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# THE $\alpha$-BERNSTEIN OPERATOR OF A LIPSCHITZ CONTINUOUS FUNCTION

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**Abstract:** The relationship between the $\alpha$-Bernstein operator and its approximation function is discussed in this paper. If the $\alpha$-Bernstein operator belongs to the class of Lipschitz functions, then its approximation function $f(x)$ also does, and vice versa. Moreover, the $\alpha$-Bernstein operator can maintain the Lipschitz constant of the original function.

**Keywords:** function approximation; $\alpha$-Bernstein operators; Lipschitz continuity; Lipschitz functions

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## 1 Introduction

In 1885, Weierstrass published his famous theorem asserting that every continuous function in a compact interval of the real line is the uniform limit of a sequence of algebraic polynomials in [1]. Several different proofs of Weierstrass's theorem are known, but a remarkable one was given by Bernstein [2] in 1912. As a generalization of the Bernstein operator, the $\alpha$-Bernstein operator proposed by Chen in [3] has almost the same approximation property as the Bernstein operator.

Suppose a function $f(x)$ is continuous on [0,1] . The $n$th $(n \geq 1)$ $\alpha$ -Bernstein operator of $f(x)$ is denoted and defined by $T_{n,\alpha}(f;x) = \sum\limits_{i=0}^{n} f_i \, p_{n,i}^{(\alpha)}(x)$, where $f_i = f(\frac{i}{n})$ and $0 \leq \alpha \leq 1$ . For $i = 0, 1, \cdots, n$ , the $\alpha$-Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree $n$ is defined by $p_{1,0}^{(\alpha)}(x) = 1 - x$, $p_{1,1}^{(\alpha)}(x) = x$, and when $n \geq 2$,

$$p_{n,i}^{(\alpha)}(x) = \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right] x^{i-1}(1-x)^{n-i-1}.$$

Obviously, $T_{n,\alpha}(f;x)$ is an algebraic polynomial of degree no greater than $n$ and its importance in approximation theory arises from the fact that $\lim\limits_{n\to\infty} T_{n,\alpha}(f;x) = f(x)$ uniformly on [0,1] . When $\alpha = 1$ , the $\alpha$-Bernstein polynomial reduces to the classical Bernstein

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polynomial, i.e.,

$$p_{n,i}^{(1)}(x) = \binom{n}{i} x^i (1-x)^{n-i}.$$

So the $\alpha$-Bernstein operator has following identity

$$T_{n,1}(f;x) = \sum_{i=0}^{n} f_i \binom{n}{i} x^i (1-x)^{n-i} = B_n(f;x),$$

which means that the class of $\alpha$-Bernstein operators contains the classical Bernstein ones.

$\alpha$-Bernstein operators are given explicitly and depend only on the values of a function for rational values of the variable. They have various shape-preserving properties and are easy to handle in computer algebra systems and these are very useful when the evaluation of $f$ is difficult and time-consuming. One of the outstanding properties of these polynomials is that they mimic the behavior of the given function $f(x)$ to a remarkable degree. Some results that hold for continuous functions, e.g., an upper bound for the error is obtained in terms of the usual modulus of continuity. However, there are special classes of functions for which better estimates can be given. Thus, we consider the case of Lipschitz functions.

Lipschitz continuity, named after German mathematician Rudolf Lipschitz, is a strong form of uniform continuity for functions. In the theory of differential equations, Lipschitz continuity is the central condition of the Picard-Lindelöf theorem which guarantees the existence and uniqueness of the solution to an initial value problem. A special type of Lipschitz continuity, called contraction, is used in the Banach fixed point theorem. We have the following chain of inclusions for functions over a closed and bounded interval: continuously differentiable $\subseteq$ Lipschitz continuous $\subseteq$ uniformly continuous $=$ continuous.

A real-valued function $f(x) : [0,1] \to \mathbb{R}$ is called Lipschitz continuous or to satisfy a Hölder condition of order $\beta \in (0,1]$ if there exists a positive real constant $M$ such that for every pair of points $x_1, x_2 \in [0,1]$ ,

$$| f(x_1) - f(x_2) | \leq M | x_1 - x_2 |^{\beta}. \tag{1}$$

Sometimes a Hölder condition of order $\beta$ is also called a uniform Lipschitz condition of order $\beta$ . Any such $M$ , which depends upon $f(x)$ and $\beta$ , referred to as a Lipschitz constant for the function $f(x)$ . The smallest constant is sometimes called the (best) Lipschitz constant of the function (or modulus of uniform continuity). For $f(x)$ satisfying (1), we usually write $f(x) \in \mathrm{Lip}_M \beta$ .

