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EIGENVALUE INEQUALITIES FOR QUADRATIC POLYNOMIAL OPERATOR OF THE HORIZONTAL LAPLACIAN ON A CARNOT GROUP

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Abstract: In this paper, we investigate the Dirichlet eigenvalue problem of quadratic polynomial operator of the horizontal Laplacian on a Carnot group and establish some inequalities for its eigenvalues, which cover the results of [10] for the biharmonic horizontal Laplacian.

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1 Introduction

In recent years, with the increasing attention to analysis and geometry in metric space, the investigations of Carnot-Carathéodory (CC) spaces as a special kind of metric space were carried out with prosperous results. In this case, the Carnot group plays a fundamental role. As we all know, it is a graded Lie group whose Lie algebra is nilpotent. Roughly speaking, the Carnot group can be regarded as a local model of CC spaces since they can be seen as the natural tangent space of the CC space, just as the Euclidean space is tangent to the manifold (see [1, 2]). Carnot groups occupy a central position in the study of harmonic analysis, partial differential equation, sub-Riemannian geometry, mechanical engineering and so on (cf. [3–5]).

As we know, the Heisenberg group $\mathbb{H}^n$ is the classical example of a non-Abelian Carnot group. Niu and Zhang [6], Sun [7, 8] established some universal inequalities of eigenvalues for the Kohn-Laplacian, the Folland-Stein operator on the Heisenberg group.

The horizontal Laplacian Δ_H on a Carnot group G is a hypoelliptic operator of Hörmander type. In 2013, Aribi and El Soufi [9] gave some universal bounds for the eigenvalues of the horizontal Laplacian on Carnot groups. In 2017, Du, Wu, Li and Xia [10] investigated the

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following eigenvalue problem of the biharmonic horizontal Laplacian on a bounded domain Ω in a Carnot group G with an d_1-dimensional sub-bundle

$$\begin{cases} \Delta_H^2 u = \lambda u, \quad u \in \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where v is the outwards unit normal vector field of $\partial\Omega$. They obtained the following inequalities for eigenvalues of this problem

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le (\frac{8d_1+16}{d_1^2})^{\frac{1}{2}}\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{1}{2}}\}^{\frac{1}{2}}\{\sum_{i=i}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{1.2}$$

and

$$\sum_{i=1}^{k}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}} \le 2\sqrt{2d_1+4}\lambda_1^{\frac{1}{2}}. \tag{1.3}$$

Quadratic polynomial operator of the Laplacian is one important kind of differential operator in the research of differential geometry and partial differential equation (see [11]). In this paper, we consider the following Dirichlet weighted eigenvalue problem of quadratic polynomial operator of the horizontal Laplacian

$$\begin{cases} \Delta_H^2 u - a\Delta_H u + bu = \lambda\rho u, \quad u \in \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where ρ is a positive function continuous on $\bar{\Omega}$ and the constants $a, b \ge 0$. It is known that this eigenvalue problem has a discrete spectrum $0 < \lambda_1 \le \cdots \le \lambda_k \le \cdots \nearrow$, where each eigenvalue is repeated with its multiplicity (see [12]). In particular, when $\rho \equiv 1$ and $b = 0$, problem (1.4) becomes the following eigenvalue problem

$$\begin{cases} \Delta_H^2 u - a\Delta u = \lambda u, \quad u \in \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

Furthermore, when $a = 0$, problem becomes (1.1).

We derive the following results for problem (1.4).

Theorem 1.1 Let Ω be a bounded domain in a Carnot group G with an d_1-dimensional sub-bundle. Denote by λ_i the i-th eigenvalue of problem (1.4). Set $\sigma = (\inf_{\bar{\Omega}} \rho)^{-1}$ and $\tau = (\max_{\bar{\Omega}} \rho)^{-1}$. Then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{2\sigma^{\frac{1}{2}}}{\tau d_1}\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\big[2(d_1+2)\xi_i + ad_1\sigma\big]\}^{\frac{1}{2}}\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\xi_i\}^{\frac{1}{2}}, \tag{1.6}$$

where

$$\xi_i = \frac{-a\sigma + \sqrt{a^2\sigma^2 + 4\sigma(\lambda_i - b\tau)}}{2}.$$

Theorem 1.2 Under the assumptions of Theorem 1.1, we have

$$\sum_{j=1}^{d_1}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}} \leq \frac{2\sigma^{\frac{1}{2}}}{\tau}[(4+2d_1)\xi_1+a\sigma d_1]^{\frac{1}{2}}\xi_1^{\frac{1}{2}}. \tag{1.7}$$

From Theorem 1.1 and Theorem 1.2, we can get the following results for problem (1.5).

