

# QUASI SURE LOCAL CHUNG'S FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR INCREMENTS OF A BROWNIAN MOTION

MO Yong-xiang, LIU Yong-hong, ZHOU Xia

(*School of Mathematics and Computing Science, Guilin University of Electronic Technology;  
Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation,  
Guilin 541004, China*)

**Abstract:** In this paper, we obtain the quasi sure local Chung's functional law of the iterated logarithm for increments of a Brownian motion. As an application, a quasi sure Chung's type functional modulus of continuity for a Brownian motion is also derived.

**Keywords:** Brownian motion; increment; local Chung's law of the iterated logarithm;  $(r, p)$ -capacity

**2010 MR Subject Classification:** 60F15; 60F17; 60J65

**Document code:** A                      **Article ID:** 0255-7797(2020)02-0155-10

## 1 Introduction and Main Result

Let  $(B, H, \mu)$  be an abstract Wiener space. The capacity is a set function on  $B$  with the property that it sometimes takes positive values even for  $\mu$ -null sets, while a set of capacity zero has always  $\mu$ -measure zero. As we know, capacity is much finer than probability. An important difference between the capacity and probability is that the second Borel-Cantelli's lemma does not hold with respect to capacity  $C_{r,p}$  while it holds with respect to probability. Therefore, an interesting problem is to find out what property holds not only almost sure but also quasi sure. In this paper, we discuss this topic.

Many basic properties of Wiener processes were studied by authors (see [1–6]), such as the functional law of iterated logarithm, the functional modulus of continuity and large increments hold not only for  $\mu$ -a.s. but also for the sense of  $C_{r,p}$ -a.s.

In recent paper [2], Gao and Liu established local functional Chung's law for increments of Brownian motion. In the present paper, we discuss similar results, but the probability is

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\* **Received date:** 2018-11-15

**Accepted date:** 2019-04-02

**Foundation item:** Supported by National Natural Science Foundation of China (11661025); Science Research Foundation of Guangxi Education Department (YB2014117); Guangxi Natural Science Foundation (2018GXNSFBA281076; 2017GXNSFBA198179) and Guangxi Programme for Promoting Young Teachers's Ability (2018KY0214).

**Biography:** Mo Yongxiang (1979–), male, born at Huanjiang, Guangxi, lecturer, major in limit theory of stochastic processes.

**Corresponding author:** Liu Yonghong

replaced by  $(r, p)$ -capacity. The exact approximation rate for the modulus of continuity of Brownian motion can be viewed as a special case of our results.

We use standard notation and concepts on the abstract Wiener space  $(B, H, \mu)$ , including the  $H$ -derivative  $D$ , its adjoint  $D^*$  and the Ornstein-Uhlenbeck operator  $\mathcal{L} = -D^*D$ . Let  $D^{r,p}, r > 0, 1 \leq p < \infty$  be Sobolev space of Wiener functionals, i.e.,

$$D^{r,p} = (1 - \mathcal{L})^{-\frac{r}{2}} L^p, \quad \|F\|_{r,p} = \|(1 - \mathcal{L})^{\frac{r}{2}} F\|_{L^p}, \quad F \in L^p,$$

where  $L^p$  denotes  $L^p$ -space of real-valued functions on  $(B, \mu)$ . For  $r > 0, p > 1$ ,  $(r, p)$ -capacity is defined by  $C_{r,p}(O) = \inf\{\|F\|_{r,p}^p; F \geq 1, \mu\text{-a.s. on } O\}$ , for open set  $O \subset W$ , and for any set  $A \subset W, C_{r,p}(A) = \inf\{C_{r,p}(O); A \subset O \subset W, O \text{ is open}\}$ .

Let us consider classical Wiener space  $(W, H, \mu)$  as follows

$$\begin{aligned} W &= \{w \in C([0, \infty), \mathbb{R}^d); w(0) = 0, \lim_{t \rightarrow \infty} \frac{|w(t)|}{t} = 0\}, \\ H &= \{h \in W; \quad h(t) = \int_0^t \dot{h}(s) ds, \dot{h} \in L^2([0, \infty), \mathbb{R}^d)\}, \\ \mu &\text{ is Wiener measure on } W; \\ \|w\|_W &= \sup_{t \geq 0} \frac{|w(t)|}{1+t}, \quad w \in W; \quad \|h\|_H^2 = \int_0^\infty \langle \dot{h}(t), \dot{h}(t) \rangle dt. \end{aligned}$$

Let  $\mathcal{C}^d$  denote the space of continuous functions from  $[0, 1]$  to  $\mathbb{R}^d$  endowed with usual supnorm  $\|f\| := \sup_{0 \leq t \leq 1} |f(t)|$  and denote by  $\mathcal{C}_0^d := \{f \in \mathcal{C}^d; f(0) = 0\}$ , by  $\mathcal{H}^d := \{f \in \mathcal{C}_0^d; f(t) = \int_0^t \dot{f}(s) ds, \|f\|_{\mathcal{H}^d}^2 := \int_0^1 |\dot{f}(t)|^2 dt < \infty\}$ ,  $K := \{f \in \mathcal{H}^d; 2I(f) \leq 1\}$ , where

$$I(f) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{f}(t)|^2 dt, & f \text{ is absolutely continuous;} \\ +\infty, & \text{otherwise.} \end{cases}$$

Throughout this paper, let  $a_u, b_u$  be two non-decreasing continuous functions from  $(0, 1)$  to  $(0, e^{-1})$  satisfying

- (i)  $a_u \leq b_u$  for any  $u \in (0, 1)$  and  $\lim_{u \rightarrow 0} a_u = 0$ ,
- (ii)  $\frac{b_u}{a_u}$  is non-increasing.