Obviously, a Lipschitz continuous function must be a continuous function for all $\beta \in (0,1]$ . Intuitively, a Lipschitz continuous function is limited in how fast it can change when $\beta = 1$ : there exists a real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number. An everywhere differentiable function is Lipschitz continuous if and only if it has bounded first derivative. If $f(x) \in \mathrm{Lip}_M \beta$ , then $f(x)$ is uniformly continuous.

Several proofs are known for the remainder in approximations of Lipschitz functions by Bernstein operators. Kac considered these classes in [4–5]. In 1987, Brown et al. [6] gave an elementary proof that the Lipschitz constant is preserved, i.e., if $f(x) \in \mathrm{Lip}_M\beta$ , then for $n = 1, 2, \cdots$, $B_n(f;x) \in \mathrm{Lip}_M\beta$ also. The case $\beta = 1$ was considered by Hajek in [7] and the general case was studied in [8] (another more complicated proof, using probabilistic methods was given in [9]).

The objective of this paper is to prove that the $\alpha$-Bernstein operator belongs to the class of Lipschitz functions, then its approximation function $f(x)$ also does, and vice versa.

## 2 Main Lemma

For convenience of description, the $\alpha$-Bernstein operator for $f(x)$ is usually expressed as $T_{n,\alpha}(f;x) = (1-\alpha)G_n(f;x) + \alpha B_n(f;x)$, where

$$G_n(f;x) = \sum_{i=0}^{n-1} g_i \binom{n-1}{i} x^i(1-x)^{n-i-1}, B_n(f;x) = \sum_{i=0}^{n} f_i \binom{n}{i} x^i(1-x)^{n-i},$$

and

$$g_i = (1-\frac{i}{n-1})f_i + \frac{i}{n-1}f_{i+1}. \tag{2}$$

**Lemma 1** If $f(x) \in \mathrm{Lip}_M\beta$, then for $n = 1, 2, \cdots$ , $i, j, i+j = 0, 1, \cdots, n-1$ ,

$$|g_{i+j} - g_i| \le M\left[(1-\frac{j}{n-1})(\frac{j}{n})^\beta + \frac{j}{n-1}(\frac{j+1}{n})^\beta\right]. \tag{3}$$

**Proof** It easily follows from (1) that

$$|f_{i+j+1} - f_{i+1}|,\ |f_{i+j} - f_i| \le M(\frac{j}{n})^\beta,\ |f_{i+j+1} - f_{i+j}| \le \frac{M}{n^\beta}. \tag{4}$$

It follows from (2) that

$$\begin{aligned}|g_{i+j} - g_i| &= \left|\left[(1-\frac{i+j}{n-1})f_{i+j} + \frac{i+j}{n-1}f_{i+j+1}\right] - \left[(1-\frac{i}{n-1})f_i + \frac{i}{n-1}f_{i+1}\right]\right| \\ &= \left|\frac{i}{n-1}(f_{i+j+1} - f_{i+1}) + (1-\frac{i}{n-1})(f_{i+j} - f_i) + \frac{j}{n-1}(f_{i+j+1} - f_{i+j})\right| \\ &\le \frac{i}{n-1}|f_{i+j+1} - f_{i+1}| + (1-\frac{i}{n-1})|f_{i+j} - f_i| + \frac{j}{n-1}|f_{i+j+1} - f_{i+j}|.\end{aligned}$$

Substituting (4) into the above inequality, we can obtain

$$\begin{aligned}|g_{i+j} - g_i| &\le \frac{i}{n-1}M(\frac{j}{n})^\beta + (1-\frac{i}{n-1})M(\frac{j}{n})^\beta + \frac{j}{n-1}M(\frac{1}{n})^\beta \\ &= M\left[(\frac{j}{n})^\beta + \frac{j}{n-1}(\frac{1}{n})^\beta\right] \le M\left[(1-\frac{j}{n-1})(\frac{j}{n})^\beta + \frac{j}{n-1}(\frac{j+1}{n})^\beta\right].\end{aligned}$$

In fact, the last inequality above is equivalent to $j^\beta + 1 \le (j+1)^\beta$ , and Lemma 1 is proved.

**Lemma 2** If $f(x)$ is convex on [0,1] , then $T_{n,\alpha}(f;x) \ge f(x)$ for all $x \in [0,1]$ and $n \ge 1$.

**Proof** For each $x \in [0,1]$ , let us define $x_r = \frac{r}{n}$ and $\lambda_r = p_{n,r}^{(\alpha)}(x), 0 \le r \le n$. We see that $\lambda_r \ge 0$ when $x \in [0,1]$ , and note from $T_{n,\alpha}(1;x) = 1$ and $T_{n,\alpha}(x;x) = x$ , respectively, that $\sum\limits_{r=0}^{n} \lambda_r = 1$ and $\sum\limits_{r=0}^{n} \lambda_r x_r = x$. Then we obtain since $f(x)$ is convex on [0,1] ,

$$T_{n,\alpha}(f;x) = \sum_{r=0}^{n} \lambda_r f(x_r) \ge f(\sum_{r=0}^{n} \lambda_r x_r) = f(x),$$

and this completes the proof.

