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GENERALIZED FRACTAL LACUNARY INTERPOLATION WITH VARIABLE SCALING PARAMETERS BASED ON EXTRAPOLATION SPLINE

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Abstract: In this paper, the generalized Birkhoff $(0, m)$ lacunary interpolation problem for the fractal function with proper perturbable parameters is investigated. An extrapolation algorithm is proposed to obtain an approximate spline polynomial solution, and convergence estimates are presented under the assumption of $\|\alpha_k^{(r)}\|_\infty \leq \frac{1}{(2N)^m}(k = 1, 2, \cdots, N;\ r = 0, 1, \cdots, m)$. The numerical results show that the interpolate perturbation method we provide works effectively.

Keywords: extrapolation spline; fractal function; lacunary interpolation; scaling factor; approximation order

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1 Introduction

Spline interpolation is often preferred over polynomial interpolation, because the interpolation error can be made small even when using low degree polynomials for the spline [1–3]. Many research results were obtained about spline lacunary interpolation, such as, spline $(0, 2, 3)$ and $(0, 2, 4)$ lacunary interpolation [4], Varma's $(0, 2)$ case of spline [5] and the minimizing error bounds in lacunary interpolation by spline for $(0, 1, 4)$ and $(0, 2)$ case given by Saceed [6] and Jawmer [7].

A fractal is a detailed, recursive and infinitely self-similar mathematical set in which Hausdorff dimension strictly exceeds its topological dimension [8]. Fractals exhibit similar patterns at increasingly small scales [9]. If this replication is exactly the same at every scale, it is called a self-similar pattern [10, 11]. Fractal was widely used as a research tool for generating natural-looking shapes such as mountains, trees, clouds, etc. There were

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increasing researches in fractal functions and their applications over the last three decades. Fractal function is a good choice for modeling natural object [12], and fractal interpolation techniques provide good deterministic representations of complex phenomena. Barnsley [13] and Hutchinson [14] are pioneers in terms of applying fractal function to interpolate sets of data. The fractal interpolation problems based on Hermite functions and cubic spline are solved in ref. [15] and ref. [16]. Viswanathan [17] gave the fractal spline $(0,4)$ and $(0,2)$ lacunary interpolation, and Viswanathan [17] also did research on the fractal $(0,2)$ lacunary interpolation by using the spline function of ref. [5]. Inspired by their research, this paper devoted to research the general fractal $(0,m)$ lacunary interpolation with the scaling factors based on the extrapolation spline.

The paper is organized as follows. In Section 2, by using extrapolation algorithm we deduce the explicit formulas of generalized $(0,m)$ lacunary interpolation spline function. The error estimate is also given and numerical example is presented to demonstrate the effectiveness of our proposed method. In Section 3, we use the spline function which has been obtained in Section 2 to interpolate the fractal function. We find that when scaling factors fulfill $\|\alpha_k\|_r \leq \frac{1}{(2N)^m}$, there is fractal function $T^{\alpha}_{\Delta,b} \in C^m[0,1]$ satisfying, ${T^{\alpha}_{\Delta,b}}^{(p)}(x_k) = S^{(p)}_{\Delta}(x_k),\ p=0,m;\ {T^{\alpha}_{\Delta,b}}^{(r)}(x_N) = S^{(r)}_{\Delta}(x_N),\ r=0,1,\cdots,m$. Approximation orders for the proposed class of fractal interpolation and their derivatives are discussed. Numerical simulations are also carried out to show the validity and efficiency of our proposed method. The concluding remarks are given in Section 4.

2 Spline $(0,m)$ Lacunary Interpolation

In this section, the spline explicit formula for the $(0,m)$ lacunary interpolation function is constructed by using extrapolation algorithm. The method we adopt is similar to those given in literature [19]. An example for illuminating the details and efficiency of the procedure is provided, and the error estimate will be shown in Theorem 2.2.

2.1 Spline Lacunary Interpolation

For a given partition $\Delta: x_1 < x_2 < \cdots < x_N, x_{k+1} - x_k = h_k,\ I = [x_1, x_N]$, and real values $\{y_1, y_2, \cdots, y_{N-1}; y_1^{(m)}, y_2^{(m)}, \cdots, y_{N-1}^{(m)};\ y_N, y_N^{(1)}, \cdots, y_N^{(m)}\}$.