Let  $w \in W$ , for  $u \in (0, 1), 0 \leq t \leq 1, \Delta(t, u)$  denotes the following path

$$\Delta(t, u)(s) = w(b_u t + a_u s) - w(b_u t), \quad s \in [0, 1].$$

Set

$$\ell_u = \log \frac{b_u \log b_u^{-1}}{a_u}, \quad \beta_u = (2a_u \ell_u)^{-1/2}, \quad u \in (0, 1)$$

and

$$K = \{\varphi \in C_0[0, 1] : 2I(\varphi) \leq 1\}.$$

The following theorems are results of this paper.

**Theorem 1.1** If conditions (i) and (ii) hold, then for any  $f \in K$  with  $2I(f) < 1$ , we have

$$\liminf_{u \rightarrow 0} \ell_u \inf_{t \in [0, 1 - \frac{au}{bu}]} \|\beta_u \Delta(t, u) - f\| = b(f), \quad C_{r,p} - \text{q.s.}$$

where  $b(f) = (\frac{c_d}{2})^{1/2} / (1 - 2I(f))^{-1/2}$ ,  $c_d$  is a positive constant.

Under the additional assumption (iii)  $\lim_{u \rightarrow 0} \frac{\log(b_u/a_u)}{\log \log b_u^{-1}} = \infty$ , we get

**Theorem 1.2** If conditions (i), (ii) and (iii) hold, then for any  $f \in K$  with  $2I(f) < 1$ , we have

$$\lim_{u \rightarrow 0} \ell_u \inf_{t \in [0, 1 - \frac{au}{bu}]} \|\beta_u \Delta(t, u) - f\| = b(f), \quad C_{r,p} - \text{q.s.}$$

**Corollary 1.1** Let  $M_{t,h}(x) = \frac{w(t+hx) - w(t)}{\sqrt{2h \log(2e^{-2}h^{-1})}}$ ,  $0 \leq x \leq 1, 0 \leq t \leq 1 - h$ . For any  $f \in K$  with  $2I(f) < 1$ , we have

$$\lim_{h \rightarrow 0} \log \frac{1}{h} \inf_{t \in [0, e^{-2} - h]} \|M_{t,h}(\cdot) - f(\cdot)\| = b(f), \quad C_{r,p} - \text{q.s.} \tag{1.1}$$

## 2 Proof of Theorem 1.1

Proof of Theorem 1.1 is completed by the following lemmas.

**Lemma 2.1** (see [3], Lemma 2.2) Let  $1 \leq k \in \mathbb{Z}$ ,  $q_1, q_2 \in (1, \infty)$  be given so that  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$ . For any  $f \in K$ , put

$$F_\varepsilon^{(i)}(w) = \|\varepsilon \left( \frac{w(t_i + \cdot h_i) - w(t_i)}{\sqrt{h_i}} \right) - f\|, \quad i = 1, 2, \dots, n, \varepsilon > 0,$$

where  $0 \leq t_i < \infty, h_i > 0, i = 1, 2, \dots, n$ . Then there exists a constant  $c = c(k, p, q_1, f, d) > 0$ , for any  $\delta \in (0, 1], \varepsilon \in (0, 1]$ , we have

$$C_{k,p} \left( \prod_{i=1}^n \{z; a_i < F_\varepsilon^{(i)}(z) < b_i\} \right)^{1/p} \leq c \delta^{-2k^2 - k} n^k \mu \left( \prod_{i=1}^n \{z; a_i - \delta < F_\varepsilon^{(i)}(z) < b_i + \delta\} \right)^{1/q_2}.$$

**Lemma 2.2** (see [3], Lemma 2.4) There exists a positive number  $c_d$  such that for any  $h > 0, \tau > 0, f \in K$ , we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log C_{r,p} \left( \left\| \frac{w(t+h\cdot) - w(t)}{\sqrt{h}} - \frac{f}{\varepsilon} \right\| \leq \varepsilon \tau \right) \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mu \left( \left\| \frac{w(t+h\cdot) - w(t)}{\sqrt{h}} - \frac{f}{\varepsilon} \right\| \leq \varepsilon \tau \right) = -c_d \tau^{-2} - I(f). \end{aligned}$$

**Lemma 2.3** For any  $f \in K$  with  $2I(f) < 1$ , we have

$$\liminf_{u \rightarrow 0} \ell_u \inf_{t \in [0, 1 - \frac{au}{bu}]} \|\beta_u \Delta(t, u) - f\| \geq b(f). \quad C_{r,p} - \text{q.s.}$$