Corollary 1.1 Let Ω be a bounded domain in a Carnot group G with an d_1-dimensional sub-bundle. Denote by λ_i the i-th eigenvalue of problem (1.5). Then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{2}{d_1}\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[2(d_1+2)\zeta_i+ad_1\right]\}^{\frac{1}{2}}\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\zeta_i\}^{\frac{1}{2}} \tag{1.8}$$

and

$$\sum_{j=1}^{d_1}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}} \leq 2[(4+2d_1)\zeta_1+ad_1]^{\frac{1}{2}}\zeta_1^{\frac{1}{2}}. \tag{1.9}$$

where $\zeta_i=\frac{-a+\sqrt{a^2+4\lambda_i}}{2}$.

Remark It is easy to find that $\xi_i=\lambda_i^{\frac{1}{2}}$ when $\rho\equiv 1$ and $a=b=0$. Thus (1.6) and (1.7) respectively become (1.2) and (1.3) when $\rho\equiv 1$ and $a=b=0$. Therefore our results cover the results of Du, Wu, Li and Xia [10] for the biharmonic horizontal Laplacian.

2 Proof of Main Results

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2.

For the convenience of readers, we fisrt give some basic knowledge about the Carnot group. A Carnot group G of step r is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g}=V_1\oplus V_2\oplus\cdots\oplus V_r$. It is r-nilpotent, i.e., $[V_1,V_j]=V_{j+1}, j=1,\cdots,r-1, [V_j,V_r]=0, j=1,\cdots,r$. We also assume that there exists a scalar product $\langle,\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, such that the $V_j's$ are mutually orthogonal. The whole $\mathfrak{g}$ is generated by the layer V_1 which induces a subbundle HG of TG of rank $d_1=\dim V_1$. We call HG the horizontal bundle of the Carnot group. Let $\{e_1^i,\cdots,e_{d_i}^i, i\leq r\}$ be an orthonormal basis of V_i and $\{X_1^i,\cdots,X_{d_i}^i\}$ denotes the system of left invariant vector fields that coincides with $\{e_1^i,\cdots,e_{d_i}^i\}$ at the identity element of G. We consider G is endowed with a left-invariant Riemannian metric $\mathfrak{g}_G$ with respect to which the family $\{X_1^1,\cdots,X_{d_r}^r\}$ constitute an orthonormal frame for TG. The corresponding Levi-Civita connection ∇ induces a connection ∇^H on HG that we call "horizontal connection": if X is a vector field and Y is a horizontal vector field on G, then $\nabla_X^H Y=\pi_H\nabla_X Y$, where $\pi_H: TG\to HG$ is the orthogonal projection. The horizontal Laplacian Δ_H is defined by

$$\Delta_H u := \mathrm{div}_H\nabla^H u=\sum_{j=1}^{d_1}\left(X_j^1\right)^2 u, \tag{2.1}$$

where $u\in C^2$.

Define $S^{2,2}(\Omega)$ by

$$S^{2,2}(\Omega):=\{f\in L^2(\Omega)||\nabla^H f|,|\nabla^H f|^2\in L^2(\Omega).\}, \tag{2.2}$$

where $|\nabla^H f|^2 = \sum\limits_{j=1}^{d_1} (X_j^1 f)^2$. Then $S^{2,2}(\Omega)$ is a Hilbert space with a Sobolev norm $\|f\|^2 = \int_\Omega (\sum\limits_{p=1}^{2} |(\nabla^H)^p f|^2 + f^2)$. Furthermore, we consider the subspace $S_0^{2,2}(\Omega)$ defined by

$$S_0^{2,2}(\Omega) := \{f \in S^{2,2}(\Omega) \Big| f\,|_{\partial\Omega} = \frac{\partial f}{\partial \upsilon}\,|_{\partial\Omega} = 0\}. \tag{2.3}$$