We define the spline function $S_\Delta(x)$ in each subinterval such that

$$S_\Delta(x) = \begin{cases} S_1(x), x_1 \leq x \leq x_2, \\ S_k(x), x_k \leq x \leq x_{k+1}, k = 2,3,\cdots,N-2, \\ S_{N-1}(x), x_{N-1} \leq x \leq x_N, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} S_k(x) &= y_k + y_k^{(m)} \frac{(x-x_k)^m}{m!} + \sum_{r=1}^{m-1} a_{k,r} \frac{(x-x_k)^r}{r!} + a_{k,m+1} \frac{(x-x_k)^{m+1}}{(m+1)!} \\ &\quad + a_{k,m+2} \frac{(x-x_k)^{m+2}}{(m+2)!}. \end{aligned} \tag{2.2}$$

$S_\Delta(x)$ has the following conditions

$$\begin{cases} S_\Delta(x_k) = f(x_k) = y_k, \ k = 1, 2, 3, \cdots, N-1 \\ S_\Delta^{(m)}(x_k) = f^{(m)}(x_k) = y_k^{(m)}, \ k = 1, 2, 3, \cdots, N-1 \\ S_\Delta^{(r)}(x_N) = f^{(r)}(x_N) = y_N^{(r)}, \ r = 0, 1, 2, \cdots, m. \end{cases} \tag{2.3}$$

and satisfying

$$\{S_\Delta(x) \in C^m(I); S(x) \in P_{(m+2)}(x), x \in [x_k, x_{k+1}], k = 1, 2, \cdots, N-1\}.$$

The coefficients of these polynomials can be determined by the following conditions

$$\begin{cases} S_k(x_{k+1}) = S_{k+1}(x_{k+1}) = y_{k+1}, k = 1, 2, \cdots, N-2 \\ S_k^{(m)}(x_{k+1}) = S_{k+1}^{(m)}(x_{k+1}) = y_{k+1}^{(m)}, k = 1, 2, \cdots, N-2, \\ S_k^{(q)}(x_{k+1}) = S_{k+1}^{(q)}(x_{k+1}), 1 \le q < m, k = 1, 2, \cdots, N-2, \\ S_{N-1}^{(r)}(x_N) = y_N^{(r)}, r = 0, 1, \cdots, m. \end{cases} \tag{2.4}$$

For $k = 1, 2, \cdots, N-1$, we denote

$$\begin{cases} \alpha_{k,0} = y_{k+1} - y_k - hy'_k - \frac{h^2}{2} y_k^{(2)} - \frac{h^3}{3!} y_k^{(3)} - \cdots - \frac{h^m}{m!} y_k^{(m)}, \\ \alpha_{k,1} = y'_{k+1} - y'_k - hy''_k - \frac{h^2}{2} y_k^{(3)} - \frac{h^3}{3!} y_k^{(4)} - \cdots - \frac{h^{m-1}}{(m-1)!} y_k^{(m)}, \\ \alpha_{k,2} = y''_{k+1} - y''_k - hy_k^{(3)} - \frac{h^2}{2} y_k^{(4)} - \frac{h^3}{3!} y_k^{(5)} - \cdots - \frac{h^{m-2}}{(m-2)!} y_k^{(m)}, \\ \quad \vdots \\ \alpha_{k,m-1} = y_{k+1}^{(m-1)} - y_k^{(m-1)} - hy_k^{(m)}, \\ \alpha_{k,m} = y_{k+1}^{(m)} - y_k^{(m)}. \end{cases} \tag{2.5}$$

To obtain the unknown coefficients $a_{k,j}$ $(k = 1, 2, \cdots, N-1, j = 1, 2, \cdots, m+2, j \ne m)$, we take the following five steps

Step 1 For $k = N-1$, we have

$$\begin{aligned} \alpha_{N-1,0} = {} & h\left(\alpha_{N-1,1} - y'_{N-1}\right) + \frac{h^2}{2}\left(\alpha_{N-1,2} - y''_{N-1}\right) + \cdots + \frac{h^{m-1}}{(m-1)!}\alpha_{N-1,m-1} \\ & + \frac{h^{m+1}}{(m+1)!}\alpha_{N-1,m+1} + \frac{h^{m+2}}{(m+2)!}\alpha_{N-1,m+2}; \\ \alpha_{N-1,1} = {} & \left(\alpha_{N-1,1} - y'_{N-1}\right) + h\left(\alpha_{N-1,2} - y''_{N-1}\right) + \cdots + \frac{h^{m-2}}{(m-2)!}\alpha_{N-1,m-1} \\ & + \frac{h^m}{m!}\alpha_{N-1,m+1} + \frac{h^{m+1}}{(m+1)!}\alpha_{N-1,m+2}; \\ \alpha_{N-1,2} = {} & \left(\alpha_{N-1,2} - y''_{N-1}\right) + h\left(\alpha_{N-1,2} - y'''_{N-1}\right) + \cdots + \frac{h^{m-3}}{(m-3)!}\alpha_{N-1,m-1} \\ & + \frac{h^{m-1}}{(m-1)!}\alpha_{N-1,m+1} + \frac{h^m}{m!}\alpha_{N-1,m+2}; \\ & \vdots \\ \alpha_{N-1,m-1} = {} & \left(\alpha_{N-1,m-1} - y_{N-1}^{(m-1)}\right) + \frac{h^2}{2}\alpha_{N-1,m+1} + \frac{h^3}{3!}\alpha_{N-1,m+2}; \\ \alpha_{N-1,m} = {} & h\alpha_{N-1,m+1} + \frac{h^2}{2}\alpha_{N-1,m+2}. \end{aligned}$$