**Proof Case I**  $\limsup_{u \rightarrow 0} \frac{\log \frac{bu}{au}}{\log \log b_u^{-1}} < \infty$ . If  $\limsup_{u \rightarrow 0} \frac{\log \frac{bu}{au}}{\log \log b_u^{-1}} < \infty$ , then  $b_u \rightarrow 0$  as  $u \rightarrow 0$  and there exists a  $0 < M < \infty$  such that  $\frac{b_u}{a_u} \leq (\log b_u^{-1})^M$ . Let  $l(u) = a_u (\log \frac{b_u \log b_u^{-1}}{a_u})^{-3}$ , and

take  $u_n$  such that  $a_{u_n} = (\exp(\frac{n}{(\log n)^3}))^{-1}$ . Set  $k_n = [\frac{b_{u_n}}{l(u_{n+1})}]$ ,  $t_i = il(u_{n+1}), i = 0, 1, \dots, k_n$ .

Then

$$\begin{aligned} & \min_{0 \leq i \leq k_n} \|\beta_{u_{n+1}}(w(t_i + a_{u_{n+1}} \cdot) - w(t_i)) - f\| \\ & \leq 2 \sup_{t \in [0, b_{u_n}]} \sup_{s \in [0, l(u_{n+1})]} |w(t+s) - w(t)| \beta_{u_{n+1}} \\ & \quad + \inf_{t \in [0, b_{u_n} - a_{u_{n+1}}]} \|\beta_{u_{n+1}}(w(t + a_{u_{n+1}} \cdot) - w(t)) - f\|. \end{aligned} \tag{2.1}$$

For any  $0 < \varepsilon < 1$ , choose  $\delta > 0$  such that  $\eta = -2\delta + 2I(f) + \frac{1-2I(f)}{(1-\varepsilon)^2} > 1$ . By Lemma 2.2, for  $n$  large enough, we have

$$\begin{aligned} & C_{r,p}(l_{u_{n+1}} \min_{0 \leq i \leq k_n} \|\beta_{u_{n+1}}(w(t_i + a_{u_{n+1}} \cdot) - w(t_i)) - f\| \leq (1 - \varepsilon)b(f)) \\ & \leq \sum_{0 \leq i \leq k_n} C_{r,p}(l_{u_{n+1}} \|\beta_{u_{n+1}}(w(t_i + a_{u_{n+1}} \cdot) - w(t_i)) - f\| \leq (1 - \varepsilon)b(f)) \\ & = \sum_{0 \leq i \leq k_n} C_{r,p}(\|\frac{w(t_i + a_{u_{n+1}} \cdot) - w(t_i)}{\sqrt{a_{u_{n+1}}}} - \sqrt{2l_{u_{n+1}}}f\| \leq \frac{2}{\sqrt{2l_{u_{n+1}}}}(1 - \varepsilon)b(f)) \\ & \leq (1 + k_n) \exp\{(2 \log \frac{b_{u_{n+1}} \log b_{u_n}^{-1}}{a_{u_{n+1}}})(-\frac{c_d}{4(1 - \varepsilon)^2(b(f))^2} - I(f) + \delta)\} \\ & = (1 + k_n) \exp\{(-\log \frac{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}{a_{u_{n+1}}})(2I(f) - 2\delta + \frac{1 - 2I(f)}{(1 - \varepsilon)^2})\} \\ & \leq \frac{b_{u_n} + l(u_{n+1})}{l(u_{n+1})} (\frac{a_{u_{n+1}}}{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}})^\eta, \end{aligned}$$

by Borel-Cantelli's lemma,

$$\liminf_{n \rightarrow \infty} l_{u_{n+1}} \min_{0 \leq i \leq k_n} \|\beta_{u_{n+1}}(w(t_i + a_{u_{n+1}} \cdot) - w(t_i)) - f\| \geq b(f), \quad C_{r,p} - \text{q.s.} \tag{2.2}$$

On the other hand, for any  $\delta_0 > 0$ ,

$$\begin{aligned} & C_{r,p}(2l_{u_{n+1}} \sup_{0 \leq t \leq b_{u_n}} \sup_{0 \leq s \leq l(u_{n+1})} |w(t+s) - w(t)| \beta_{u_{n+1}} \geq \delta_0) \\ & = C_{r,p}(\sqrt{\frac{2l_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{0 \leq t \leq b_{u_n}} \sup_{0 \leq s \leq l(u_{n+1})} |w(t+s) - w(t)| \geq \delta_0) \\ & = C_{r,p}(\sqrt{\frac{2l_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{0 \leq t \leq \frac{b_{u_n}}{l(u_{n+1})}} \sup_{0 \leq s \leq 1} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0) \\ & \leq \sum_{i=0}^{[\frac{b_{u_n}}{l(u_{n+1})}]} C_{r,p}(\sqrt{\frac{2l_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{i \leq t \leq i+1} \sup_{0 \leq s \leq 1} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0), \end{aligned}$$

moreover

$$\mu(\sqrt{\frac{2l_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{i \leq t \leq i+1} \sup_{0 \leq s \leq 1} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0)$$