For every $f, g \in S_0^{2,2}(\Omega)$, we have

$$\int_\Omega f X_j^1 g = -\int_\Omega g X_j^1 f. \tag{2.4}$$

Proof of Theorem 1.1 In order to construct a good trial function, we use some functions introduced by Danielli, Garofalo and Nhieu [4]. Since the Carnot group G of step r is nilpotent, the exponential exp: $\mathfrak{g} \to G$ is a global diffeomorphism. Setting $\{e_1, \cdots, e_{d_1}\}$ and $\{X_1^1, \cdots, X_{d_1}^1\}$ be an orthonormal basis of V_1 and the system of left invariant vector fields, respectively. We can define a smooth map $x_i : G \to R$ by

$$x_i(g) := \langle \exp^{-1}(g), e_i \rangle_{\mathfrak{g}}, \quad i = 1, \cdots, d_1. \tag{2.5}$$

These functions satisfy

$$X_j^1(x_i) = \delta_{ij}, \quad \Delta_H x_i = 0 \quad \text{for} \quad i, j = 1, \cdots, d_1. \tag{2.6}$$

Denote by u_i the i-th weighted orthonormal eigenfunction of problem (1.5) corresponding to the eigenvalue λ_i, namely, we have

$$\begin{cases} \Delta_H^2 u_i - a\Delta_H u_i + bu_i = \lambda_i \rho u_i, \quad u_i \in \Omega, \\ u_i|_{\partial\Omega} = \dfrac{\partial u_i}{\partial \nu}|_{\partial\Omega} = 0, \\ \displaystyle\int_\Omega \rho u_i u_j = \delta_{ij}. \end{cases} \tag{2.7}$$

Set $\varphi_{ij} = x_i u_j - \sum\limits_{l=1}^{k} q_{ij}^l u_l$, where $q_{ij}^l = \int_\Omega \rho x_i u_j u_l$. It is easy to find

$$\varphi_{ij}\,|_{\partial\Omega} = \frac{\partial \varphi_{ij}}{\partial \nu}\,|_{\partial\Omega} = 0, \quad \int_\Omega \rho \varphi_{ij} u_l = 0 \quad \text{for} \quad l = 1, \cdots, k. \tag{2.8}$$

Using the Rayleigh-Ritz inequality, we have

$$\begin{aligned} \lambda_{k+1} \int_\Omega \rho \varphi_{ij}^2 &\le \int_\Omega \varphi_{ij} \left[\Delta_H^2 \varphi_{ij} - a\Delta_H \varphi_{ij} + b\varphi_{ij}\right] \\ &= \lambda_j \int_\Omega \rho \varphi_{ij}^2 + \int_\Omega \varphi_{ij} \left[(\Delta_H^2 \varphi_{ij} - a\Delta_H \varphi_{ij} + b\varphi_{ij}) - \lambda_j \rho x_i u_j\right] \\ &= \lambda_j \int_\Omega \rho \varphi_{ij}^2 + \int_\Omega \varphi_{ij} (\Delta_H^2 - a\Delta_H + b - \lambda_j \rho)(x_i u_j) \\ &= \lambda_j \int_\Omega \rho \varphi_{ij}^2 + \int_\Omega x_i u_j (\Delta_H^2 - a\Delta_H + b - \lambda_j \rho)(x_i u_j) - \sum_{l=1}^{k} q_{ij}^l s_{ij}^l. \end{aligned} \tag{2.9}$$

Substituting

$$\begin{aligned} s_{ij}^l &= \int_\Omega u_l(\Delta_H^2 - a\Delta_H + b - \lambda_j\rho)(x_iu_j) \\ &= \int_\Omega x_iu_j\big(\Delta_H^2 - a\Delta_H + b\big)u_l - \lambda_j\int_\Omega \rho x_iu_ju_l \\ &= (\lambda_l - \lambda_j)q_{ij}^l \end{aligned} \tag{2.10}$$

into (2.9), we derive

$$(\lambda_{k+1} - \lambda_j)\int_\Omega \rho\varphi_{ij}^2 \le \int_\Omega x_iu_j(\Delta_H^2 - a\Delta_H + b - \lambda_j\rho)(x_iu_j) + \sum_{l=1}^k(\lambda_j - \lambda_l)(q_{ij}^l)^2. \tag{2.11}$$

By direct calculations, we have

$$\Delta_H(x_iu_j) = 2X_i^1u_j + x_i\Delta_Hu_j \tag{2.12}$$

and

$$\Delta_H^2(x_iu_j) = 2\Delta_HX_i^1u_j + 2X_i^1(\Delta_Hu_j) + x_i\Delta_H^2u_j. \tag{2.13}$$