Step 2 Solving the equation systems of Step 1, we obtain

$$a_{N-1,r} - y_{N-1}^{(r)} = A_{0r}\alpha_{N-1,0} + A_{1r}\alpha_{N-1,1} + \cdots A_{mr}\alpha_{N-1,m}, \tag{2.6}$$

$$a_{N-1,s} = B_{0s}\alpha_{N-1,0} + B_{1s}\alpha_{N-1,1} + \cdots B_{ms}\alpha_{N-1,m}, \tag{2.7}$$

where $r = 1, 2, \cdots, m-1; s = m+1, m+2$.

Step 3 For $k = 1, 2, \cdots, N-2$, establishing the equations systems

$$\begin{aligned}
\alpha_{k,0} &= h(a_{k,1} - y'_k) + \tfrac{h^2}{2}(a_{k,2} - y''_k) + \cdots + \tfrac{h^{m-1}}{(m-1)!}(a_{k,m-1} - y_k^{(m-1)}) \\
&\quad + \tfrac{h^{m+1}}{(m+1)!}a_{k,m+1} + \tfrac{h^{m+2}}{(m+2)!}a_{k,m+2}, \\
\alpha_{k,1} &= (a_{k,1} - y'_k) - (a_{k+1,1} - y'_{k+1}) + h(a_{k,2} - y''_k) + \tfrac{h^2}{2}(a_{k,3} - y_k^{(3)}) \\
&\quad + \cdots + \tfrac{h^{m-2}}{(m-2)!}(a_{k,m-1} - y_k^{(m-1)}) + \tfrac{h^m}{m!}a_{k,m+1} + \tfrac{h^{m+1}}{(m+1)!}a_{k,m+2}, \\
\alpha_{k,2} &= (a_{k,2} - y''_k) - (a_{k+1,1} - y''_{k+1}) + h(a_{k,3} - y_k^{(3)}) + \tfrac{h^2}{2}(a_{k,4} - y_k^{(4)}) \\
&\quad + \cdots + \tfrac{h^{m-3}}{(m-3)!}(a_{k,m-1} - y_k^{(m-1)}) + \tfrac{h^{m-1}}{(m-1)!}a_{k,m+1} + \tfrac{h^m}{m!}a_{k,m+2}, \\
&\vdots \\
\alpha_{k,m-1} &= (\alpha_{k,m-1} - y_k^{(m-1)}) - (\alpha_{k+1,m-1} - y_k^{(m-1)}) + \tfrac{h^2}{2}a_{k,m+1} + \tfrac{h^3}{3}a_{k,m+2}, \\
\alpha_{k,m} &= ha_{k,m+1} + \tfrac{h^2}{2}a_{k,m+2}.
\end{aligned}$$

Step 4 Combinating the equations of $\alpha_{k,0}$ and $\alpha_{k,m}$ in Step 3, we obtain

$$\begin{aligned}
a_{k,m+1} &= C_{1,m+1}(a_{k,1} - y'_k) + \cdots + C_{m-1,m+1}(a_{k,m-1} - y_k^{(m-1)}), \\
a_{k,m+2} &= D_{1,m+1}(a_{k,1} - y'_k) + \cdots + D_{m-1,m+1}(a_{k,m-1} - y_k^{(m-1)}).
\end{aligned}$$

Step 5 Solving the rest parts of Step 3.

The solutions from step 4 will be substituted by the remaining equations $a_{k,m+1}, a_{k,m+2}$, starting with $(a_{k,j} - y_k^{(j)}) - (a_{k+1,j} - y_{k+1}^{(j)}) + \cdots$. In view of the coefficient matrix of the equation systems (2.6) is non-singular, thus the unknowns $a_{N-1,1}, a_{N-1,2}, \cdots, a_{N-1,m-1}$ be solved and obtained. Repeating above iteration, all undetermined coefficients of $S_\Delta(x)$ can be calculated. Therefore, we obtain the following conclusion for spline $(0, m)$ Birkhoff interpolation problem.

Theorem 2.1 Assume that $f(x) \in C^m(I)$. For a uniform partition $\Delta := \{x_1, x_2, \cdots, x_N : x_k < x_{k+1},\ k = 1, 2, \cdots, N-1\}$ of the compact interval $I = [x_1, x_N]$, there is a unique spline function $S_\Delta(x) \in S^m_{(m+2)}(\Delta)$, satisfying the $(0, m)$ lacunary interpolation conditions

$$S_\Delta(x_k) = f(x_k),\ S_\Delta^{(m)}(x_k) = f^{(m)}(x_k)\ (1 \le k \le N-1);$$

$$S_\Delta^{(r)}(x_N) = f^{(r)}(x_N),\ r = 0, 1, \cdots, m.$$

2.2 Approximation Theory

From the Theorem 2 of ref. [20], we can get the following approximation theory.

