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一类脉冲分数阶偏微分方程解的振动性

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摘 要: 本文研究了一类带 Neumann 边界条件的脉冲分数阶偏微分方程解的振动性质. 利用修正后的 Riemann-Liouille 分数阶定义下的相关性质及 Riccati 变换和不等式技巧, 获得了一些判别解振动的充分条件, 并给出相关例子说明了主要结论, 推广了文献 [12] 中的结果.

关键词: 脉冲; 分数阶偏微分方程; 修正后的 Riemann-Liouille 分数阶导数; 振动
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1 引言

脉冲微分方程理论的出现使得对理论物理、化学、生物技术和人口动力学等学科中的某些过程和现象的精确模拟成为可能. 近年来, 由于分数阶微分方程在反常扩散、多孔介质力学、非牛顿流体力学、粘弹性力学、软物质力学、生物医学以及控制系统等领域的广泛应用, 其研究越来越受到人们的关注. 不论在理论还是应用中, 人们主要研究分数阶微分方程的定性性质, 研究方向众多, 如解的存在性以及带有初值条件或边界条件的分数阶微分方程解的稳定性, 其相关定性理论研究迅速发展, 并得到了一些研究成果. 偏微分方程解的振动问题已有一些学者通过研究得到了一些结论 [1]–[3]. 近年来, 分数阶微分方程的振动性问题受到广泛关注, 一些学者研究带阻尼项的分数阶微分方程的振动性 [4]–[10], 并陆续有很好的研究成果发表 [11]–[17]. 但关于脉冲分数阶微分方程的振动性研究却很少, 本篇论文借鉴现有的一些研究结果以及方法来进一步讨论脉冲分数阶偏微分方程的性质.

本文将讨论如下方程

$$\begin{cases} D^{\alpha}_{+,t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right)+p(t)D^{\alpha}_{+,t}u(x,t)=a(t)h(u)\Delta u(x,t) \\ +\sum_{i=1}^{j}a_i(t)h_i(u(x,t-\tau_i))\Delta u(x,t-\tau_i)-m(x,t,u(x,t))-q(x,t)F(u(x,t)), \quad t\neq t_k, \\ D^{\alpha}_{+,t}u(x,t_k^+)-D^{\alpha}_{+,t}u(x,t_k^-)=\alpha_k D^{\alpha}_{+,t}u(x,t_k),\ k=1,2,3,\cdots,(x,t)\in\Omega\times R_+\equiv E, \\ u(x,t_k^+)-u(x,t_k^-)=\beta_k u(x,t_k),\ k=1,2,3,\cdots,(x,t)\in\Omega\times R_+\equiv E, \end{cases} \tag{1.1}$$

边界条件

$$\frac{\partial u(x,t)}{\partial N}=0, \qquad (x,t)\in\partial\Omega\times R_+, \qquad t\neq t_k, \tag{1.2}$$

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其中 $\alpha \in (0,1)$, Δ 为拉普拉斯算子, Ω 是 R^n 内的有界域, $\partial\Omega$ 充分光滑, $\overline{\Omega} = \Omega \cup \partial\Omega$, N 为 $\partial\Omega$ 的单位外法向量, $0 < t_1 < \cdots < t_i < \cdots$ 且 $\lim\limits_{i\to\infty} t_i = +\infty$.

如下是本文的基本假设

(H1) $a(t), p(t), a_i(t) \in C(R_+; R_+)$, $r(t) \in C^{\alpha}(R_+; R_+)$ 且 $\tau_i \geq 0$ 是常数, $i \in I_j = \{1, 2 \cdots, j\}$; $\alpha_k > -1$, $\beta_k > -1$.

(H2) $q(x,t) \in C(\overline{E}; R_+)$, $q(t) = \min\limits_{x\in\overline{\Omega}} q(x,t)$; $h'(u), h_i^{'}(u), F(u) \in C(R;R)$, 对于 $x \neq 0$, 存在一个正常数 c, 使得 $\frac{F(x)}{x} \geq c > 0$; $u \neq 0$, $uh'(u) > 0$, $uh_i^{'}(u) > 0$.