**Lemma 3** For $f(x) = x^\beta, 0 < \beta \le 1$ and for $n = 1,2,\cdots$ , $T_{n,\alpha}(x^\beta;h) \le h^\beta$, $0 \le h \le 1$.

**Proof** As a consequence of Lemma 2, we have, since the function $f(x) = -x^\beta, 0 < \beta \le 1$, is convex on [0,1], that for $n = 1,2,\cdots$ , $T_{n,\alpha}(-x^\beta;x) \ge -x^\beta$. It follows from $T_{n,\alpha}(-x^\beta;x) = -T_{n,\alpha}(x^\beta;x)$ that $T_{n,\alpha}(x^\beta;x) \le x^\beta, n = 1,2,\cdots$ . This completes the proof.

## 3 Main Result

One of the most interesting connections between direct and converse results for $\alpha$-Bernstein operator like the classic operator is given in Theorem 1. The following theorem is the main result of this paper.

**Theorem 1** Fix $\beta \in (0,1]$, $M > 0$, and $f \in C[0,1]$. The following assertions are equivalent

(i) $f(x) \in \mathrm{Lip}_M\beta$ .

(ii) For any $n \ge 1$ , $T_{n,\alpha}(f;x) \in \mathrm{Lip}_M\beta$ .

**Proof** (i) $\Rightarrow$(ii) Fix points $x_1, x_2 \in [0,1]$ , $x_1 \le x_2$. We use the representation

$$\begin{aligned} G_n(f;x_2) &= \sum_{i=0}^{n-1} g_i \binom{n-1}{i} x_2^i (1-x_2)^{n-i-1} \\ &= \sum_{i=0}^{n-1} g_i \binom{n-1}{i} (1-x_2)^{n-i-1} \sum_{k=0}^{i} \binom{i}{k} x_1^k (x_2-x_1)^{i-k} \\ &= \sum_{i=0}^{n-1} \sum_{k=0}^{i} g_i \binom{n-1}{i} \binom{i}{k} x_1^k (x_2-x_1)^{i-k} (1-x_2)^{n-i-1}. \end{aligned}$$

Let us put $i = j + k$ . We can write $\sum\limits_{i=0}^{n-1} \sum\limits_{k=0}^{i} = \sum\limits_{k=0}^{n-1} \sum\limits_{j=0}^{n-k-1}$ . We also have

$$\binom{n-1}{j+k} \binom{j+k}{k} = \binom{n-1}{k} \binom{n-k-1}{j}.$$

So, we can write the double summation as

$$\begin{aligned} G_n(f;x_2) &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} g_{j+k} \binom{n-1}{j+k} \binom{j+k}{k} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1} \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} g_{j+k} \binom{n-1}{k} \binom{n-k-1}{j} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1}. \end{aligned}$$

And, in a similar way,

$$\begin{aligned}G_n(f;x_1) &= \sum_{k=0}^{n-1} g_k \binom{n-1}{k} x_1^k (1-x_1)^{n-k-1} \\ &= \sum_{k=0}^{n-1} g_k \binom{n-1}{k} x_1^k \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} (x_2-x_1)^j (1-x_2)^{n-k-j-1} \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} g_k \binom{n-1}{k} \binom{n-k-1}{j} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1}.\end{aligned}$$

From the representations obtained above, for $x_1 \le x_2$, one has

$$\begin{aligned}&|G_n(f;x_2) - G_n(f;x_1)| \\ =&\left|\sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \binom{n-1}{k} \binom{n-k-1}{j} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1} (g_{k+j} - g_k)\right| \\ \le&\sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \binom{n-1}{k} \binom{n-k-1}{j} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1} |g_{k+j} - g_k|.\end{aligned}$$

Because we can write

$$\begin{aligned}&\sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} = \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1}, \\ &|G_n(f;x_2) - G_n(f;x_1)| \\ \le&\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} \binom{n-1}{k} \binom{n-k-1}{j} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1} |g_{k+j} - g_k|.\end{aligned}$$