Theorem 2.2 Assume that $f(x) \in C^m[-1, 1]$, $S_\Delta(x)$ is the $(0, m)$ lacunary spline function defined by eq.(2.1). When $0 \le r \le m$ and $-1 \le x \le 1$, we have

$$\|f(x) - S_\Delta(x)\|_\infty \le C\Delta^m_{m+2}(x)w(f^{(m)}, \Delta_{m+2}(x)),$$

where C is constant independent of x, $\Delta_{m+2}(x) = \frac{\sqrt{1-x^2}}{m+2} + \frac{1}{(m+2)^2}$, $\|f\|_\infty = \max\{|f(x)| : x \in [-1, 1]\}$, $w(f^{(r)}, \delta) = \max\{|f^{(r)}(x_1) - f^{(r)}(x_2)| : |x_1 - x_2| \le \delta\}$ is the maximum norm modulus of continuity of $f^{(r)}(x)$ on the interval $[-1, 1]$.

2.3 Numerical Example

Assume $f(x) = x * \cos(x) - \sin(x)$. For a uniform partition $\Delta := \{x_1, x_2, \cdots, x_N : x_k < x_{k+1}, k = 1, 2, \cdots, N\}$ of the interval $I = [-1, 1]$, applying the result of spline $(0, m)$ Birkhoff interpolation, we take $m = 2$, and consider the absolute error $e_{k,p} = |a_{k,p} - y_k^{(p)}|$ and the error curve $R(x) = |f(x) - S_\Delta(x)|$ for the nodes x_k. We list a table and present error curve graphs. The numerical results are listed in Table 1, and the function fitting graphs are listed in Fig.1-2. It is observed that our proposed spline function is effective and stable.

Table 1 The absolute error $e_{k,p}$

N	$e_{1,1}$	$e_{2,1}$	$e_{3,1}$	$e_{4,1}$	$e_{5,1}$	$e_{6,1}$
$N = 4$	$3.082(-2)$	$3.715(-3)$	$3.127(-3)$	$4.164(-4)$	$-$	$-$
$N = 5$	$5.200(-2)$	$3.782(-3)$	$6.481(-3)$	$2.617(-3)$	$1.567(-4)$	$-$
$N = 6$	$6.582(-2)$	$1.620(-2)$	$7.452(-4)$	$1.879(-2)$	$1.879(-3)$	$7.085(-5)$

(a): The error curve, $N = 5$ (b): The error curve, $N = 7$

Figure 1 The error curve of $R(x) = f(x) - S_\Delta(x)$

(c): The error curve, $N = 10$ (d): The error curve, $N = 20$

Figure 2 The error curve of $R(x) = f(x) - S_\Delta(x)$

3 Fractal (0,m) Lacunary Interpolation

For positive integer $N > 2$, consider a data set $\{(x_n, y_n) \in \mathbb{R}^2 : n = 0, 1, 2, ..., N\}$, where $x_0 < x_1 < x_2 < \cdots < x_N$. Let $I = [x_0, x_N]$, $I_n = [x_{n-1}, x_n]$, $n \in J = \{1, 2, \cdots, N\}$ and $L_n : I \to I_n$ be homeomorphic affinities such as

$$|L_n(x) - L_n(x^*)| \leq l_n |x - x^*|$$

for all $x, x^* \in I$ and $0 \le l_n < 1$.

Consider $N-1$ continuous maps $F_n : I \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

$$|F_n(x,y) - F_n(x,y^*)| \le r_n|y-y^*|;\ F_n(x_0,\ y_0) = y_{n-1},\ F_n(x_N,\ y_N) = y_n (n \in J)$$

for all $x \in I, y, y^* \in \mathbb{R}$, and for some $0 \le r_n < 1$.

Defining functions $w_n:\ I \times \mathbb{R} \to I_n \times \mathbb{R} \subseteq I \times \mathbb{R},\ w_n(x,y) = (L_n(x),\ F_n(x,y)),\ n \in J$. The $\{I \times \mathbb{R},\ w_n:\ n = 1,2,...,N\}$ is called an Iterated Function System (IFS) [13]. From ref. [17], we know that the IFS has a unique attractor $G(g)$ which is the graph of a continuous function $g : I \to \mathbb{R}$ satisfying $g(x_n) = y_n$ $(n = 0,1,2,\cdots,N)$, and function g is the fixed point of the Read-Bajraktarević (RB) operator $T : C_{y_0,y_N}(I) \to C_{y_0,y_N}(I)$ defined