Using (2.12) and (2.13), we obtain

$$(\Delta_H^2 - a\Delta_H + b)(x_iu_j) = 2\left[\Delta_HX_i^1u_j + X_i^1(\Delta_Hu_j) - aX_i^1u_j\right] + \lambda_j\rho x_iu_j. \tag{2.14}$$

Substituting (2.14) into (2.11), we get

$$\begin{aligned} (\lambda_{k+1} - \lambda_j)\int_\Omega \rho\varphi_{ij}^2 \le& 2\int_\Omega x_iu_j[\Delta_HX_i^1u_j + X_i^1(\Delta_Hu_j) - aX_i^1u_j] + \sum_{l=1}^k(\lambda_j - \lambda_l)(q_{ij}^l)^2 \\ =& 2\int_\Omega [X_i^1u_j\Delta_H(x_iu_j) - X_i^1(x_iu_j)\Delta_Hu_j] - 2a\int_\Omega x_iu_jX_i^1u_j \\ &+ \sum_{l=1}^k(\lambda_j - \lambda_l)(q_{ij}^l)^2 \\ =& \int_\Omega [4(X_i^1u_j)^2 - 2u_j\Delta_Hu_j + au_j^2] + \sum_{l=1}^k(\lambda_j - \lambda_l)(q_{ij}^l)^2. \end{aligned} \tag{2.15}$$

From

$$-2\int_\Omega x_iu_jX_i^1u_j = 2\int_\Omega u_jX_i^1(x_iu_j) = 2\int_\Omega u_j^2 + 2\int_\Omega x_iu_jX_i^1u_j \tag{2.16}$$

and

$$\tau \le \int_\Omega u_j^2 = \int_\Omega \frac{1}{\rho}\rho u_j^2 \le \sigma, \tag{2.17}$$

we can get

$$-2\int_\Omega x_iu_jX_i^1u_j = \int_\Omega u_j^2 \ge \tau. \tag{2.18}$$

Set $p_{ij}^l = \int_\Omega u_lX_i^1u_j$. Then we have

$$-2\int_\Omega \varphi_{ij}X_i^1u_j = -2\int_\Omega x_iu_jX_i^1u_j + 2\sum_{l=1}^k q_{ij}^lp_{ij}^l \ge \tau + 2\sum_{l=1}^k q_{ij}^lp_{ij}^l. \tag{2.19}$$

Multiplying the both sides of (2.19) by $(\lambda_{k+1}-\lambda_j)^2$, using the Cauchy-Schwarz inequality and (2.15), we have

$$\begin{aligned}
&(\lambda_{k+1}-\lambda_j)^2(\tau+2\sum_{l=1}^{k} q_{ij}^l p_{ij}^l)\\
\le& -2(\lambda_{k+1}-\lambda_j)^2\int_\Omega \sqrt{\rho}\varphi_{ij}(\frac{1}{\sqrt{\rho}}X_i^1 u_j-\sum_{l=1}^{k} p_{ij}^l\sqrt{\rho}u_l)\\
\le&\delta(\lambda_{k+1}-\lambda_j)^3\int_\Omega \rho\varphi_{ij}^2+\frac{(\lambda_{k+1}-\lambda_j)}{\delta}\int_\Omega(\frac{1}{\sqrt{\rho}}X_i^1 u_j-\sum_{l=1}^{k} p_{ij}^l\sqrt{\rho}u_l)^2\\
\le&\delta(\lambda_{k+1}-\lambda_j)^2\{\int_\Omega[4(X_i^1u_j)^2-2u_j\Delta_H u_j+au_j^2]+\sum_{l=1}^{k}(\lambda_j-\lambda_l)(q_{ij}^l)^2\}\\
&+\frac{\sigma}{\delta}(\lambda_{k+1}-\lambda_j)\int_\Omega(X_i^1u_j)^2-\frac{1}{\delta}(\lambda_{k+1}-\lambda_j)\sum_{l=1}^{k}(p_{ij}^l)^2,
\end{aligned}\tag{2.20}$$

where δ is any positive number. Taking sum on j from 1 to k in (2.20), and noticing $q_{ij}^l=q_{il}^j, p_{ij}^l=-p_{il}^j$, we get

