)}{2k+m-(p+q)}}\\ &+\frac{nm_4\varepsilon}{(n-2)m_2}\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x+\frac{nm_4\varepsilon}{(n-2)m_2}\int_\Omega |u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x.\end{aligned}\tag{2.5}$$

By Young's inequality to the last term in the right of (2.5), we get

$$\begin{aligned}
&\frac{nm_4\varepsilon}{(n-2)m_2}\int_\Omega |u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x=\int_\Omega \frac{nm_4\varepsilon}{(n-2)m_2}u^{nk+m-1}\mathrm{d}x\\
\le&\int_\Omega\left[\frac{nk+m-1}{nk+p+q-1}(u^{nk+m-1})^{\frac{nk+p+q-1}{nk+m-1}}+\frac{p+q-m}{nk+p+q-1}\left(\frac{nm_4\varepsilon}{(n-2)m_2}\right)^{\frac{nk+p+q-1}{p+q-m}}\right]\mathrm{d}x\\
=&\frac{nk+m-1}{nk+p+q-1}\int_\Omega u^{nk+p+q-1}\mathrm{d}x+|\Omega|\frac{p+q-m}{nk+p+q-1}\left(\frac{nm_4\varepsilon}{(n-2)m_2}\right)^{\frac{nk+p+q-1}{p+q-m}}.
\end{aligned}\tag{2.6}$$

Substituting (2.6) into (2.5), we obtain

$$\begin{aligned}
&\varepsilon\left(\int_\Omega u^{\frac{nk+m-1}{2}\cdot\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{m_4}{m_2}}\\
\le&\varepsilon\left[1-\frac{nm_4}{(n-2)m_2}\right](C(\Omega))^{\frac{2(2k+m)(p+q-1)}{2k+m-(p+q)}}+\frac{nm_4\varepsilon}{(n-2)m_2}\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x \qquad (2.7)\\
&+\frac{nk+m-1}{nk+p+q-1}\int_\Omega u^{nk+p+q-1}\mathrm{d}x+|\Omega|\frac{p+q-m}{nk+p+q-1}\left(\frac{nm_4\varepsilon}{(n-2)m_2}\right)^{\frac{nk+p+q-1}{p+q-m}}.
\end{aligned}$$

Substituting (2.7) into (2.4), we have

$$\begin{aligned}
&\frac{p+q-m}{nk+p+q-1}\int_\Omega u^{nk+p+q-1}\mathrm{d}x\\
\le&C(\varepsilon)\left(\int_\Omega u^{nk}\mathrm{d}x\right)^{\frac{m_3}{m_1}}+\frac{nm_4\varepsilon}{(n-2)m_2}\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x+\left[1-\frac{nm_4}{(n-2)m_2}\right]\varepsilon(C(\Omega))^{\frac{2(2k+m)(p+q-1)}{2k+m-(p+q)}}\\
&+|\Omega|\frac{p+q-m}{nk+p+q-1}\left(\frac{nm_4\varepsilon}{(n-2)m_2}\right)^{\frac{nk+p+q-1}{p+q-m}}.
\end{aligned}\tag{2.8}$$

Substituting (2.8) into (2.3), we get

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}t}\le&-\left[\frac{4nkm(nk-1)}{(nk+m-1)^2}-\frac{nm_4\varepsilon}{(n-2)m_2}\frac{|\Omega|ank(nk+p+q-1)}{p+q-m}\right]\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x\\
&+\frac{|\Omega|ank(nk+p+q-1)}{p+q-m}C(\varepsilon)\phi^{\frac{m_3}{m_1}}+|\Omega|^2ank\left(\frac{nm_4\varepsilon}{(n-2)m_2}\right)^{\frac{nk+p+q-1}{p+q-m}}\\
&+\frac{|\Omega|ank(nk+p+q-1)}{p+q-m}\left[1-\frac{nm_4}{(n-2)m_2}\right]\varepsilon(C(\Omega))^{\frac{2(2k+m)(p+q-1)}{2k+m-(p+q)}}.
\end{aligned}$$

Now we can choose

$$\varepsilon=\frac{4m(p+q)(nk-1)(p+q-m)(2k+m-1)}{a|\Omega|(nk+m-1)^2(nk+p+q-1)(2k+m)(p+q-1)}$$

to make the coefficient of $\int_\Omega|\nabla u^{\frac{nk+m-1}{2}}|^2\mathrm{d}x$ vanishes. Then we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}\le C_1+C_2\phi^M,\tag{2.9}$$

where

$$
\begin{aligned}
M =& \frac{(2k+m)[n(2k+m-1)-(n-2)(p+q-1)]}{n[2k+m-(p+q)](2k+m-1)}, \\
C_1 =& \frac{|\Omega| ank(nk+p+q-1)[2k+m-(p+q)]}{(p+q-m)(2k+m)} \\
& \times \left( \frac{4m(nk-1)(p+q-m)(2k+m-1)}{|\Omega| a(nk+m-1)^2(nk+p+q-1)(p+q-1)} \right)^{-\frac{p+q}{2k+m-(p+q)}}, \qquad (2.10) \\
C_2 =& |\Omega|^2 ank \left( \frac{4m(nk-1)(p+q-m)}{(nk+m-1)^2(nk+p+q-1)} \right)^{\frac{nk+p+q-1}{p+q-m}} \\
& + \frac{4nkm[2k+m-(p+q)](nk-1)}{(nk+m-1)^2(2k+m)(p+q-1)} (C(\Omega))^{\frac{2(2k+m)(p+q-1)}{2k+m-(p+q)}}.
\end{aligned}
$$

Integrating (2.9), we have

$$
t^* \geq \int_{\phi(0)}^{\infty} \frac{\mathrm{d}\eta}{C_1 + C_2 \eta^M} \qquad (2.11)
$$

with

$$
\phi(0) = \int_\Omega u_0^{nk} \mathrm{d}x.
$$

By the analysis above, we can get the proof of Theorem 1.1.

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# 一类拟线性抛物型方程解的爆破时间的下界

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**摘 要:** 本文巧妙应用广义Sobolev不等式, 研究了一类拟线性抛物型方程解的爆破时间的下界, 该结果推广了文献[1]中的定理2.1和定理3.1的结论, 同样完善了文献[2]中的模型(4.1)的结论.

**关键词:** 拟线性抛物型方程; 初边值问题; 爆破时间的下界

MR(2010)**主题分类号:** 35K59 **中图分类号:** O29