(H3) $m \in C(\overline{E} \times R; R)$,

$$m(x,t,\eta) = \begin{cases} \geq 0, & \eta \in (0,+\infty), \\ \leq 0, & \eta \in (-\infty, 0). \end{cases}$$

(H4) $u(x,t)$ 和它的分数阶导数 $D_{+,t}^{\alpha}u(x,t)$ 是分段连续函数, $t = t_k, k = 1,2,3,\cdots$ 为第一类间断点, 且在 $t = t_k$ 处左连续, 即 $D_{+,t}^{\alpha}u(x,t_k^-) = D_{+,t}^{\alpha}u(x,t_k), u(x,t_k^-) = u(x,t_k)$.

定义 1.1 [18] $f: R_+ \to R$, 阶数为 $\alpha > 0$ 的 Riemann-Liouville 左侧分数阶积分定义如下

$$(I_{0+}^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1} f(v)dv. \tag{1.3}$$

上式在 R_+ 是逐点定义的, Γ 是 gamma 函数.

定义 1.2 [19] 修正后的 Riemann-Liouville 分数阶导数定义如下

$$\begin{cases} D_t^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \ 0 < \alpha < 1, \\ D_t^{\alpha}f(t) = (f^n(t))^{\alpha-n}, \ 1 \leq n \leq \alpha < n+1. \end{cases} \tag{1.4}$$

下面给出关于 α 阶修正后的 Riemann-Liouville 分数阶导数的一些计算公式

$$\begin{array}{rcl} D_t^{\alpha}t^r & = & \dfrac{\Gamma(1+\alpha)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \\ D_t^{\alpha}\left(f(t)h(t)\right) & = & h(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}h(t), \\ D_t^{\alpha}f[h(t)] & = & f_h' D_t^{\alpha}h(t) = D_h^{\alpha}f[h(t)](h'(t))^{\alpha}. \end{array}$$

及一些在本文的证明中要用到的记号

$$U(t) = \int_{\Omega} u(x,t)dx, \quad R(t) = I_+^{\alpha}\left(\frac{p(t)}{r(t)}\right), \quad \xi = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad \widetilde{r}(\xi) = r(t),$$

$$\widetilde{p}(\xi) = p(t), \quad \widetilde{q}(\xi) = q(t), \quad \widetilde{\psi}(\xi) = \psi(t), \quad \widetilde{R}(\xi) = R(t).$$

2 主要结果和证明

定理 2.1 如果下列脉冲分数阶微分不等式

$$\begin{cases} D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}U(t)\right] + p(t)D_{+,t}^{\alpha}U(t) \leq -cq(t)U(t) \\ D_{+,t}^{\alpha}U(t^+) = (1+\alpha_k)D_{+,t}^{\alpha}U(t), \qquad k = 1,2,3,\cdots, \\ U(t^+) = (1+\beta_k)U(t), \qquad k = 1,2,3,\cdots \end{cases} \tag{2.1}$$

没有最终正解, 且下列脉冲分数阶微分不等式

$$\begin{cases} D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}U(t)\right]+p(t)D_{+,t}^{\alpha}U(t)\geq -cq(t)U(t) \\ D_{+,t}^{\alpha}U(t^{+})=(1+\alpha_k)D_{+,t}^{\alpha}U(t), \qquad k=1,2,3,\cdots, \\ U(t^{+})=(1+\beta_k)U(t), \qquad k=1,2,3,\cdots \end{cases} \tag{2.2}$$

没有最终负解, 那么方程 (1.1) 和 (1.2) 的每个非平凡解 $u(x,t)$ 在 E 内都是振动的.

证 设 $u(x,t)$ 是方程 (1.1) 和 (1.2) 的一个非振动解. 不妨设存在 $t_0\geq 0$, 使得 $u(x,t)>0$, $u(x,t-\tau_i)>0$, $(x,t)\in\Omega\times[t_0,+\infty)$.