$$\begin{aligned}
&\tau\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^2-2\sum_{j,l=1}^{k}(\lambda_{k+1}-\lambda_j)(\lambda_j-\lambda_l)q_{ij}^l p_{ij}^l\\
\le&\sum_{j=1}^{k}\delta(\lambda_{k+1}-\lambda_j)^2\int_\Omega\left[4(X_i^1u_j)^2-2u_j\Delta_H u_j+au_j^2\right]+\frac{\sigma}{\delta}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)\int_\Omega(X_i^1u_j)^2\\
&-\delta\sum_{j,l=1}^{k}(\lambda_{k+1}-\lambda_j)(\lambda_j-\lambda_l)^2(q_{ij}^l)^2-\frac{1}{\delta}\sum_{j,l=1}^{k}(\lambda_{k+1}-\lambda_j)(p_{ij}^l)^2.
\end{aligned}\tag{2.21}$$

It implies that

$$\begin{aligned}
\tau\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^2\le\delta\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega[4(X_i^1u_j)^2-2u_j\Delta_H u_j+au_j^2]\\
+\frac{\sigma}{\delta}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)\int_\Omega(X_i^1u_j)^2.
\end{aligned}\tag{2.22}$$

Taking sum on i from 1 to d_1 in (2.22), we get

$$\begin{aligned}
\tau d_1\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^2\le\delta\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega\left[2(d_1+2)|\nabla^H u_j|^2+ad_1u_j^2\right]\\
+\frac{\sigma}{\delta}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)\int_\Omega|\nabla^H u_j|^2.
\end{aligned}\tag{2.23}$$

Using the Schwarz inequality,

$$\begin{aligned}
\int_\Omega|\nabla^H u_j|^2=-\int_\Omega u_j\Delta_H u_j\le(\int_\Omega u_j^2)^{\frac12}(\int_\Omega(\Delta_H u_j)^2)^{\frac12}\\
\le\sigma^{\frac12}(\lambda_j-a\int_\Omega|\nabla^H u_j|^2-b\tau)^{\frac12}.
\end{aligned}\tag{2.24}$$

Then we have

$$(\int_{\Omega} |\nabla^H u_j|^2)^2 + a\sigma \int_{\Omega} |\nabla^H u_j|^2 - \sigma(\lambda_j - b\tau) \leq 0. \tag{2.25}$$

This is a quadratic inequality of $\int_{\Omega} |\nabla^H u_j|^2$. Thus we obtain

$$\int_{\Omega} |\nabla^H u_j|^2 \leq \xi_j. \tag{2.26}$$

Substituting (2.26) into (2.23), we get

$$\tau d_1 \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)^2 \leq \delta \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)^2 [2(d_1 + 2)\xi_j + a d_1 \sigma] + \frac{\sigma}{\delta} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)\xi_j. \tag{2.27}$$

Taking

$$\delta = \left\{ \frac{\sigma \sum\limits_{j=1}^{k} (\lambda_{k+1} - \lambda_j)\xi_j}{\sum\limits_{j=1}^{k} (\lambda_{k+1} - \lambda_j)^2 [2(d_1 + 2)\xi_j + a d_1 \sigma]} \right\}^{\frac{1}{2}}$$

in (2.27), we can obtain (1.6). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 Similar to the proof of Theorem 1.1, let x_i's be the functions satisfying (2.5) and (2.6). Denote by u_i the i-th weighted orthonormal eigenfunction of problem (1.5) corresponding to the eigenvalue λ_i. Define a $d_1 \times d_1$-matrix $Q = (q_{ij})_{d_1 \times d_1}$, where $q_{ij} = \int_{\Omega} \rho x_i u_1 u_{j+1}$. According to the QR-factorization theorem, we know that there exists an orthogonal matrix $T = (t_{ij})_{d_1 \times d_1}$ such that $R = TQ$ is an upper triangle matrix, namely, we have

$$R_{ij} = \sum_{k=1}^{d_1} t_{ik} c_{kj} = \sum_{k=1}^{d_1} \int_{\Omega} t_{ik} \rho x_k u_1 u_{j+1} = 0 \quad \text{for} \quad 0 < j < i \leq d_1. \tag{2.28}$$

Set $y_i = \sum\limits_{k=1}^{d_1} t_{ik} x_k$. Then (2.28) can be written as

$$\int_{\Omega} \rho y_i u_1 u_{j+1} = 0 \quad \text{for} \quad 0 < j < i \leq d_1. \tag{2.29}$$

