

SHARP ESTIMATES FOR EIGENVALUES OF BI-DRIFTING LAPLACIAN

LI Yan-li^{1,2}, DU Feng^{2,3}

(1.School of Electronic and Information Science, Jingchu University of Technology,
Jingmen 448000, China)

(2.Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, China)

(3.School of Mathematics and Physics, Jingchu University of Technology,
Jingmen 448000, China)

Abstract: In this paper, we study the four types of eigenvalue problems for the bi-drifting Laplacian. By using the weighted Reilly formula, we get some sharp lower bounds for the first nonzero eigenvalue for these eigenvalue problems on compact smooth metric measure spaces with boundary and under some condition on the m -weighted Ricci curvature, which generalize the corresponding results for the eigenvalues of biharmonic operator.

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1 Introduction

In 1977, Reilly [19] obtained a sharp lower bound for the first Dirichlet eigenvalue of the Laplacian as following.

Reilly's Theorem Let M be an $n(\geq 2)$ -dimensional compact Riemannian manifold with Ricci curvature bounded below by $(n-1)\kappa > 0$ and boundary. If the mean curvature of ∂M is nonnegative then the first Dirichlet eigenvalue of the Laplacian of M satisfies

$$\lambda_1 \geq n\kappa,$$

and equality holds if and only M is isometric to an n -dimensional Euclidean semi-sphere of radius $\frac{1}{\sqrt{\kappa}}$.

For the Neumann boundary case, a similar result was proven by Escobar [10] and Xia [25] independently. In [13], Li-Wei generalize these results including Dirichlet boundary and Neumann boundary to the first eigenvalue of the drifting Laplacian $\Delta_f := \Delta - \langle \nabla f, \nabla \cdot \rangle$

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Biography: Li Yanli (1981-), female, born at Huanggang, Hubei, associate professor, major in differential equations and applications. E-mail: liyanli1891@163.com .

on compact smooth metric measure spaces $(M, \langle, \rangle, e^{-f} d\nu)$, here (M, \langle, \rangle) is n -dimensional Riemannian manifold with a metric \langle, \rangle , f is a smooth real-valued function on M and $d\nu$ is the Riemannian volume element related to \langle, \rangle (sometimes, we also call $d\nu$ the volume density). The drifting Laplacian is also called weighted Laplacian (or Witten Laplacian), some interesting results concerning eigenvalues of the drifting Laplacian can be found in [7-9, 11, 13, 17, 26]. Compared with Ricci curvature of the Riemannian manifolds, we can define the so-called weighted Ricci curvature Ric^f on smooth metric measure spaces as following

$$\text{Ric}^f = \text{Ric}_M + \text{Hess}f,$$

which is also called the ∞ -Bakry-Émery Ricci tensor. The equation $\text{Ric}^f = \kappa \langle, \rangle$ for some constant κ is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow. For $\kappa = 0$, $\kappa > 0$, and $\kappa < 0$, the gradient Ricci soliton $(M, \langle, \rangle, e^{-f} d\nu, \kappa)$ is called steady, shrinking, and expanding, respectively. Set

$$\text{Ric}^m = \text{Ric}^f - \frac{1}{m-n} \nabla f \otimes \nabla f \quad (m > n), \quad (1.1)$$

which is called m -weighted Ricci curvature [1] of M (also called the m -Bakry-Émery Ricci tensor). When $m = n$, let f be a constant and $\text{Ric}^m = \text{Ric}_M$. A smooth metric measure space is not necessarily compact when $\text{Ric}^f \geq \lambda > 0$, unlike in the case of Riemannian manifolds where such a complete one is compact if its Ricci curvature is bounded from below uniformly by some positive constant [21], but an interesting fact is that when $\text{Ric}^m \geq (m-1)\kappa > 0$, a complete smooth metric measure space $(M, \langle, \rangle, e^{-f} d\nu)$ is automatically compact and the diameter of M satisfies $\text{diam}(M) \leq \pi/\sqrt{\kappa}$ [18].

In [5], Chen-Cheng-Wang-Xia gave some lower bounds for the first eigenvalue of four kinds of eigenvalue problems of the biharmonic operator on compact manifolds with boundary and positive Ricci curvature. In [6], Du-Bezerra generalize these results to the bi-drifting Laplacian on compact manifolds with boundary and positive m -weighted Ricci curvature. Recently, Wang-Xia [24] gave some new sharp lower bounds for the first eigenvalues of the biharmonic operator on compact manifolds with boundary and positive Ricci curvature. Inspired by these investigations, in this paper, we first get.