当 $t\neq t_k$. 将方程 (1.1) 的第一个式子关于 x 在 Ω 内积分, 得到

$$\begin{aligned} &D_{+,t}^{\alpha}\int_{\Omega}\left(r(t)D_{+,t}^{\alpha}u(x,t)\right)dx+p(t)\int_{\Omega}D_{+,t}^{\alpha}u(x,t)dx \\ =&a(t)\int_{\Omega}h(u)\Delta u(x,t)dx+\sum_{i=1}^{j}a_i(t)\int_{\Omega}h_i(u(x,t-\tau_i))\Delta u(x,t-\tau_i)dx \\ &-\int_{\Omega}m(x,t,u(x,t))dx-\int_{\Omega}q(x,t)F(u(x,t))dx. \end{aligned} \tag{2.3}$$

由 Green 公式、边值条件 (1.2) 及条件 (H2), 容易得到

$$\begin{aligned} &\int_{\Omega}h(u)\Delta u(x,t)dx=\int_{\partial\Omega}h(u)\frac{\partial u(x,t)}{\partial N}dS-\int_{\Omega}h'(u)\mid \mathrm{grad}\ u\mid^2 dx \\ =&-\int_{\Omega}h'(u)\mid \mathrm{grad}\ u\mid^2 dx\leq 0, \qquad t\geq t_0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\int_{\Omega}h_i(u(x,t-\tau_i))\Delta u(x,t-\tau_i)dx \\ =&-\int_{\Omega}h_i'(u(x,t-\tau_i))\mid \mathrm{grad}\ u(x,t-\tau_i)\mid^2 dx\leq 0, \qquad t\geq t_0, \end{aligned} \tag{2.5}$$

及

$$\int_{\Omega}m(x,t,u(x,t))dx\geq 0, \qquad t\geq t_0, \tag{2.6}$$

$$\int_{\Omega}q(x,t)F(u(x,t))dx\geq\int_{\Omega}cq(t)u(x,t)dx\geq cq(t)U(t), \qquad t\geq t_0. \tag{2.7}$$

由 (2.3)–(2.7) 式, 得到

$$D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}U(t)\right]+p(t)D_{+,t}^{\alpha}U(t)\leq -cq(t)U(t), \qquad t\geq t_0. \tag{2.8}$$

当 $t=t_k$. 分别对方程 (1.1) 的第二个式子和第三个式子关于 x 在 Ω 内积分, 得到

$$D_{+,t}^{\alpha}U(t_k^{+})=D_{+,t}^{\alpha}\int_{\Omega}u(x,t_k^{+})dx=(1+\alpha_k)D_{+,t}^{\alpha}\int_{\Omega}u(x,t_k)dx=(1+\alpha_k)D_{+,t}^{\alpha}U(t_k), \quad k=1,2,3,\cdots. \tag{2.9}$$

$$U(t_k^+)=\int_\Omega u(x,t_k^+)dx=(1+\beta_k)\int_\Omega u(x,t_k)dx=(1+\beta_k)U(t_k),\ k=1,2,3,\cdots. \tag{2.10}$$

由 (2.8)–(2.10) 式知, $U(t)=\int_\Omega u(x,t)dx$ 是脉冲分数阶微分不等式 (2.1) 的最终正解, 这与假设矛盾.

若 $u(x,t)$ 是脉冲分数阶微分方程 (1.1) 和 (1.2) 的最终负解. 用类似方法, 容易得到 $U(t)=\int_\Omega u(x,t)dx$ 是脉冲分数阶微分不等式 (2.2) 的最终负解, 这与假设矛盾. 定理得证.

引理 2.2 [18] 若 $G(t)=\int_0^t(t-v)^{-\alpha}U(v)dv,\ \ \alpha\in(0,1),\ \ t>0$, 则 $G'(t)=\Gamma(1-\alpha)(D_+^\alpha U)(t),\ \ \alpha\in(0,1),\ \ t>0$.

引理 2.3 若 $0<\alpha<1$, 则 $(D_+^\alpha I_{0+}^\alpha f)(x)=f(x)$.