$$\begin{aligned}
 &= \mu\left(\sqrt{\frac{2}{(\ell_{u_{n+1}})^2}} \sup_{i \leq t \leq i+1} \sup_{0 \leq s \leq 1} |w(t+s) - w(t)| \geq \delta_0\right) \\
 &= \mu\left(\sqrt{\frac{4}{(\ell_{u_{n+1}})^2}} \sup_{0 \leq t \leq 1} \|w(\frac{t}{2} + \frac{\cdot}{2}) - w(\frac{t}{2})\| \geq \delta_0\right) = \mu\left(\sqrt{\frac{4}{(\ell_{u_{n+1}})^2}} w \in A\right),
 \end{aligned}$$

where  $A = \{f \in W; \sup_{0 \leq t \leq 1} \|f(\frac{t}{2} + \frac{\cdot}{2}) - f(\frac{t}{2})\| \geq \delta_0\}$ . For  $f \in A$ ,  $\sup_{0 \leq t \leq 1} \int_{\frac{t}{2}}^{\frac{1+t}{2}} |\dot{f}(s)|^2 ds \geq \delta_0^2$ , thus  $\inf_{f \in A} \frac{1}{2} \|f\|_H^2 \geq \frac{\delta_0^2}{2}$ . By Theorem 1.1 in [5], we have

$$\begin{aligned}
 &C_{r,p} \left(\sqrt{\frac{2\ell_{u_{n+1}}}{a_{u_{n+1}}} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq 1} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0}\right) \\
 &\leq \left(\frac{a_{u_{n+1}}}{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}\right)^{\frac{1}{8} \delta_0^2 \log \frac{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}{a_{u_{n+1}}}}.
 \end{aligned}$$

Taking into account  $\log \frac{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}{a_{u_{n+1}}} \rightarrow \infty$ , as  $n \rightarrow \infty$ , thus

$$\sum_n \frac{b_{u_n} + l(u_{n+1})}{l(u_{n+1})} \left(\frac{a_{u_{n+1}}}{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}\right)^{\frac{1}{8} \delta_0^2 \log \frac{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}{a_{u_{n+1}}}} < \infty,$$

by Borel-Cantelli's lemma

$$\limsup_{n \rightarrow \infty} 2\ell_{u_{n+1}} \sup_{0 \leq t \leq b_{u_n}} \sup_{0 \leq s \leq l(u_{n+1})} |w(t+s) - w(t)| \beta_{u_{n+1}} = 0, \quad C_{r,p} - \text{q.s.} \quad (2.3)$$

By (2.1)–(2.3), we get

$$\lim_{n \rightarrow \infty} \ell_{u_{n+1}} \inf_{0 \leq t \leq b_{u_n} - a_{u_{n+1}}} \|\beta_{u_{n+1}}(w(t + a_{u_{n+1}} \cdot) - w(t)) - f\| \geq b(f), \quad C_{r,p} - \text{q.s.} \quad (2.4)$$

Remark that  $u_n \rightarrow 0$ , so for any small enough  $u$ , there is a unique  $n$  such that  $u \in (u_{n+1}, u_n]$ . Let  $\phi_{t,u}(s) = \beta_u(w(t + a_u s) - w(t))$ ,  $s \in [0, 1], t \in [0, b_u - a_u]$ . We define  $X(u) = \ell_u \inf_{t \in [0, u - a_u]} \|\phi_{t,u}(\cdot) - f(\cdot)\|, X_n = \inf_{u_{n+1} < u \leq u_n} X(u)$ . By the definition of infimum, for any  $\varepsilon > 0$ , there exists  $T_n \in (u_{n+1}, u_n]$  such that  $X_n \geq X(T_n) - \varepsilon$ .

For any  $r \in [0, 1]$ , let  $x = \frac{r a_{u_{n+1}}}{a_{T_n}}$ . Then we have, for  $0 \leq x \leq 1$ .

$$\begin{aligned}
 &\inf_{t \in [0, b_{u_n} - a_{u_{n+1}}]} \|\beta_{u_{n+1}}[w(t + a_{u_{n+1}} \cdot) - w(t)] - f\| \\
 &\leq \inf_{t \in [0, b_{T_n} - a_{T_n}]} \sup_{0 \leq r \leq 1} |\phi_{t,u_n}(r) - f(r)| \\
 &= \inf_{t \in [0, b_{T_n} - a_{T_n}]} \sup_{0 \leq x \leq \frac{a_{u_{n+1}}}{a_{T_n}}} \left| \phi_{t,u_n}\left(\frac{a_{T_n}}{a_{u_{n+1}}} x\right) - f\left(\frac{a_{T_n}}{a_{u_{n+1}}} x\right) \right| \\
 &\leq \inf_{t \in [0, b_{T_n} - a_{T_n}]} \sup_{0 \leq x \leq 1} \left| \left[\frac{\beta_{u_{n+1}}}{\beta_{T_n}} \phi_{t,T_n}(x) - f\left(\frac{a_{T_n}}{a_{u_{n+1}}} x\right)\right] \right| \tag{2.5} \\
 &\leq \inf_{t \in [0, b_{T_n} - a_{T_n}]} \frac{\beta_{u_{n+1}}}{\beta_{T_n}} \|\phi_{t,T_n}(\cdot) - f(\cdot)\| + \left| \frac{\beta_{u_{n+1}}}{\beta_{T_n}} - 1 \right| \|f(\cdot)\| + \|f(\cdot) - f\left(\frac{a_{T_n}}{a_{u_{n+1}}} \cdot\right)\| \\
 &\leq \frac{\beta_{u_{n+1}}}{\beta_{T_n}} \left(\frac{b_{u_n} \log b_{u_n}^{-1}}{a_{u_n}}\right)^{-1} X(T_n) + \left| \frac{\beta_{u_{n+1}}}{\beta_{T_n}} - 1 \right| \|f(\cdot)\| + \|f(\cdot) - f\left(\frac{a_{T_n}}{a_{u_{n+1}}} \cdot\right)\|.
 \end{aligned}$$

