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RESEARCH ANNOUNCEMENT ON “LAGRANGE ELEMENTS HOLD DISCRETE COMPACTNESS BASED ON THE CLASSICAL VARIATIONAL FORMULATION OF THE MAXWELL EIGENPROBLEM”

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1 Introduction and Main Results

Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded Lipschitz domain, multiply connected. Let Γ denote the boundary of Ω, with a finite number of connected components Γ_i, $1 \leq i \leq N_\Gamma$, where N_Γ is an integer number. Let Ω be occupied with discontinuous, anisotropic and inhomogeneous media of $\boldsymbol{\mu}$(magnetic permeability) and $\boldsymbol{\varepsilon}$ (electric permittivity). Let $\mathbf{curl} := \nabla\times$ and $\mathrm{div} := \nabla\cdot$, with ∇ being the gradient operator, and let $\mathbf{n}$ be the outward unit normal vector to Γ. The Maxwell eigenproblem reads as follows:

Find eigenvalue $\omega^2 > 0$ and eigenfunction $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}) = \omega^2\boldsymbol{\varepsilon}\mathbf{u} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{1.1c}$$

$$\int_{\Gamma_i} (\boldsymbol{\varepsilon}\mathbf{u}) \cdot \mathbf{n} = 0, \quad 2 \leq i \leq N_\Gamma. \tag{1.1d}$$

We study the Lagrange finite element method for numerically solving the above problem (1.1). As is well-known, the Maxwell eigenproblem is usually solved by the so-called edge elements in the finite element discretization. The classic edge elements are Nédélec elements ([1, 2]). Although the Lagrange elements have been the conventional and classical finite elements in the numerical solution of partial differential equations, they have been relatively not as popular as the edge elements in computational electromagnetism; as a matter of fact, in general meshes, some lower-order Lagrange elements such as the linear element have been widely recognized and well-known to generate spurious and incorrect discrete eigenmodes of the Maxwell eigenproblem. For this reason, there has

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been highly interesting in studying the Lagrange finite element method, typically for the Maxwell's equations including the Maxwell eigenproblem, on how the Lagrange finite element method is used for obtaining correct and spurious-free approximations, cf. [3–13], and the references cited therein. However, all the existing Lagrange finite element methods in the literature that are valid for the correct numerical solutions of the Maxwell's equations, and in particular, that are spurious-free and spectral-correct for the spectral approximations of the Maxwell eigenproblem, are based on some non-classical continuous or discrete variational formulations; all the discrete variational formulations have modifications, different from the classical variational formulation (1.2).

The classical variational formulation for the Maxwell eigenproblem reads as follows:

Find $(\omega^2 > 0, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times H_0(\mathbf{curl}\,;\Omega)$ such that for all $\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)$,

$$(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \omega^2(\varepsilon\mathbf{u}, \mathbf{v}). \tag{1.2}$$

The classical variational formulation (1.2) naturally incorporates all the constraints (1.1b) and (1.1d) of the Maxwell eigenproblem (1.1), while the perfect-conductor Dirichlet boundary condition is kept in the solution space $H_0(\mathbf{curl}\,;\Omega) := \{\mathbf{v} \in (L^2(\Omega))^d : \mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^{2d-3}, \mathbf{n} \times \mathbf{v}|_\Gamma = \mathbf{0}\}$, where in two dimensions, $\mathbf{t} \cdot \mathbf{v}|_\Gamma = 0$ with $\mathbf{t}$ being the counterclockwise unit tangential vector along Γ. Meanwhile, the classical variational formulation (1.2) is much simpler, particularly in implementations, than other variational formulations, and has been employed in practice. The well-known fact is that many of the Nédélec elements are spurious-free and spectral-correct for the classical variational formulation (1.2) in the finite element discretization. However, the issue has been open on whether the Lagrange elements could be spurious-free and spectral-correct for the classical variational formulation (1.2). Now, we have proven that all the Lagrange elements of any order are indeed spurious-free and spectral-correct for the classical variational formulation (1.2). The key and crucial property is the so-called discrete compactness, which is also the key and crucial property for those edge elements ([14]) which are spurious-free and spectral-correct for approximating the Maxwell eigenproblem with the use of the classical variational formulation (1.2). We have proven that all the Lagrange elements of any order hold the discrete compactness based on the classical variational formulation (1.2) in the finite element discretization. The meshes on which the Lagrange elements live may be required to be refined in some suitable manner for lower-order Lagrange elements and may be further required to have no singular vertices, edges, etc, for higher-order Lagrange elements.