$$T(g)(x) = F_n(L_n^{-1}(x),\ g(L_n^{-1}(x))),\ x \in I_n,\ n = 1,2,\cdots,N.$$

The above function g is called Fractal Interpolation Function(FIF) corresponding to the data $\{(x_n,y_n):\ n = 0,1,2,\cdots,N\}$ and it satisfies

$$g(x) = F_n(L_n^{-1}(x),\ g(L_n^{-1}(x))),\ x \in I_n,\ n \in J \tag{3.1}$$

For a partition $\Delta := \{x_0,x_1,x_2,\cdots,x_N : x_{n-1} < x_n, n \in J\}$ of $I = [x_0,x_N]$, $x_n - x_{n-1} := h_n$, and the data set $\{(x_n, f(x_n)),\ n = 0,1,2,\cdots,N\}$, suppose that $F_n(x,y) = \alpha_n(y - b(x)) + f(L_n(x))$, where α_n is called scaling factors, satisfying $|\alpha_n| < 1$, and $b : I \to \mathbb{R}$ is a continuous function, such as $b \ne f$, $b(x_0) = f(x_0)$, $b(x_N) = f(x_N)$.

Thus, we ensure the existence of a fractal function $(Tf)(x_n) = f(x_n)$ $(n = 0,1,2,\cdots,N)$. From (3.1), we can obtain

$$(Tf)(x) = f(x) + \alpha_n(L_n^{-1}(x)) \cdot (f-b)(L_n^{-1}(x)), x \in I_n, n \in J.$$

The most widely studied functions $L_n(x)$ so far are defined by the following,

$$L_n(x) = d_n x + b_n$$

with

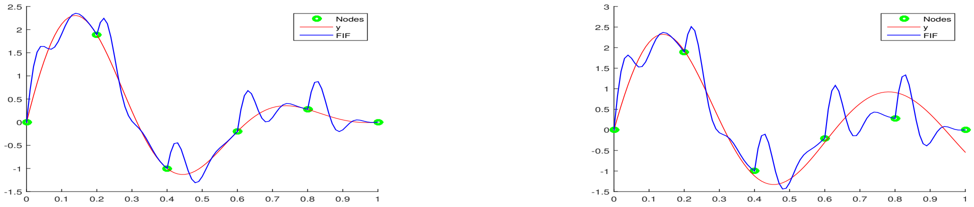
$$d_n = \frac{x_n - x_{n-1}}{x_N - x_0},\quad b_n = \frac{x_N x_{n-1} - x_n x_0}{x_N - x_0}$$

In many cases, the data are evenly sampled $h = x_n - x_{n-1}, x_N - x_0 = Nh$.

Example 1 Consider function $f(x) = (2x^2 - 5x + 3)\sin(10x)$. For a uniform partition $\Delta := \{x_0, x_1, \cdots, x_6 : x_{n-1} < x_n, n = 1,2,\cdots,5\}$ of $[0,1]$ with step size $h = \frac{1}{5}$, $b(x) = x^2 f(x)$. The left graph of Figure 3 shows the fractal function with $\alpha_n = 0.3$ $(n = 1,2,\cdots,5)$. The right graph presents the fractal function with scaling variable $\alpha_1 = 0.4 + \frac{x}{8}$, $\alpha_2 = 0.3 - \frac{\sin(x)}{4}$, $\alpha_3 = 0.6 - \frac{x^2}{4}$, $\alpha_4 = 0.3 + \frac{\cos(x)}{4}$, $\alpha_5 = 0.3 + \frac{x}{5}$.

3.1 Interpolation Theorem

We denote $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$, $\|\alpha\|_r = \max\{\|\alpha_k^{(p)}\|_\infty : p = 0,1,r\}$.



(a) The fractal function with $\alpha_n \equiv 0.3$ (b) The fractal function with different α_n

Figure 3 $y = (2x^2 - 5x + 3)\sin(10x)$ with $N = 5$

Theorem 3.1 Assume that $f \in C^m(I)$, $\Delta := \{x_0, x_1, \cdots, x_N : x_{n-1} < x_n, n \in J\}$ be an arbitrary partition of $I = [x_0, x_N]$. There are suitable smooth functions b and α_n, when scaling factors α_n fulfill $\|\alpha_n\|_\infty < \max\limits_{1\le n\le N}\{|\frac{d_n}{2}|^m\}$, the corresponding fractal function $T^\alpha_{\Delta,b} \in C^m(I)$ satisfies

$$T^\alpha_{\Delta,b}(x_n) = S_\Delta(x_n);\ {T^\alpha_{\Delta,b}}^{(m)}(x_n) = S^{(m)}_\Delta(x_n),\ 1 \le n \le N-1$$

with boundary conditions,

$${T^\alpha_{\Delta,b}}^{(p)}(x_0) = S^{(p)}_\Delta(x_0)\ (p = 0, m);\ {T^\alpha_{\Delta,b}}^{(r)}(x_N) = S^{(r)}_\Delta(x_N),\ r = 0, 1, \cdots, m.$$