Noting that

$$1 \geq \frac{\log \frac{b_{u_n} \log b_{u_n}^{-1}}{a_{u_n}}}{\log \frac{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}{a_{u_{n+1}}}} = \frac{a_{u_n} \log \frac{b_{u_n} \log b_{u_n}^{-1}}{a_{u_n}}}{a_{u_{n+1}} \log \frac{b_{u_{n+1}} \log b_{u_{n+1}}^{-1}}{a_{u_{n+1}}}} \frac{a_{u_{n+1}}}{a_{u_n}} \geq \frac{a_{u_{n+1}}}{a_{u_n}} \rightarrow 1, \tag{2.6}$$

$$\frac{\beta_{u_{n+1}}}{\beta_{T_n}} \leq \frac{\beta_{u_{n+1}}}{\beta_{u_n}} \leq \frac{a_{u_n}}{a_{u_{n+1}}} \leq \exp\left(\frac{1}{(\log n)^3}\right) = 1 + \frac{1}{(\log n)^3} + o\left(\frac{1}{(\log n)^3}\right), \tag{2.7}$$

$$\|f\left(\frac{a_u}{a_{u_n}} \cdot\right) - f(\cdot)\| \leq \sqrt{2} \sqrt{1 - \frac{a_{u_{n+1}}}{a_{u_n}}} \leq \sqrt{2} \left(\frac{1}{(\log n)^3} + o\left(\frac{1}{(\log n)^3}\right)\right)^{1/2}, \tag{2.8}$$

$$\log \frac{b_{u_n} \log b_{u_n}^{-1}}{a_{u_n}} \leq (M + 1) \log \log a_{u_n}^{-1}. \tag{2.9}$$

by (2.4),(2.5), (2.7)–(2.9), we get

$$\liminf_{n \rightarrow \infty} X(T_n) \geq b(f), \quad C_{r,p} \text{- q.s..}$$

Since

$$\liminf_{u \rightarrow 0} X(u) \geq \liminf_{n \rightarrow \infty} X_n \geq \liminf_{n \rightarrow \infty} X(T_n) \geq b(f) - \varepsilon,$$

which ends the proof.

**Case II**  $\limsup_{u \rightarrow 0} \frac{\log \frac{bu}{au}}{\log \log b_u^{-1}} = \infty$ . If  $\limsup_{u \rightarrow 0} \frac{\log \frac{bu}{au}}{\log \log b_u^{-1}} = \infty$ , then we can choose a non-increasing sequence  $\{u_n; n \geq 1\}$  with  $\frac{b_{u_n}}{a_{u_n}} = n^d, d > 1$ . Let  $h(n) = \frac{\log \frac{b_{u_n}}{a_{u_n}}}{\log \log b_{u_n}^{-1}} = \frac{\log n^d}{\log \log b_{u_n}^{-1}}$ , then  $b_{u_n}^{-1} = \exp\{n^{\frac{d}{h(n)}}\}$  and  $h(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Let  $l(u), k_n$  and  $t_i, i = 1, 2, \dots, k_n$  be defined by Case I. Then for some constant  $C > 0$ , if  $d$  is chosen in a suitable way, then

$$\sum_{n=1}^{\infty} \left(1 + \frac{b_{u_n}}{l(u_{n+1})}\right) \left(\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}}\right)^n \leq C \sum_{n=1}^{\infty} n^{-d(\eta-1)} (\log n)^3 < \infty.$$

Furthermore,

$$\begin{aligned} \frac{a_{u_n}}{a_{u_{n+1}}} &= \frac{b_{u_n}}{n^d} \frac{(n+1)^d}{b_{u_{n+1}}} = \frac{(n+1)^d}{n^d} \exp\left((n+1)^{d/h(n+1)} - n^{d/h(n)}\right) \\ &\leq \left(1 + \frac{1}{n}\right)^d \exp\left(n^{(d/h(n))^{-1}}\right) \\ &= \left(1 + \frac{1}{n}\right)^d \left(1 + \frac{1}{n^{1-(d/h(n))}} + o\left(\frac{1}{n^{1-(d/h(n))}}\right)\right), \end{aligned}$$

which implies that

$$\log \frac{b_{u_n} \log b_{u_n}^{-1}}{a_{u_n}} \left(1 - \frac{a_{u_{n+1}}}{a_{u_n}}\right) = d \left(\log n + \frac{\log n}{h(n)}\right) \left(1 - \frac{a_{u_{n+1}}}{a_{u_n}}\right) \rightarrow 0, \quad n \rightarrow \infty. \tag{2.10}$$

Similarly to the proof of Case I, the proof of Lemma 2.3 is completed.