证 由定义 1.1 和 1.2 可得

$$\begin{aligned}(D_+^\alpha I_{0+}^\alpha f)(x)&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{dt}{(x-t)^\alpha}(\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)ds}{(t-s)^{1-\alpha}}-I_{0+}^\alpha f(0))\\&=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{dt}{(x-t)^\alpha}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds\\&=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x f(s)ds\int_s^x(x-t)^{-\alpha}(t-s)^{\alpha-1}dt.\end{aligned}$$

令 $t=s+\mu(x-s)$, 利用 Beta 函数的定义得

$$\int_s^x(x-t)^{-\alpha}(t-s)^{\alpha-1}dt=\int_0^1\mu^{\alpha-1}(1-\mu)^{-\alpha}d\mu=B(\alpha,1-\alpha),$$

$$(D_+^\alpha I_{0+}^\alpha f)(x)=\frac{B(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x f(s)ds=f(x).$$

引理 2.4 [20] 假设 $w\in PC^1[R_+,R]$,

$$w'(t)\le g_1(t)w(t)+g_2(t),\qquad t\ne t_k,\quad t\ge t_0,$$

$$w(t_k^+)\le(1+\delta_k)w(t_k),\qquad k=1,2,3,\cdots,$$

其中 $g_1,g_2\in C[R_+,R]$, δ_k 是常数, $PC^1[R_+,R]=\{x(t):R_+\to R,\ x(t)$在除 $t=t_k,k=1,2,\cdots$ 以外的点连续可微, $x(t_k^+),x(t_k^-),x'(t_k^+),x'(t_k^-)$存在, 且 $x(t_k)=x(t_k^-),x'(t_k)=x'(t_k^-)\}$. 则

$$\begin{aligned}w(t)\le&w(t_0)\prod_{t_0<t_k<t}(1+\delta_k)\exp(\int_{t_0}^t g_1(s)ds)\\&+\int_{t_0}^t\prod_{s<t_k<t}(1+\delta_k)\exp(\int_s^t g_1(\sigma)d\sigma)g_2(s)ds,\qquad t\ge t_0.\end{aligned}$$

定理 2.5 假设存在 $t^*\ge0$,

$$\int_{t^*}^\infty\frac{1}{r(s)e^{R(s)}}ds=+\infty, \tag{2.11}$$

且

$$\lim_{\xi\to\infty}\int_{t^*}^{\xi}\prod_{t^*<\xi_k<s}\frac{1+\beta_k}{1+\alpha_k}\widetilde{\psi}(s)ds=+\infty, \tag{2.12}$$

其中 $\widetilde{\psi}(s)=ce^{\widetilde{R}(s)}\widetilde{q}(s)$. 那么方程 (1.1)–(1.2) 每个非平凡解 $u(x,t)$ 在 E 内振动.

证 用反证法. 假设 $U(t)$ 是脉冲分数阶微分不等式 (2.1) 的非振动解. 不失一般性, 假设 $U(t)$ 是脉冲分数阶微分不等式 (2.1) 的最终正解, 存在 $t^*\geq 0$, 使得 $U(t)>0, U(t-\tau_i)>0, G(t)>0$, $t\geq t^*$. 由 (2.1) 式及引理 2.3, 有

$$\begin{aligned}
D_+^\alpha\left[e^{R(t)}r(t)D_+^\alpha U(t)\right] &= e^{R(t)}D_+^\alpha\left[r(t)D_+^\alpha U(t)\right]+r(t)D_+^\alpha U(t)D_+^\alpha e^{R(t)}\\
&= e^{R(t)}D_+^\alpha\left[r(t)D_+^\alpha U(t)\right]+r(t)D_+^\alpha U(t)e^{R(t)}D_+^\alpha I_{0+}^\alpha\left(\frac{p(t)}{r(t)}\right)\\
&= e^{R(t)}D_+^\alpha\left[r(t)D_+^\alpha U(t)\right]+e^{R(t)}p(t)D_+^\alpha U(t)\\
&= e^{R(t)}\left[D_+^\alpha[r(t)D_+^\alpha U(t)]+p(t)D_+^\alpha U(t)\right]\\
&< -ce^{R(t)}q(t)U(t)<0.
\end{aligned} \tag{2.13}$$