Proof For convenience, in the following $k = 0, 1, 2, 3, \cdots, N-1$. Let $S_\Delta(x)$ be prescribed in eq.(2.1), we choose smooth function $b \in C^m(I)$ to satisfy

$$b^{(r)}(x_0) = S^{(r)}_\Delta(x_0),\ b^{(r)}(x_N) = S^{(r)}_\Delta(x_N),\ r = 0, 1, \cdots, m.$$

Consider the operator $T : C^m(I) \to C^m(I)$

$$(Tf)(x) = S_\Delta(x) + \alpha_n(L_n^{-1}(x))[f(L_n^{-1}(x)) - b(L_n^{-1}(x))],\ x \in I_n,\ n \in J. \tag{3.2}$$

We can deduce that

$$(Tf)^{(r)}(x) = S^{(r)}_\Delta(x) + d_n^{-r}\sum_{i=0}^{r}\binom{r}{i}(f-b)^{(i)}(L_n^{-1}(x))\alpha_n^{(r-i)}(L_n^{-1}(x)). \tag{3.3}$$

Using the conditions on b and the properties $L_n(x_N) = L_{n+1}(x_0) = x_n (n \in J)$ of L_n, we obtain $(Tf)^{(r)}(x_n^+) = (Tf)^{(r)}(x_n^-) = S^{(r)}_\Delta(x_n)$ for $r = 0, 1, \cdots, m$. From (3.2) and (3.3), we get

$$(Tf)^{(r)}(x_n) = S^{(r)}_\Delta(x_n),\ (Tf)^{(r)}(x_N) = S^{(r)}_\Delta(x_N);\ r = 0, 1, \cdots, m.$$

For $f, g \in C^m(I)$, $r = 1, 2, \cdots, m$, $x \in I_n$, we have

$$\begin{aligned} |(Tf)^{(r)} - (Tg)^{(r)}| &= d_n^{-r}|\sum_{i=0}^{r}\binom{r}{i}(f-g)^{(i)}(L_n^{-1}(x))\alpha_n^{(r-i)}(L_i^{-1}(x))| \\ &\le |d_n^{-r}|\|\alpha\|_r\|f-g\|_r\sum_{i=0}^{r}\binom{r}{i}. \end{aligned} \tag{3.4}$$

When $r = 0$,

$$\|(Tf)(x) - (Tg)(x)\|_\infty \leq \|\alpha\|_\infty \|f - g\|_\infty,$$

Therefore, when $\|\alpha\|_\infty < \max\{|\frac{d_n}{2}|^m : n \in J\}$, T is a contraction, by the Banach fixed point theorem, T has a unique fixed point $T^\alpha_{\Delta,b}$ that satisfies

$$(T^\alpha_{\Delta,b})(x) = S_\Delta(x) + \alpha_n(L_n^{-1}(x))[T^\alpha_{\Delta,b}(L_n^{-1}(x)) - b(L_n^{-1}(x))],\ x \in I_n,\ n \in J.$$

It is apparently seen from the above discussion that

$$T^\alpha_{\Delta,b}{}^{(p)}(x_n) = S_\Delta^{(p)}(x_n),\ p = 0, m;\ T^\alpha_{\Delta,b}{}^{(r)}(x_N) = S_\Delta^{(r)}(x_N),$$

$$r = 0, 1, \cdots, m;\ n = 0, 1, 2, \cdots, N-1.$$

3.2 Approximation Estimates

we will give error estimates for $\|(T^\alpha_{\Delta,b} - S_\Delta)^r\|_\infty$ and obtain convergence results while choosing suitable perturbation parameter α_n. Consider $I = [0,1]$, $0 = x_0 < x_1 < \cdots < x_N = 1$, and $h = x_n - x_{n-1} = \frac{1}{N}$, $N \geq 3, n = 0, 1, 2, \cdots, N$. $I_n = [\frac{n-1}{N}, \frac{n}{N}]$, $n \in \{1, 2, \cdots, N\}$. Let homeomorphic affinities L_n be $L_n(x) = \frac{x+(n-1)}{N}$. We have the following convergence estimates.