**Lemma 2.4** For any  $f \in K$  with  $2I(f) < 1$ , we have

$$\liminf_{u \rightarrow 0} \ell_u \inf_{t \in [0, 1 - \frac{au}{u}]} \|\beta_u \Delta(t, u) - f\| \leq b(f), \quad C_{r,p} \text{- q.s..}$$

**Proof** Set  $\rho := \lim_{u \rightarrow 0^+} \frac{a_u}{b_u}$ . If  $\rho < 1$  and  $b_u \rightarrow b \neq 0$  as  $u \rightarrow 0$ , then  $\lim_{u \rightarrow 0^+} \frac{\log \frac{b_u}{a_u}}{\log \log b_u^{-1}} = \infty$ . In this case, see Lemma 3.2. Therefore, we only consider the following two cases

- (I)  $\rho < 1$  and  $b_u \rightarrow 0$  as  $u \rightarrow 0$ ,
- (II)  $\rho = 1$ .

**Case I**  $\rho < 1$  and  $b_u \rightarrow 0$  as  $u \rightarrow 0$  If  $\rho < 1$  and  $b_u \rightarrow 0$  as  $u \rightarrow 0$ , then we can choose  $\{u_k, k \geq 1\}$  such that  $b_{u_{k+1}} = b_{u_k} - a_{u_k}$ ,  $k \geq 1$ . For any  $\varepsilon > 0$ , choose  $\delta > 0$  such that  $\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta < 1$ . Set  $k = [r] + 1$ , by Lemma 2.1, we have

$$\begin{aligned} & C_{r,p} \left( \prod_{n=m_0}^l (\ell_{u_n} \|\beta_{u_n} \Delta \left(1 - \frac{a_{u_n}}{b_{u_n}}, u_n\right) - f\| \geq b(f)(1 + 2\varepsilon)) \right)^{1/p} \\ \leq & cl^k \left( \frac{b(f)\varepsilon}{\ell_{u_1}} \right)^{-2k^2-k} \mu \left( \prod_{n=m_0}^l \left( \left\| \frac{\Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n)}{\sqrt{2a_{u_n} \ell_{u_n}}} - f \right\| \geq \frac{b(f)(1 + \varepsilon)}{\ell_{u_n}} \right) \right)^{1/q_2} \\ \leq & cl^k \left( \frac{b(f)\varepsilon}{\ell_{u_1}} \right)^{-2k^2-k} \prod_{n=m_0}^l \left\{ 1 - \mu \left( \left\| \frac{\Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n)}{\sqrt{2a_{u_n} \ell_{u_n}}} - f \right\| < \frac{b(f)(1 + \varepsilon)}{\ell_{u_n}} \right) \right\}^{1/q_2}, \end{aligned}$$

moreover, by small deviation,

$$\begin{aligned} & \mu \left( \left\| \frac{1}{\sqrt{2\ell_{u_n}}} \frac{\Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n)}{\sqrt{a_{u_n}}} - f \right\| < \frac{b(f)(1 + \varepsilon)}{\ell_{u_n}} \right) \\ = & \mu \left( \|w - \sqrt{2\ell_{u_n}} f\| < \frac{2b(f)(1 + \varepsilon)}{\sqrt{2\ell_{u_n}}} \right) \geq \left( \frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta}, \end{aligned}$$

thus

$$\begin{aligned} & C_{r,p} \left( \prod_{n=m_0}^l (\ell_{u_n} \|\beta_{u_n} \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f\| \geq b(f)(1 + 2\varepsilon)) \right)^{1/p} \\ \leq & cl^k \left( \frac{\ell_{u_1}}{\varepsilon b(f)} \right)^{\frac{2k^2+k}{2}} \exp \left( -\frac{1}{q_2} \sum_{n=m_0}^l \left( \frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} \right). \end{aligned}$$

Noting that, there exists  $A = A(m_0) > 0$  such that

$$\sum_{n=m_0}^l \left( \frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} > A(\log b_{u_1}^{-1})^{\varepsilon'},$$

where  $\varepsilon' = 1 - \frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta > 0$ .

We discuss as follows

- (a) If  $\limsup_{u \rightarrow 0} \frac{\log(b_u/a_u)}{\log \log b_u^{-1}} < \infty$ , then for some  $0 < M < \infty$ , we have  $\frac{b_u}{a_u} \leq (\log b_u^{-1})^M$ . Take  $\theta > 2/\varepsilon', u_0 = e^{-(\log l_0)^\theta}$ , there exists  $l_0$  large enough, we can prove that

$$\log b_{u_l}^{-1} \geq (\log l)^\theta, \quad l \geq l_0,$$

we get

$$(\log b_{u_l}^{-1})^{\varepsilon'} \geq (\log l)^2.$$

When  $l$  is large enough, for constant  $c' > 0$ , we can also prove that  $c'l \geq \log b_{u_l}^{-1}$ . Thus we have