Theorem 1.1 Let $(M, \langle, \rangle, e^{-f} d\nu)$ be an $n(\geq 2)$ -dimensional smooth metric measure space with boundary ∂M and denote by ν the outward unit normal vector field of ∂M . Assume that the m -weighted Ricci curvature of M is bounded below by $(m-1)\kappa > 0$. Let λ_1 be the first nonzero eigenvalue with Dirichlet boundary condition of the drifting Laplacian of M and let Γ_1 be the first eigenvalue of the clamped plate problem on M ,

$$\Delta_f^2 g = \Gamma_1 g \text{ in } M, \quad g = \frac{\partial^2 g}{\partial \nu^2} = 0 \text{ on } \partial M. \quad (1.2)$$

Then we have

$$\Gamma_1 \geq \lambda_1 \left(\frac{1}{m} \lambda_1 + (m-1)\kappa \right) \quad (1.3)$$

with equality holds if and only if M is isometric to an n -dimensional euclidean unit semi-sphere of radius of $1/\sqrt{\kappa}$ and f is constant.

Theorem 1.2 Under the assumption of Theorem 1.2, let Λ_1 be the first nonzero eigenvalue of the following buckling problem

$$\Delta_f^2 \phi = -\Lambda_1 \Delta_f \phi \text{ in } M, \quad \phi = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial M. \quad (1.4)$$

Then we have

$$\Lambda_1 \geq \frac{1}{m} \lambda_1 + (m-1)\kappa \quad (1.5)$$

with equality holds if and only if M is isometric to an n -dimensional Euclidean unit semi-sphere of radius of $1/\sqrt{\kappa}$ and f is constant.

Remark 1.3 Compared with Theorems 1.7 and 1.8 in [6], we don't need to assume that the weighted mean curvature of ∂M is bounded in Theorems 1.1 and 1.2.

The study of Steklov eigenvalue problem was started by Steklov [20], his motivation came from physics. From then on, for the Steklov eigenvalue problem, many interesting results were obtained in [2, 3, 15, 16, 22, 23]. Recently, some results for the Steklov eigenvalues of the drifting Laplacian were given in [4, 12]. In this paper, we consider the following two fourth order Steklov eigenvalue problems for the drifting Laplacian

$$\Delta_f^2 u = 0 \text{ in } M, \quad u = \Delta_f u - p \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M \quad (1.6)$$

and

$$\Delta_f^2 u = 0 \text{ in } M, \quad u = \frac{\partial^2 u}{\partial \nu^2} - q \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M, \quad (1.7)$$

where ν denotes the outward unit normal vector field of ∂M . Then, we obtain

Theorem 1.4 Let M be an $n(\geq 2)$ -dimensional compact connected smooth metric measure space with boundary ∂M and non-negative m -weighted Ricci curvature. Let p_1, q_1 be the first nonzero eigenvalue of problems (1.6) and (1.7), respectively. Then we have

$$q_1 \geq \frac{p_1}{m}, \quad (1.8)$$

the equality holds in (1.8) if and only if M is isometric to an Euclidean ball and f is constant.

Theorem 1.5 Let M be an $n(\geq 2)$ -dimensional compact connected smooth metric measure space with boundary ∂M . Assume that m -weighted Ricci curvature is bounded below by $-(m-1)\kappa$ for some nonnegative constant κ , and the weighted mean curvature of M is bounded below by $\frac{(m-1)c}{n-1}$ with a positive constant c . Let λ_1 be the first Dirichlet eigenvalue of the drifting Laplacian, and let p_1 is the first nonzero eigenvalue of the Steklov problem (1.6), then we have

$$p_1 \geq \frac{mc\lambda_1}{\lambda_1 + m\kappa}, \quad (1.9)$$

the equality holds in (1.9) if and only if M is isometric to an Euclidean ball of radius $1/c$ and f is constant.

2 Proofs of Theorem 1.1–Theorem 1.2

In this section, we give the proofs of Theorem 1.1–Theorem 1.2 listed in Section 1. Before proving these results, we first recall some notations. Let M be an n -dimensional compact manifold with boundary. We often write $\langle \cdot, \cdot \rangle$ the Riemannian metric on M as well as that induced on ∂M . Let ∇ and Δ_f be the connection and the drifting Laplacian on M , respectively. Let ν be the unit outward normal vector of ∂M . The shape operator of ∂M is given by $S(X) = \nabla_X \nu$ and the second fundamental form of ∂M is defined as $II(X, Y) = \langle S(X), Y \rangle$, here $X, Y \in T\partial M$. The eigenvalues of S are called the principal curvatures of ∂M and the mean curvature H of ∂M is given by $H = \frac{1}{n-1} \text{tr} S$, here $\text{tr} S$ denotes the trace of S . We can now state Reilly-type formula (see [17, Theorem 1]). For a smooth function g defined on M , the following identity holds if $h = \frac{\partial g}{\partial \nu} \Big|_{\partial M}$, $z = g \Big|_{\partial M}$,