Moreover, from (2.6), we get

$$X_j^1(y_i) = \sum_{k=1}^{d_1} t_{ik} \delta_{kj} = t_{ij}, \quad \Delta_H y_i = 0 \quad \text{for} \quad i, j = 1, \cdots, d_1. \tag{2.30}$$

Consider the function $\phi_i = (y_i - a_i)u_1$, where $a_i = \int_{\Omega} \rho y_i u_1^2$. It is easy to find that $\int_{\Omega} \rho \phi_i u_{j+1} = 0 \quad \text{for} \quad 0 \leq j < i \leq d_1$. Using the Rayleigh-Ritz inequality, we have

$$\begin{aligned} \lambda_{i+1} \int_{\Omega} \rho \phi_i^2 &\leq \int_{\Omega} \phi_i (\Delta_H^2 \phi_i - a\Delta_H \phi_i + b\phi_i) \\ &= \lambda_1 \int_{\Omega} \rho \phi_i^2 + \int_{\Omega} \phi_i \left[(\Delta_H^2 \phi_i - a\Delta_H \phi_i + b\phi_i) - \lambda_1 \rho y_i u_1 \right] \\ &= \lambda_1 \int_{\Omega} \rho \phi_i^2 + \int_{\Omega} \phi_i \left[\Delta_H^2 (y_i u_1) - a\Delta_H (y_i u_1) + b y_i u_1 - \lambda_1 \rho y_i u_1 \right] \\ &= \lambda_1 \int_{\Omega} \rho \phi_i^2 + \int_{\Omega} y_i u_1 \left[\Delta_H^2 (y_i u_1) - a\Delta_H (y_i u_1) + b y_i u_1 - \lambda_1 \rho y_i u_1 \right] - a_i s_i. \end{aligned} \tag{2.31}$$

Since

$$\begin{aligned} s_i =& \int_\Omega u_1 \left[\left(\Delta_H^2(y_iu_1) - a\Delta_H(y_iu_1) + by_iu_1\right) - \lambda_1\rho y_iu_1\right] \\ =& \int_\Omega y_iu_1\left(\Delta_H^2u_1 - a\Delta_Hu_1 + bu_1\right) - \lambda_1\int_\Omega \rho y_iu_1^2 \\ =&0, \end{aligned} \tag{2.32}$$

we get

$$(\lambda_{i+1}-\lambda_1)\int_\Omega \rho\phi_i^2 \le \int_\Omega y_iu_1[\Delta_H^2(y_iu_1) - a\Delta_H(y_iu_1) + by_iu_1 - \lambda_1\rho y_iu_1]. \tag{2.33}$$

On the one hand, using (2.30), we obtain

$$\begin{aligned} \Delta_H(y_iu_1) &= \sum_{j=1}^{d_1}(X_j^1)^2(y_iu_1) = \sum_{j=1}^{d_1}X_j^1\left(t_{ij}u_1 + y_iX_j^1u_1\right) \\ &= 2\sum_{j=1}^{d_1}t_{ij}X_j^1u_1 + y_i\Delta_Hu_1 \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} \Delta_H^2(y_iu_1) &= \Delta_H(2\sum_{j=1}^{d_1}t_{ij}X_j^1u_1 + y_i\Delta_Hu_1) \\ &= 2\sum_{j=1}^{d_1}t_{ij}\Delta_HX_j^1u_1 + 2\sum_{j=1}^{d_1}t_{ij}X_j^1(\Delta_Hu_1) + y_i\Delta_H^2u_1. \end{aligned} \tag{2.35}$$

It follows from (2.34) and (2.35) that

$$\begin{aligned} &(\Delta_H^2 - a\Delta_H + b)(y_iu_1) \\ =&2[\sum_{j=1}^{d_1}t_{ij}\Delta_H(X_j^1u_1) + \sum_{j=1}^{d_1}t_{ij}X_j^1(\Delta_Hu_1) - a\sum_{j=1}^{d_1}t_{ij}X_j^1u_1] + \lambda_1\rho y_iu_1. \end{aligned} \tag{2.36}$$