$$\begin{aligned} & C_{r,p} \left( \bigcap_{n=m_0}^l (\ell_{u_n} \|\beta_{u_n} \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f\| \geq b(f)(1 + 2\varepsilon)) \right)^{1/p} \\ & \leq c l^k (\log \log b_{u_l}^{-1})^{2k^2+k} \exp\left(-\frac{A}{q_2} (\log b_{u_l}^{-1})^{\varepsilon'}\right) \leq \frac{c_0 l^k (\log l)^{2k^2+k}}{\exp\left(\frac{A}{q_2} (\log l)^2\right)} \rightarrow 0 (l \rightarrow \infty), \end{aligned}$$

where  $c_0$  is a constant. We get

$$C_{r,p} \left( \bigcup_{l=1}^{\infty} \bigcap_{n=l}^{\infty} (\ell_{u_n} \|\beta_{u_n} \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f\| \geq b(f)(1 + 2\varepsilon)) \right) = 0,$$

consequently

$$\liminf_{n \rightarrow \infty} \ell_{u_n} \|\beta_{u_n} \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f\| \leq b(f), \quad C_{r,p} - \text{q.s.},$$

we have

$$\liminf_{u \rightarrow 0} \ell_u \|\beta_u \Delta(1 - \frac{a_u}{b_u}, u) - f\| \leq b(f), \quad C_{r,p} - \text{q.s.}.$$

(b) If  $\limsup_{u \rightarrow 0} \frac{\log(b_u/a_u)}{\log \log b_u^{-1}} = \infty$ , then see Lemma 3.2.

**Case II** Set  $\rho = 1$ . By applying Lemma 2.1, similarly to the corresponding that of (3.3) in [2].

### 3 Proof of Theorem 1.2

Proof of Theorem 1.2 is completed by the following lemmas.

**Lemma 3.1** If condition (iii) also holds, then there exists an non-increasing  $\{u_n, n \geq 1\}$  for any  $f \in K$  with  $2I(f) < 1$ , we have

$$\limsup_{n \rightarrow \infty} \ell_{u_n} \inf_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \|\beta_{u_n} (B(t + a_{u_n} \cdot) - B(t)) - \varphi\| \leq b(f), \quad C_{r,p} - \text{q.s.}.$$

**Proof** Owing to  $\lim_{u \rightarrow 0+} \frac{\log \frac{b_u}{a_u}}{\log \log b_u^{-1}} = \infty$ , there exists a subsequence  $\{u_n; n \geq 1\}$  such that  $\frac{b_{u_n}}{a_{u_n}} = n^d$ . Let

$$t_i = ia_{u_n}, i = 0, 1, \dots, k_n = \left[ \frac{b_{u_{n+1}}}{a_{u_n}} \right] - 1 \quad \text{and} \quad h(n) = \frac{\log \frac{b_{u_n}}{a_{u_n}}}{\log \log b_{u_n}^{-1}} = \frac{d \log n}{\log \log b_{u_n}^{-1}},$$

then  $b_{u_n}^{-1} = \exp\{n^{\frac{d}{h(n)}}\}$  and  $h(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . And for any small enough  $\alpha > 0$ ,  $\frac{(n+1)^\alpha}{\log b_{u_{n+1}}^{-1}} \rightarrow \infty$ ,  $1 \leq \frac{b_{u_n}}{b_{u_{n+1}}} = \exp\left(\frac{d}{h(n+1)} - \frac{d}{h(n)}\right) \leq \exp\left(n^{\frac{d}{h(n)} - 1}\right) \rightarrow 1$  and  $\frac{a_{u_n}}{a_{u_{n+1}}} \rightarrow 1$

as  $n \rightarrow \infty$ . Choose  $\delta > 0$  such that  $\eta_0 := \frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta < 1$ . Take  $k = [r] + 1$ , by Lemma 2.1 and small deviation, for  $n$  large enough, we have

$$\begin{aligned} & C_{r,p}(\ell_{u_n}) \inf_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \|\beta_{u_n}(w(t + a_{u_n} \cdot) - w(t)) - f\| \geq b(f)(1 + 2\varepsilon)^{1/p} \\ & \leq C_{r,p}(\min_{0 \leq i \leq k_n} \|\beta_{u_n}(w(t_i + a_{u_n} \cdot) - w(t_i)) - f\| \geq \frac{b(f)(1 + 2\varepsilon)}{\ell_{u_n}})^{1/p} \\ & \leq c(1 + k_n)^k \left(\frac{\ell_{u_n}}{b(f)\varepsilon}\right)^{2k^2+k} \mu\left(\min_{0 \leq i \leq k_n} \|\beta_{u_n}(w(t_i + a_{u_n} \cdot) - w(t_i)) - f\| \geq \frac{b(f)(1 + \varepsilon)}{\ell_{u_n}}\right)^{1/q_2} \\ & = c(1 + k_n)^k \left(\frac{\ell_{u_n}}{b(f)\varepsilon}\right)^{2k^2+k} \mu\left(\left\|\frac{1}{\sqrt{a_{u_n}}}(w(a_{u_n} \cdot)) - \sqrt{2\ell_{u_n}}f\right\| \geq \frac{2b(f)(1 + \varepsilon)}{\sqrt{2\ell_{u_n}}}\right)^{\frac{1+k_n}{q_2}} \\ & = c(1 + k_n)^k \left(\frac{\ell_{u_n}}{b(f)\varepsilon}\right)^{2k^2+k} \left(1 - \mu\left(\|w(\cdot) - \sqrt{2\ell_{u_n}}f\| < \frac{2b(f)(1 + \varepsilon)}{\sqrt{2\ell_{u_n}}}\right)\right)^{\frac{1+k_n}{q_2}} \\ & \leq c(1 + k_n)^k \left(\frac{\ell_{u_n}}{b(f)\varepsilon}\right)^{2k^2+k} \left(1 - \left(\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}}\right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta}\right)^{\frac{1+k_n}{q_2}} \\ & \leq c_0 n^{kd} (\log n)^{2(2k^2+k)} \exp\left\{-\frac{1}{q_2} \left(\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}}\right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} \left(\left[\frac{b_{u_{n+1}}}{a_{u_n}}\right]\right)\right\} \\ & = c_0 n^{kd} (\log n)^{2(2k^2+k)} \exp\left\{-\frac{1}{q_2} \left(\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}}\right)^{\eta_0} \left(\left[\frac{b_{u_{n+1}}}{a_{u_n}}\right]\right)\right\}, \end{aligned}$$