Let $\mathcal{T}_h$ denote the shape-regular triangulations into triangles in two dimensions and tetrahedra in three dimensions. Let $\mathcal{S}_h$ denote the triangulations of Ω into triangles in two dimensions and tetrahedra in three dimensions; $\mathcal{S}_h$ is obtained possibly from $\mathcal{T}_h$ by some suitable refinements. Let $\mathbf{V}_h \subset H_0(\mathbf{curl}\,;\Omega) \cap (H^1(\Omega))^d$ denote the finite element space defined with respect to $\mathcal{S}_h$. Each component of the finite element function $\mathbf{v}_h \in \mathbf{V}_h$ is $H^1(\Omega)$-conforming, called C^0 elements or Lagrange elements; all components of $\mathbf{v}_h$ are independent of each other. The Lagrange finite element method based on the classical variational formulation (1.2) reads as follows:

Find $(\omega_h^2 > 0, \mathbf{u}_h \neq \mathbf{0}) \in \mathbb{R} \times \mathbf{V}_h$, such that for all $\mathbf{v}_h \in \mathbf{V}_h$,

$$(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) = \omega_h^2(\varepsilon\mathbf{u}_h, \mathbf{v}_h). \tag{1.3}$$

For any element (triangle or tetrahedron) $S \in \mathcal{S}_h$, let $P_\ell(S)$ denote the usual space of polynomials of total degree less than or equal to $\ell \geq 1$ for the integer ℓ. The Lagrange finite element space

$$\mathbf{V}_h := \{\mathbf{v}_h \in H_0(\mathbf{curl}\,;\Omega) \cap (H^1(\Omega))^d : \mathbf{v}_h|_S \in \mathbf{P}(S), \forall S \in \mathcal{S}_h\} \tag{1.4}$$

is taken as either of the following Lagrange elements which include all Lagrange elements of any order:

- $\star$ Two dimensions($d = 2$):

$$\mathbf{P}(S) = \begin{cases} (P_\ell(S))^2 \text{ on Powell-Sabin refinement } \mathcal{S}_h \text{ of } \mathcal{T}_h, \text{ where } \ell = 1, \\ (P_\ell(S))^2 \text{ on Clough-Tocher refinement } \mathcal{S}_h \text{ of } \mathcal{T}_h, \text{ where } \ell = 2, 3, \\ (P_\ell(S))^2 \text{ on } \mathcal{S}_h := \mathcal{T}_h \text{ itself, where } \ell \geq 4. \end{cases} \tag{1.5}$$

- $\star$ Three dimensions($d = 3$):

$$\mathbf{P}(S) = \begin{cases} (P_\ell(S))^3 \text{ on Worsey-Piper refinement } \mathcal{S}_h \text{ of } \mathcal{T}_h, \text{ where } \ell = 1, \\ (P_\ell(S))^3 \text{ on Worsey-Farin refinement } \mathcal{S}_h \text{ of } \mathcal{T}_h, \text{ where } \ell = 2, 3, \\ (P_\ell(S))^3 \text{ on Alfeld refinement } \mathcal{S}_h \text{ of } \mathcal{T}_h, \text{ where } \ell = 4, 5, 6, 7, \\ (P_\ell(S))^3 \text{ on } \mathcal{S}_h := \mathcal{T}_h \text{ itself, where } \ell \geq 8. \end{cases} \tag{1.6}$$

In the above, all the refinements $\mathcal{S}_h$ can be found in the monograph [15]. For the purpose of theoretical analysis, we introduce a scalar Hilbert space

$$Q := \{q \in H^1(\Omega) : q|_{\Gamma_1} = 0, \quad q|_{\Gamma_i} = \text{constant}, 2 \leq i \leq N_\Gamma\}$$

and let $Q_h \subset Q \cap H^2(\Omega)$ denote the C^1 element on the same meshes $\mathcal{S}_h$, which is a scalar $H^2(\Omega)$-conforming C^1 element. All the scalar $H^2(\Omega)$-conforming C^1 finite element spaces Q_h of lower-order can be found in the monograph [15]; higher-order C^1 elements can also be similarly defined. Define the so-called discrete divergence-free space