Theorem 3.2 For the uniform equidistance partition of $I = [0,1]$, the following bounds for the fractal function and its derivatives hold:

$$\|T^\alpha_{\Delta,b} - S_\Delta\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|S_\Delta - b\|_\infty,$$

$$\|((T^\alpha_{\Delta,b}) - S_\Delta)^{(r)}\|_\infty \leq \frac{(2^{r+1} - 1)N^r\|\alpha\|_r}{1 - N^r\|\alpha\|_\infty}\|S_\Delta - b\|_r,\ r = 1, 2, \cdots, m.$$

Proof From the equation $(T^\alpha_{\Delta,b})(x) = S_\Delta(x) + \alpha_k(L_n^{-1}(x))[T^\alpha_{\Delta,b}(L_n^{-1}(x)) - b(L_n^{-1}(x))],\ x \in I_n,\ n \in J$, we have

$$|(T^\alpha_{\Delta,b}) - S_\Delta| \leq \|\alpha\|_\infty[\|T^\alpha_{\Delta,b} - S_\Delta]\|_\infty + \|S_\Delta - b\|_\infty].$$

Then

$$\|(T^\alpha_{\Delta,b}) - S_\Delta\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|S_\Delta - b\|_\infty. \tag{3.5}$$

It's evident that the following equality holds

$$(T^\alpha_{\Delta,b})^{(r)}(x) = S_\Delta^{(r)}(x) + N^r \sum_{i=0}^{r} \binom{r}{i} ((T^\alpha_{\Delta,b}) - b)^{(i)}(L_n^{-1}(x))\alpha_n^{(r-i)}(L_n^{-1}(x)),\ r = 0, 1, \cdots, m.$$

Therefore, we have

$$\begin{aligned} |(T^\alpha_{\Delta,b})^{(r)} - S_\Delta^{(r)}| &\leq \{N^r(\|\alpha_n\|_r\|T^\alpha_{\Delta,b} - b\|_{r-1}(2^r - 1) + \|\alpha_n\|_\infty\|\|(T^\alpha_{\Delta,b})^{(r)} - b^{(r)}\|_\infty) \\ &\leq N^r(\|\alpha\|_r[\|T^\alpha_{\Delta,b} - S_\Delta\|_{r-1} + \|S_\Delta - b\|_{r-1}](2^r - 1) + \|\alpha\|_\infty\|[\|(T^\alpha_{\Delta,b})^{(r)} \\ &\quad -S_\Delta^{(r)}\|_\infty + \|S_\Delta^{(r)} - b^{(r)}\|_\infty]\}. \end{aligned} \tag{3.6}$$

It can be easy deduced that

$$|(T^{\alpha}_{\Delta,b})^{(r)} - S^{(r)}_{\Delta}| \leq \frac{N^r\|\alpha\|_r}{1-N^r\|\alpha\|_\infty}[(2^r-1)\|T^{\alpha}_{\Delta,b} - S_{\Delta}\|_{r-1} + 2^r\|S_{\Delta} - b\|_r]. \tag{3.7}$$

When $r=1$, using (3.7) and (3.5), we have

$$|(T^{\alpha}_{\Delta,b})^{(1)} - S^{(1)}_{\Delta}| \leq \frac{N\|\alpha\|_1}{1-N\|\alpha\|_\infty}[\|T^{\alpha}_{\Delta,b} - S_{\Delta}\|_\infty + 2\|S_{\Delta} - b\|_1] \leq \frac{3N\|\alpha\|_1}{1-N\|\alpha\|_\infty}\|S_{\Delta} - b\|_1. \tag{3.8}$$

On account of $\|\alpha\|_\infty \leq 3N\|\alpha\|_1$, it follows that

$$\|T^{\alpha}_{\Delta,b} - S_{\Delta}\|_1 = \max\{\|(T^{\alpha}_{\Delta,b})^{(1)} - S^{(1)}_{\Delta}\|_\infty, \|T^{\alpha}_{\Delta,b} - S_{\Delta}\|_\infty\} \leq \frac{3N\|\alpha\|_1}{1-N\|\alpha\|_\infty}\|S_{\Delta} - b\|_1.$$

Similarly, when $r=2$, we obtain

$$|(T^{\alpha}_{\Delta,b})^{(2)} - S^{(2)}_{\Delta}| \leq \frac{N^2\|\alpha\|_2}{1-N^2\|\alpha\|_\infty}[3\|T^{\alpha}_{\Delta,b} - S_{\Delta}\|_1 + 4\|S_{\Delta} - b\|_2] \leq \frac{7N^2\|\alpha\|_2}{1-N^2\|\alpha\|_\infty}\|S_{\Delta} - b\|_2.$$

$$\|(T^{\alpha}_{\Delta,b}) - S_{\Delta}\|_2 \leq \frac{7N^2\|\alpha\|_2}{1-N^2\|\alpha\|_\infty}\|S_{\Delta} - b\|_2.$$

Summarizing the above process, we can obtain the result.

Theorem 3.3 Assume that $f \in C^m([0,1])$. $T^{\alpha}_{\Delta,b}$ is the fractal interpolation function given in Theorem 3.1. For uniform equidistance partition on $[0,1]$, when the scaling factor α_n satisfies $\|\alpha^{(r)}_n\|_\infty \leq \frac{1}{(2N)^m}$, $r=0,1,\cdots,m$, we have

$$\|(T^{\alpha}_{\Delta,b} - f)^{(r)}\|_\infty \leq C_r N^{-(m-r)} + C_r \triangle^{m-r}_{m+2}(x)\omega(f^{(m)}, \triangle_{m+2}(x))$$

as $N \to \infty, r=0,1,\cdots,m$.