由上式知 $e^{R(t)}r(t)D_+^\alpha U(t)$ 在 $t\geq t^*$ 上是减函数. 不妨设 $D_+^\alpha U(t)>0$, $t\in[t^*,\infty)$. 否则, 存在 $T\in[t^*,\infty)$, 使得 $D_+^\alpha U(T)<0$, $e^{R(t)}r(t)D_+^\alpha U(t)\leq e^{R(T)}r(T)D_+^\alpha U(T)=c_1<0$, 其中 c_1 在 $t\in[T,\infty)$ 是一个常数. 根据引理 2.2, 有

$$\frac{G'(t)}{\Gamma(1-\alpha)}=D_+^\alpha U(t)\leq\frac{c_1}{e^{R(t)}r(t)}. \tag{2.14}$$

对上述不等式从 T 到 t 积分, 得

$$\int_T^t\frac{G'(s)}{\Gamma(1-\alpha)}ds\leq\int_T^t\frac{c_1}{e^{R(s)}r(s)}ds, \tag{2.15}$$

$$G(t)\leq G(T)+\Gamma(1-\alpha)c_1\int_T^t\frac{1}{e^{R(s)}r(s)}ds. \tag{2.16}$$

当 $t\to+\infty$, $\lim\limits_{t\to\infty}G(t)\leq-\infty$, 这与 $G(t)>0$ 矛盾. 因此 $D_+^\alpha U(t)>0$, $t>T$.

令

$$w(t)=e^{R(t)}\frac{r(t)D_+^\alpha U(t)}{U(t)},$$

则 $w(t)>0$, 由 (2.1) 式及引理 2.3, 容易得到

$$\begin{aligned}
D_+^\alpha w(t) &= e^{R(t)}D_+^\alpha[\frac{r(t)D_+^\alpha U(t)}{U(t)}]+\frac{r(t)D_+^\alpha U(t)}{U(t)}D_+^\alpha e^{R(t)}\\
&= e^{R(t)}\frac{D_+^\alpha[r(t)D_+^\alpha U(t)]}{U(t)}-\frac{e^{R(t)}r(t)D_+^\alpha U(t)D_+^\alpha U(t)}{U^2(t)}+\frac{p(t)}{r(t)}w(t)\\
&\leq e^{R(t)}\frac{-p(t)D_{+,t}^\alpha U(t)-cq(t)U(t)}{U(t)}-\frac{w^2(t)}{e^{R(t)}r(t)}+\frac{p(t)}{r(t)}w(t)\\
&\leq -ce^{R(t)}q(t)-\frac{p(t)}{r(t)}w(t)-\frac{w^2(t)}{e^{R(t)}r(t)}+\frac{p(t)}{r(t)}w(t)\\
&\leq -ce^{R(t)}q(t).
\end{aligned} \tag{2.17}$$

即

$$D_+^\alpha w(t) \le -\psi(t), \tag{2.18}$$

其中 $\psi(t) = ce^{R(t)}q(t)$. 利用 (H1) 和 $w(t)$ 的定义式, 不等式 (2.1) 的第二个式子和第三个式子变为

$$w(t_k^+) = \frac{1+\alpha_k}{1+\beta_k} w(t_k), \qquad k = 1, 2, 3, \cdots . \tag{2.19}$$

令 $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $\widetilde{w}(\xi) = w(t)$, $\widetilde{\psi}(\xi) = \psi(t)$, 则 $\widetilde{w}(\xi) > 0$, 可以得到

$$D_+^\alpha w(t) = D_+^\alpha \widetilde{w}(\xi) = \widetilde{w}'(\xi) D_+^\alpha(\xi) = \widetilde{w}'(\xi), \tag{2.20}$$

则 (2.18) 式变为

$$\widetilde{w}'(\xi) \le -\widetilde{\psi}(\xi). \tag{2.21}$$

由 (2.19)–(2.21) 式, 容易得到

$$\begin{cases} \widetilde{w}'(\xi) \le -\widetilde{\psi}(\xi), & \xi \ne \xi_k, \quad \xi \ge t^*, \\ \widetilde{w}(\xi_k^+) = \frac{1+\alpha_k}{1+\beta_k} \widetilde{w}(\xi_k), & k = 1, 2, 3, \cdots . \end{cases} \tag{2.22}$$