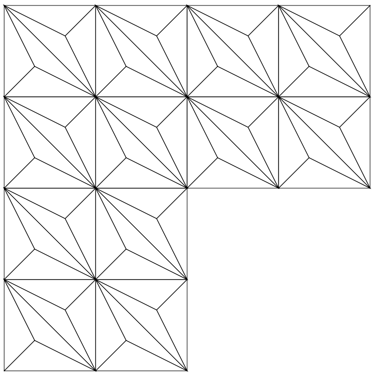
$$\mathbf{K}_h = \{\mathbf{v}_h \in \mathbf{V}_h : (\boldsymbol{\varepsilon}\mathbf{v}_h, \nabla q_h) = 0, \forall q_h \in Q_h\}. \tag{1.7}$$

Theorem 1.1 [Discrete Compactness] Let $\mathbf{K}_h$ be defined by (1.7), while $\mathbf{V}_h$ is defined by (1.4), with either of $\mathbf{P}(S)$ given in the above (1.5) and (1.6). Then, $\mathbf{K}_h$ is compactly imbedded into $(L^2(\Omega))^d$, i.e., the discrete compactness holds.

Remark 1.1 In two dimensions, for Clough-Tocher refinement $\mathcal{S}_h$ of $\mathcal{T}_h$, Theorem 1.1 has been essentially proven in [13], the first theory and the first result which have proved the discrete compactness of the Lagrange elements P_ℓ of any order $\ell \geq 2$ on the Clough-Tocher meshes (of course, including the Lagrange elements P_ℓ given by (1.5) for all $\ell \geq 2$). The theory in [13] is also straightforwardly applicable to the Lagrange elements P_ℓ given by (1.5) for all $\ell \geq 1$.

2 Numerical Results

Figure 1 Clough-Tocher mesh



We report some numerical results using the P_2 quadratic element on the Clough-Tocher refinement. The L-shaped domain $(-1,1)^2 \setminus ([0,1] \times [-1,0])$. The first five eigenvalues are available as follows (`https://perso.univ-rennes1.fr/monique.dauge/benchmax.html`):

$$1.47562182408, 3.53403136678, 9.86960440109, 9.86960440109, 11.3894793979.$$

The regularity for the eigenfunctions of the above five eigenvalues are as follows: for any $\epsilon > 0$, the 1rst Maxwell eigenfunction has $(H^{2/3-\epsilon}(\Omega))^2$, the 2nd one belongs to $(H^{4/3-\epsilon}(\Omega))^2$, the 3rd and 4th ones are analytic (exact value of the eigenvalue $\pi^2 = 9.86960440108936$), and the 5th one is $(H^{4/3-\epsilon}(\Omega))^2$. The numerical results of the quadratic element on the Clough-Tocher mesh in Figure 1 are reported in Table 1.

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Table 1 Discrete eigenvalues and convergence rate in L-shaped domain

ω^2	$1/h$	ω_h^2	$\lvert\omega^2-\omega_h^2\rvert/\lvert\omega^2\rvert$	Conv. rate
1.47562182408	2	1.26681348398191	1.4151E-01	-
	4	1.38962063231217	5.8281E-02	1.28
	8	1.44097107456963	2.3482E-02	1.31
	16	1.46178932082432	9.3740E-03	1.32
	32	1.47011963434306	3.7287E-03	1.33
	64	1.47343627187179	1.4811E-03	1.33
3.53403136678	2	3.52498803221372	2.5589E-03	-
	4	3.53253603986291	4.2312E-04	2.60
	8	3.53378944123257	6.8456E-05	2.63
	16	3.53399284521416	1.0900E-05	2.65
	32	3.53402527360988	1.7241E-06	2.66
	64	3.53403040550787	2.7200E-07	2.66
9.86960440109	2	9.88660746614568	1.7228E-03	-
	4	9.87079931009014	1.2107E-04	3.83
	8	9.86968257808880	7.9210E-06	3.93
	16	9.86960935618144	5.0206E-07	3.98
	32	9.86960471195374	3.1497E-08	3.99
	64	9.86960442053050	1.9697E-09	4.00
9.86960440109	2	9.88689421410594	1.7518E-03	-
	4	9.87080223812718	1.2137E-04	3.85
	8	9.86968260534025	7.9237E-06	3.94
	16	9.86960935666377	5.0210E-07	3.98
	32	9.86960471196398	3.1498E-08	3.99
	64	9.86960442053312	1.9700E-09	4.00
11.3894793979	2	11.39646630279200	6.1345E-04	-
	4	11.38926835095360	1.8530E-05	5.05
	8	11.38926511169730	1.8814E-05	-0.02
	16	11.38943266189000	4.1034E-06	2.20
	32	11.38947119689670	7.2005E-07	2.51
	64	11.38947805334070	1.1805E-07	2.61

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