Using the binomial expansion theorem and

$$\binom{n-1}{k} \binom{n-k-1}{j} = \binom{n-1}{j} \binom{n-j-1}{k},$$

we have

$$\begin{aligned}&|G_n(f;x_2) - G_n(f;x_1)| \\ \le&\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} \binom{n-1}{j} \binom{n-j-1}{k} x_1^k (x_2-x_1)^j (1-x_2)^{n-k-j-1} |g_{k+j} - g_k| \\ =&\sum_{j=0}^{n-1} |g_{k+j} - g_k| \binom{n-1}{j} (x_2-x_1)^j \sum_{k=0}^{n-j-1} \binom{n-j-1}{k} x_1^k (1-x_2)^{n-j-1-k} \\ =&\sum_{j=0}^{n-1} |g_{k+j} - g_k| \binom{n-1}{j} (x_2-x_1)^j [(1-x_2)+x_1]^{n-j-1}.\end{aligned}$$

It follows from (3) that

$$
\begin{aligned}
&|G_n(f;x_2) - G_n(f;x_1)| \\
\leq &M \sum_{j=0}^{n-1} \left[ (1 - \frac{j}{n-1})(\frac{j}{n})^\beta + \frac{j}{n-1}(\frac{j+1}{n})^\beta \right] \binom{n-1}{j} (x_2 - x_1)^j [1 - (x_2 - x_1)]^{n-1-j} \\
= &MG_n(x^\beta; x_2 - x_1).
\end{aligned}
$$

In the same way, we also have

$$|B_n(f;x_2) - B_n(f;x_1)| \leq MB_n(x^\beta; x_2 - x_1).$$

From the two inequalities obtained above, for $x_1 \leq x_2$, one has

$$
\begin{aligned}
& |T_{n,\alpha}(f;x_2) - T_{n,\alpha}(f;x_1)| \\
= & |(1-\alpha)[G_n(f;x_2) - G_n(f;x_1)] + \alpha[B_n(f;x_2) - B_n(f;x_1)]| \\
\leq & (1-\alpha)|G_n(f;x_2) - G_n(f;x_1)| + \alpha|B_n(f;x_2) - B_n(f;x_1)| \\
\leq & (1-\alpha)MG_n(x^\beta; x_2 - x_1) + \alpha MB_n(x^\beta; x_2 - x_1) \\
= & MT_{n,\alpha}(x^\beta; x_2 - x_1).
\end{aligned}
$$

It follows from Lemma 3 that $|T_{n,\alpha}(f;x_2) - T_{n,\alpha}(f;x_1)| \leq M(x_2 - x_1)^\beta$, where $M$ is the Lipschitz constant of $f$. Thus, we see that $T_{n,\alpha}(f;x) \in \mathrm{Lip}_M\beta$.

(ii)$\Rightarrow$ (i) Fix $f \in C[0,1]$ and assume that, for $x_1, x_2 \in [0,1]$ and any $n \in \mathbb{N}$,

$$|T_{n,\alpha}(f;x_1) - T_{n,\alpha}(f;x_2)| \leq M|x_1 - x_2|^\beta.$$

And given $\varepsilon > 0$, there must be $N \in \mathbb{N}$, such that for any $n > N$, $|f - T_{n,\alpha}(f)| < \frac{\varepsilon}{2}$. Then

$$
\begin{aligned}
|f(x_1) - f(x_2)| = &|f(x_1) - T_{n,\alpha}(f;x_1) + T_{n,\alpha}(f;x_1) - T_{n,\alpha}(f;x_2) + T_{n,\alpha}(f;x_2) - f(x_2)| \\
\leq &|f(x_1) - T_{n,\alpha}(f;x_1)| + |T_{n,\alpha}(f;x_1) - T_{n,\alpha}(f;x_2)| + |T_{n,\alpha}(f;x_2) - f(x_2)| \\
< &\frac{\varepsilon}{2} + M|x_1 - x_2|^\beta + \frac{\varepsilon}{2} \\
= &M|x_1 - x_2|^\beta + \varepsilon.
\end{aligned}
$$

Thus we see $f(x) \in \mathrm{Lip}_M\beta$. This completes the proof.

The interesting thing about this result is that each of the $\alpha$-Bernstein operator $T_{n,\alpha}(f;x)$, for $n = 1, 2, \cdots$, has the same Lipschitz constant as the given function $f$ when considered as being in the class of functions $\mathrm{Lip}_M\beta$ .

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# Lipschitz连续函数的$\alpha$-Bernstein算子

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**摘要**: 本文讨论了$\alpha$-Bernstein算子与其逼近函数间的一个关系: 若$\alpha$-Bernstein算子满足Lipschitz连续, 那么其逼近的函数也满足Lipschitz连续, 反之亦然, 而且$\alpha$-Bernstein 算子保持原来函数的Lipschitz常数.

**关键词**: 函数逼近; $\alpha$-Bernstein算子; Lipschitz连续; Lipschitz函数

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