根据引理 2.4, 得

$$\begin{aligned} \widetilde{w}(\xi) &\le \widetilde{w}(t^*) \prod_{t^* < \xi_k < \xi} \frac{1+\alpha_k}{1+\beta_k} - \int_{t^*}^{\xi} \prod_{s < \xi_k < \xi} \frac{1+\alpha_k}{1+\beta_k} \widetilde{\psi}(s) ds \\ &= \prod_{t^* < \xi_k < \xi} \frac{1+\alpha_k}{1+\beta_k} \left[\widetilde{w}(t^*) - \int_{t^*}^{\xi} \prod_{t^* < \xi_k < s} \frac{1+\beta_k}{1+\alpha_k} \widetilde{\psi}(s) ds \right] < 0, \end{aligned} \tag{2.23}$$

这与 $\widetilde{w}(\xi) > 0$ 矛盾, 定理 2.5 的第一部分证毕.

若 $U(t)$ 是不等式 (2.2) 的一个非振动解. 不妨设 $U(t)$ 是脉冲分数阶微分不等式 (2.2) 的最终负解, 则 $G(t) < 0, t \in [t^*, \infty)$. 用类似方法, 容易得到 $D_+^\alpha U(t) < 0$, $t \ge t^*$. 令 $w(t) = -e^{R(t)} \frac{r(t) D_+^\alpha U(t)}{U(t)}$, 则 $w < 0$. 根据 (2.2) 式, 可以得到 $D_+^\alpha w(t) \ge ce^{R(t)} q(t) = \psi(t)$ 和 $w(t_k^+) = \frac{1+\alpha_k}{1+\beta_k} w(t_k), k = 1, 2, 3, \cdots$. 即

$$\begin{cases} \widetilde{w}'(\xi) \ge \widetilde{\psi}(\xi), & \xi \ne \xi_k, \quad \xi \ge t^*, \\ \widetilde{w}(\xi_k^+) = \frac{1+\alpha_k}{1+\beta_k} \widetilde{w}(\xi_k), & k = 1, 2, 3, \cdots . \end{cases} \tag{2.24}$$

若 $\widetilde{w}(\xi) = -\widetilde{v}(\xi)$

$$\begin{cases} \widetilde{v}'(\xi) \le -\widetilde{\psi}(\xi), & \xi \ne \xi_k, \quad \xi \ge t^*, \\ \widetilde{v}(\xi_k^+) = \frac{1+\alpha_k}{1+\beta_k} \widetilde{v}(\xi_k), & k = 1, 2, 3, \cdots . \end{cases} \tag{2.25}$$

由引理 2.4, 得

$$\widetilde{v}(\xi) \le \widetilde{v}(t^*) \prod_{t^* < \xi_k < \xi} \frac{1+\alpha_k}{1+\beta_k} - \int_{t^*}^{\xi} \prod_{s < \xi_k < \xi} \frac{1+\alpha_k}{1+\beta_k} \widetilde{\psi}(s) ds, \tag{2.26}$$

则

$$\begin{aligned}\widetilde{w}(\xi) &\geq \widetilde{w}(t^*)\prod_{t^*<\xi_k<\xi}\frac{1+\alpha_k}{1+\beta_k}+\int_{t^*}^{\xi}\prod_{s<\xi_k<\xi}\frac{1+\alpha_k}{1+\beta_k}\widetilde{\psi}(s)ds\\ &=\prod_{t^*<\xi_k<\xi}\frac{1+\alpha_k}{1+\beta_k}[\widetilde{w}(t^*)+\int_{t^*}^{\xi}\prod_{t^*<\xi_k<s}\frac{1+\beta_k}{1+\alpha_k}\widetilde{\psi}(s)ds]>0,\end{aligned}\tag{2.27}$$

这与结论 $\widetilde{w}(\xi)<0$ 矛盾. 定理 2.5 证毕.