where  $c_0$  is a constant. If  $d$  is chosen in a suitable way, then

$$\sum_n c_0^p n^{pkd} (\log n)^{2p(2k^2+k)} \exp\left\{-\frac{p}{q_2} \left(\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}}\right)^{\eta_0} \left(\left[\frac{b_{u_{n+1}}}{a_{u_n}}\right]\right)\right\} < \infty,$$

by Borel-Cantelli's lemma

$$\limsup_{n \rightarrow \infty} \ell_{u_n} \inf_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \|\beta_{u_n}(w(t + a_{u_n} \cdot) - w(t)) - f\| \leq b(f)(1 + 2\varepsilon), \quad C_{r,p} - \text{q.s.}$$

**Lemma 3.2** If conditions (i), (ii) and (iii) hold, then for any  $f \in K$  with  $2I(f) < 1$ , we have

$$\limsup_{u \rightarrow 0} \ell_u \inf_{t \in [0, b_u - a_u]} \|\beta_u(w(t + a_u \cdot) - w(t)) - f\| \leq b(f), \quad C_{r,p} - \text{q.s.}$$

**Proof** Let  $\phi_{t,u}(s) = \beta_u(w(t + a_u s) - w(t))$ ,  $u_n$  is defined as in Lemma 3.1. Since  $\phi_{t,u}(s) = \frac{\beta_u}{\beta_{u_n}} \phi_{t,u_n}(\frac{a_u}{a_{u_n}} s)$ , we have

$$\begin{aligned} & \inf_{t \in [0, u - a_u]} \|\phi_{t,u}(\cdot) - f(\cdot)\| \leq \inf_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \|\phi_{t,u_n}\left(\frac{a_u}{a_{u_n}} \cdot\right) - f\left(\frac{a_u}{a_{u_n}} \cdot\right)\| \\ & + \sup_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \left| \frac{\beta_u}{\beta_{u_n}} - 1 \right| \|\phi_{t,u_n}\left(\frac{a_u}{a_{u_n}} \cdot\right)\| + \sup_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \left\| f\left(\frac{a_u}{a_{u_n}} \cdot\right) - f(\cdot) \right\| \\ & \leq \inf_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \|\phi_{t,u_n}(\cdot) - f(\cdot)\| + \sup_{t \in [0, b_{u_{n+1}} - a_{u_n}]} \left| 1 - \frac{a_{u_{n+1}}}{a_{u_n}} \right| \|\phi_{t,u_n}\left(\frac{a_u}{a_{u_n}} \cdot\right)\| \\ & + \left\| f\left(\frac{a_u}{a_{u_n}} \cdot\right) - f(\cdot) \right\|. \end{aligned} \tag{3.1}$$

Moreover

$$\|f(\frac{a_u}{a_{u_n}} \cdot) - f(\cdot)\| \leq 2|1 - \frac{a_{u_{n+1}}}{u_n}|^{1/2}, \quad (3.2)$$

$$(\log \frac{b_{u_n} \log b_{u_n}^{-1}}{a_{u_n}})(1 - \frac{a_{u_{n+1}}}{a_{u_n}}) \leq d(\log n + \frac{\log n}{h(n)})(1 - \frac{a_{u_{n+1}}}{a_{u_n}}). \quad (3.3)$$

We can conclude Lemma 3.2 from (3.1)–(3.3) and Lemma 3.1.

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## Brown运动增量的拟必然局部钟泛函重对数律

莫永向, 刘永宏, 周霞

(桂林电子科技大学数学与计算科学学院; 广西高校数据分析与计算重点实验室, 广西 桂林 541004)

**摘要:** 本文获得了Brown运动增量的拟必然局部钟泛函重对数律. 应用这一结果, 推广了Brown运动拟必然钟泛函连续模.

**关键词:** Brown运动; 增量; 局部钟重对数律;  $(r, p)$ -容度

MR(2010)主题分类号: 60F15; 60F17; 60J65      中图分类号: O211.4