Substituting (2.36) into (2.33), we obtain

$$\begin{aligned} &(\lambda_{i+1}-\lambda_1)\int_\Omega \rho\phi_i^2 \\ \le&2\int_\Omega y_iu_1[\sum_{j=1}^{d_1}t_{ij}\Delta_H(X_j^1u_1) + \sum_{j=1}^{d_1}t_{ij}X_j^1(\Delta_Hu_1) - a\sum_{j=1}^{d_1}t_{ij}X_j^1u_1] \\ =&2\int_\Omega [\sum_{j=1}^{d_1}t_{ij}X_j^1u_1\Delta_H(y_iu_1) - \sum_{j=1}^{d_1}t_{ij}\Delta_Hu_1X_j^1(y_iu_1) + a\sum_{j=1}^{d_1}t_{ij}u_1X_j^1(y_iu_1)] \\ =&\int_\Omega [4(\sum_{j=1}^{d_1}t_{ij}X_j^1u_1)^2 - 2u_1\Delta_Hu_1 + a\sum_{j=1}^{d_1}t_{ij}^2u_1^2]. \end{aligned} \tag{2.37}$$

On the other hand, we have

$$\begin{aligned} \int_\Omega \phi_i\sum_{j=1}^{d_1}t_{ij}X_j^1u_1 &= -\sum_{j=1}^{d_1}t_{ij}^2\int_\Omega u_1^2 - \sum_{j=1}^{d_1}\int_\Omega (y_i-a_i)u_1t_{ij}X_j^1u_1 \\ &= -\int_\Omega u_1^2 - \int_\Omega \phi_i\sum_{j=1}^{d_1}t_{ij}X_j^1u_1. \end{aligned}$$

It implies

$$2\int_\Omega \phi_i \sum_{j=1}^{d_1} t_{ij} X_j^1 u_1 = -\int_\Omega u_1^2. \tag{2.38}$$

Multiplying the both sides of (2.38) by $(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}$, and using the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2 &= -2(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega \phi_i \sum_{j=1}^{d_1} t_{ij} X_j^1 u_1\\ &\le \delta(\lambda_{i+1}-\lambda_1)\int_\Omega \rho\phi_i^2 + \frac{1}{\delta}\int_\Omega \frac{1}{\rho}(\sum_{j=1}^{d_1} t_{ij} X_j^1 u_1)^2,\end{aligned} \tag{2.39}$$

where δ is any positive number. Substituting (2.37) into (2.39), and taking sum on i from 1 to d_1, we have

$$\begin{aligned}&\sum_{i=1}^{d_1}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2\\ \le&\delta\int_\Omega \sum_{i=1}^{d_1}[4(\sum_{j=1}^{d_1} t_{ij}X_j^1u_1)^2 - 2u_1\Delta_H u_1 + a\sum_{j=1}^{d_1} t_{ij}^2 u_1^2] + \frac{1}{\delta}\int_\Omega \frac{1}{\rho}\sum_{i=1}^{d_1}(\sum_{j=1}^{d_1} t_{ij}X_j^1 u_1)^2\\ =&4\delta\int_\Omega \sum_{j=1}^{d_1}(X_j^1u_1)^2 - 2d_1\delta\int_\Omega u_1\Delta_H u_1 + ad_1\delta\int_\Omega u_1^2 + \frac{1}{\delta}\int_\Omega\frac{1}{\rho}\sum_{j=1}^{d_1}(X_j^1u_1)^2\\ =&(4+2d_1)\delta\int_\Omega |\nabla^H u_1|^2 + \frac{1}{\delta}\int_\Omega \frac{1}{\rho}|\nabla^H u_1|^2 + ad_1\delta\int_\Omega u_1^2.\end{aligned} \tag{2.40}$$

Using (2.17) and (2.26) and (2.40), we get

$$\tau\sum_{j=1}^{d_1}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}} \le [(4+2d_1)\xi_1 + a\sigma d_1]\delta + \frac{\sigma}{\delta}\xi_1. \tag{2.41}$$

Taking

$$\delta = [\frac{\sigma\xi_1}{(4+2d_1)\xi_1 + a\sigma d_1}]^{\frac{1}{2}}$$

in (2.41), we obtain (1.7). This concludes the proof of Theorem 1.2.

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Carnot群上水平Laplace算子的二次多项式算子的特征值不等式

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摘 要: 本文研究了Carnot群上水平Laplace算子的二次多项式算子的Dirichlet特征值问题, 并建立了一些特征值不等式. 特别地, 我们的结果涵盖了文献 [10] 对双调和水平Laplace算子所获得的结果.

关键词: 特征值; 不等式; 水平Laplace算子; Carnot群

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