3 举例

例 1 考虑如下问题

$$\left\{\begin{array}{l} D_{+,t}^{\frac{1}{2}}[e^{-t-\frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}}}D_{+,t}^{\frac{1}{2}}u(x,t)]+\frac{1}{2}te^{-t-\frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}}}D_{+,t}^{\frac{1}{2}}u(x,t)\\ =e^tu^2\Delta u(x,t)+t^{\frac{2}{3}}\Delta u(x,t-\frac{\pi}{3})-\frac{u^3(x,t)}{1+t^2+x^2}-(x^2+t^2)u(x,t),\quad t\neq t_k,\\ D_{+,t}^{\frac{1}{2}}u(x,t_k^+)-D_{+,t}^{\frac{1}{2}}u(x,t_k^-)=D_{+,t}^{\frac{1}{2}}u(x,t_k),k=1,2,3,\cdots,(x,t)\in\Omega\times R_+\equiv E,\\ u(x,t_k^+)-u(x,t_k^-)=u(x,t_k),\ k=1,2,3,\cdots,(x,t)\in\Omega\times R_+\equiv E,\end{array}\right.\tag{3.1}$$

边界条件为

$$\frac{\partial u(x,t)}{\partial N}=0,\qquad (x,t)\in\partial\Omega\times R_+,\qquad t\neq t_k,\tag{3.2}$$

证 在 (3.1) 式中, $r(t)=e^{-t-\frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}}}$, $p(t)=\frac{1}{2}te^{-t-\frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}}}$, $a(t)=e^t,a_1(t)=t^{\frac{2}{3}}$, $m(x,t,u)=\frac{u^3(x,t)}{1+t^2+x^2}$, $h(u)=u^2$, $h_1(u)=1$, $q(x,t)=x^2+t^2$, $q(t)=\min(x^2+t^2)=t^2,F(u)=u,\frac{F(u)}{u}\geq c=1$, $j=1$, $\alpha_k=\beta_k=1$. 那么

$$\begin{aligned}&R(t)=I_{0+}^{\frac{1}{2}}(\frac{p(t)}{r(t)})\ =\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-v)^{\frac{1}{2}-1}(\frac{v}{2})dv=\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(-v)d(t-v)^{\frac{1}{2}}=\frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}},\\ &\int_{t^*}^{\infty}\frac{1}{r(s)e^{R(s)}}ds=\int_{t^*}^{\infty}\frac{1}{e^{-s-\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}}ds=\int_{t^*}^{\infty}e^sds=+\infty,\end{aligned}$$

则

$$\begin{aligned}\lim_{\xi\to\infty}\int_{t^*}^{\xi}e^{\widetilde{R(s)}}\widetilde{q(s)}ds\ &=\lim_{\xi\to\infty}\int_{t^*}^{\xi}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}s^2ds=\lim_{\xi\to\infty}[\sqrt{\pi}s^{\frac{3}{2}}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}\ |_{t^*}^{\xi}-\int_{t^*}^{\xi}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}\sqrt{\pi}ds^{\frac{3}{2}}]\\ &=\lim_{\xi\to\infty}[\sqrt{\pi}s^{\frac{3}{2}}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}\ |_{t^*}^{\xi}-\int_{t^*}^{\xi}\frac{3\sqrt{\pi}}{2}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}s^{\frac{1}{2}}ds]\\ &=\lim_{\xi\to\infty}[\sqrt{\pi}s^{\frac{3}{2}}e^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}\ |_{t^*}^{\xi}-\frac{3\pi}{2}\int_{t^*}^{\xi}de^{\frac{2}{3\sqrt{\pi}}s^{\frac{3}{2}}}]\\ &=\lim_{\xi\to\infty}[\sqrt{\pi}\xi^{\frac{3}{2}}e^{\frac{2}{3\sqrt{\pi}}\xi^{\frac{3}{2}}}-\frac{3\pi}{2}e^{\frac{2}{3\sqrt{\pi}}\xi^{\frac{3}{2}}}-\sqrt{\pi}(t^*)^{\frac{3}{2}}e^{\frac{2}{3\sqrt{\pi}}(t^*)^{\frac{3}{2}}}-\frac{3\pi}{2}e^{\frac{2}{3\sqrt{\pi}}(t^*)^{\frac{3}{2}}}]\\ &=+\infty,\end{aligned}$$

$$\lim_{\xi\to\infty}\int_{t^*}^{\xi}\prod_{t^*<\xi_k<s}\frac{1+\beta_k}{1+\alpha_k}\widetilde{\psi(s)}ds=+\infty.$$

则由定理 2.5 可知, 问题 (3.1) 和 (3.2) 的解都是振动的.