Proof Using the triangle inequality $\|(T^{\alpha}_{\Delta,b} - f)^{(r)}\|_\infty \leq \|(T^{\alpha}_{\Delta,b} - S_{\triangle})^{(r)}\|_\infty + \|(S_{\Delta} - f)^{(r)}\|_\infty$, and from result of ref. [17], for $n \geq j$, $k=0,1,2,\cdots,j$, $|f^{(k)}(x) - P^{(k)}_n(x)| \leq C_j \triangle^{j-k}_n(x)\omega(f^{(j)}, \triangle_n(x))$, we have

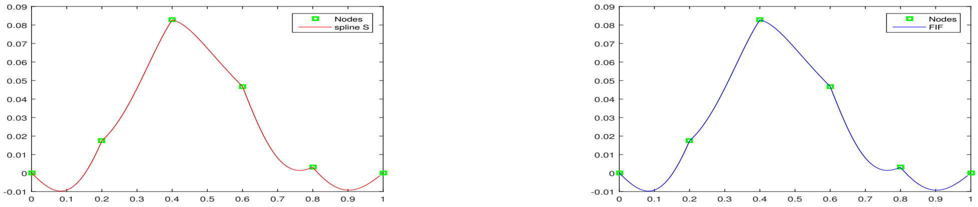
$$|f^{(r)}(x) - S^{(r)}_{\triangle}(x)| \leq C\triangle^{m-r}_{m+2}(x)\omega(f^{(m)}, \triangle_{m+2}(x)),$$

where $\triangle_{m+2}(x) = \frac{\sqrt{1-x^2}}{m+2} + \frac{1}{(m+2)^2}$. When $\|\alpha^{(r)}_n\|_\infty < \frac{1}{(2N)^m}(r=0,1,\cdots,m)$, from Theorem 3.2, we get

$$\|(T^{\alpha}_{\Delta,b} - S_{\triangle})^{(r)}\|_\infty \leq C_r \frac{1}{N^{m-r}},$$

here C_r is the constant only dependent on r.

Example 2 Taking $m=2$, consider function $f(x) = (2x^2-4x+3)^N(\sin(x))^N$, $b(x) = \cos(x)f(x)$, and the uniform partition of $[0,1]$, $\Delta: 0 = x_0 < x_1 < \cdots < x_N = 1$ with step $h = x_n - x_{n-1} = 1/N, n=1,2,\cdots,N$, $S_{\Delta}(x)$ is the spline function defined by eq.(2.1). We only consider $N=5$, and take scaling vector $\alpha_1 = 0.01x$, $\alpha_2 = 0.005x^2$,



(a) The curve of spline function (b) The curve of FIF $T^{\alpha}_{\Delta,b}(x)$

Figure 4 The comparison between the spline function and FIF, $N = 5$

$\alpha_3 = 0.01(x - \sin(x))$, $\alpha_4 = 0.01\sin(x)$, $\alpha_5 = 0.01(x - \frac{x^2}{2})$. Obviously, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_5)$ satisfies Theorem 3.2. The left graph of Figure 4 is spline function $S_{\Delta}(x)$, and the right graph is the fractal function $T^{\alpha}_{\Delta,b}(x)$.

4 Conclusion

In this paper, we use spline function to solve the fractal $(0, m)$ lacunary interpolation problem which is the general case of $(0, 2)$, $(0, 4)$ and so on, the explicit formulas of spline function are deduced by extrapolation algorithm. We find that there is fractal lacunary interpolation function with proper perturbation parameters, satisfying interpolation problem. The error estimates and convergence analysis were presented. The numerical examples demonstrate that our proposed method is efficient and viable. A more general theory of fractal Birkhoff interpolation and numerical simulations appear in our later work. The non-constant case of scaling function is expected to be resolved in future researches.

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基于外推样条下伴有压缩因子的分形函数的推广缺项插值问题的研究

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摘 要: 本文研究了分形函数的一般$(0, m)$缺项插值问题. 利用外推样条函数作为插值基函数的方法, 得到了当压缩因子满足$\|\alpha_k^{(r)}\|_\infty \leq \frac{1}{(2N)^m}$ $(k = 1, 2, \cdots, N;\ r = 0, 1, \cdots, m)$时的$(0, m)$分形缺项插值函数, 并得到了对应条件下的收敛结果. 用数值算例验证了外推样条函数作为分形缺项插值基函数是可行和有效的.

关键词: 外推样条; 分形函数; 缺项插值; 压缩因子; 逼近阶

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