参考文献

[1] Yang Jichen, Liu Anping, Liu Guangjie. Oscillation of solutions to neutral nonlinear impulsive hyperbolic equations with several delays[J]. Electronic Journal of Differential Equations, 2013, 2013(27): 207–211.

[2] Liu Anping, Ma Qingxia, He Mengxing. Oscillation of nonlinear impulsive parabolic equations of neutral type[J]. Rocky Mountain Journal of Mathematics, 2010, 36(3): 1011–1026.

[3] Liu Anping, Xiao Li, Liu Ting, Zou Min. Oscillation of nonlinear impulsive hyperbolic equation with several delays[J]. Rocky Mountain Journal of Mathematics, 2007, 37(5): 1669–1684.

[4] Prakash P, Harikrishnan S, Benchohra M. Oscillation of certain nonlinear fractional partial differential equation with damping term[J]. Applied Mathematics Letters, 2015, 43: 72–79.

[5] Li Weinian. Forced oscillation criteria for a class of fractional partial differential equations with damping term[J]. Mathematical Problems in Engineering, 2015, Article ID 410904.

[6] Zheng Bin. Oscillation for a class of nonlinear fractional differential equations with damping term [J]. Journal of Advanced Mathematical Studies, 2013, 6(1): 107–115.

[7] Chen Daxue. Oscillatory behavior of a class of fractional differential equations with damping[J]. UPB Scientific Bulletin (Series A), 2013, 75(1): 107–118.

[8] Qin Huizeng, Zheng Bin. Oscillation of a class of fractional differential equations with damping term[J]. Scientific World Journal, 2013, 2013: 1–9.

[9] Yang Jichen, Liu Anping, Liu Ting. Forced oscillation of nonlinear fractional differential equations with damping term[J]. Advances in Difference Equations, 2015, 2015(1): 1–7

[10] 马晴霞, 刘安平. 带阻尼项的非线性分数阶微分方程的振动性 (英文)[J]. 应用数学, 2016, 29(2): 291–297.

[11] Chen Daxue. Oscillation criteria of fractional differential equations[J]. Advances in Difference Equation, 2012, 2012(1): 33–42

[12] Harikrishnan S, Prakash P, Nieto J J. Forced oscillation of solutions of a nonlinear fractional partial differential equation[J]. Applied Mathematics and Computation, 2015, 254(254): 14–19.

[13] Li Weinian. On the forced oscillation of certain fractional partial differential equations[J]. Applied Mathematics Letters, 2015, 50: 5–9.

[14] Feng Qinghua, Meng Fanwei. Oscillation of solutions to nonlinear forced fractional differential equations[J]. Electronic Journal of Differential Equations, 2013, 2013(169): 1–10.

[15] Li Weinian. Oscillation of solutions for certain fractional partial differential equations[J]. Advances in Difference Equations, 2016, 2016(1): 16.

[16] Prakash P, Harikrishnan S. Oscillation of solutions of impulsive vector hyperbolic differential equations with delays[J]. Applicable Analysis, 2012, 91(3): 459–473.

[17] 熊永福, 朱思瑛, 刘安平. 分数阶偏微分方程解的强迫振动性质 (英文)[J]. 生物数学学报, 2017, 32(3): 304–310.

[18] Abbas S, Benchohra M, N'Guérékata G M. Topics in fractional differential equations[M]. New York: Springer, 2012.

[19] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results[J]. Computers and Mathematics with Applications, 2006, 51(9–10): 1367–1376.

[20] Raheem A, Maqbul M. Oscillation criteria for impulsive partial fractional differential equations[J]. Computers and Mathematics with Applications, 2017, 73(8): 1781–1788.

OSCILLATION FOR A CLASS OF IMPULSIVE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we study the oscillation criteria for a class of impulsive fractional partial differential equations with Neumann boundary condition. By using the properties of the modified Riemann-Liouville fractional partial differential equations and a generalized Riccati technique and the differential inequality methods, some sufficient conditions for the oscillatory behavior of the solution are obtained. As an application, the relevant example is given to illustrate the main conclusions, which generalize the results in [12].

Keywords: impulsive; fractional partial differential equations; modified Riemann-Liouville fractional partial derivative; oscillation

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