$$\begin{aligned} & \int_M ((\Delta_f g)^2 - |\nabla^2 g|^2 - \text{Ric}^f(\nabla g, \nabla g)) d\mu \\ &= \int_{\partial M} ((n-1)H^f h + 2\overline{\Delta}_f z) h d\mu + \int_{\partial M} \left(II(\overline{\nabla} z, \overline{\nabla} z) - \left\langle \overline{\nabla} z, \overline{\nabla} \frac{\partial z}{\partial \nu} \right\rangle \right) d\mu, \end{aligned} \quad (2.1)$$

here $H^f = H + \frac{1}{n-1} \frac{\partial f}{\partial \nu}$ denotes f -mean curvature [21] (also called weighted mean curvature), $\nabla^2 g$ is the Hessian of g ; $\overline{\Delta}$, $\overline{\Delta}_f$ and $\overline{\nabla}$ represent the Laplacian, drifting Laplacian and the gradient on ∂M with respect to the induced metric on ∂M , respectively. Using the inequalities of $|\nabla^2 g|^2 \geq \frac{1}{n}(\Delta g)^2$ and $\frac{1}{n}(a+b)^2 \geq \frac{1}{m}a^2 - \frac{1}{m-n}b^2$, then from $\Delta_f g = \Delta g - \langle \nabla f, \nabla g \rangle$, we can get

$$|\nabla^2 g|^2 \geq \frac{1}{n}(\Delta g)^2 = \frac{1}{n}(\Delta_f g + \langle \nabla f, \nabla g \rangle)^2 \geq \frac{(\Delta_f g)^2}{m} - \frac{\langle \nabla f, \nabla g \rangle^2}{m-n}. \quad (2.2)$$

Substituting (2.2) into (2.1), and it then follows from the definition of the m -Bakry-Émery Ricci tensor that

$$\begin{aligned} & \int_M \left(\frac{m-1}{m} (\Delta_f g)^2 - \text{Ric}^m(\nabla g, \nabla g) \right) d\mu \\ & \geq \int_{\partial M} ((n-1)H^\phi h + 2\overline{\Delta}_f z) h d\mu + \int_{\partial M} \left(II(\overline{\nabla} z, \overline{\nabla} z) - \left\langle \overline{\nabla} z, \overline{\nabla} \frac{\partial z}{\partial \nu} \right\rangle \right) d\mu. \end{aligned} \quad (2.3)$$

Remark 2.6 (i) When $m = n$, we know that f is a constant, then $\Delta_f = \Delta$, (2.1) becomes classic Reilly formula [19], the equality holds in (2.3) if and only if $\nabla^2 g = \frac{\Delta g}{n} \langle \cdot, \cdot \rangle$.

(ii) When $m > n$, the equality holds in (2.3) if and only if $\nabla^2 g = \frac{\Delta g}{n} \langle \cdot, \cdot \rangle$ and $\Delta_f g + \frac{m}{m-n} \langle \nabla f, \nabla g \rangle = 0$. Using inequality (2.3), we give the proofs of Theorem 1.1–Theorem 1.2.

Proof of Theorem 1.1 Let g be an eigenfunction of problem (1.2) corresponding to the first eigenvalue Γ_1 . That is

$$\Delta_f^2 g = \Gamma_1 g \text{ in } M, \quad g = \frac{\partial^2 g}{\partial \nu^2} = 0 \text{ on } \partial M. \quad (2.4)$$

Multiplying (2.4) by g and integrating on M , we infer from the divergence theorem that

$$\Gamma_1 \int_M g^2 d\mu = \int_M g \Delta_f^2 g d\mu = - \int_M \langle \nabla g, \nabla(\Delta_f g) \rangle d\mu = \int_M (\Delta_f g)^2 d\mu - \int_{\partial M} h \Delta_f g d\mu, \quad (2.5)$$

where $h = \frac{\partial g}{\partial \nu} \Big|_{\partial M}$. On the boundary ∂M , we can write

$$\nabla g = (\nabla g)^\top + (\nabla g)^\perp = (\nabla g)^\top + \langle \nabla g, \nu \rangle \nu,$$

where $(\nabla g)^\top$ is tangent to ∂M and $(\nabla g)^\perp$ is normal to ∂M . Then from $g \Big|_{\partial M} = \frac{\partial^2 g}{\partial \nu^2} \Big|_{\partial M} = 0$, we have

$$\begin{aligned} \langle \nabla f, \nabla g \rangle \Big|_{\partial M} &= \langle (\nabla f)^\top + \langle \nabla f, \nu \rangle \nu, (\nabla g)^\top + \langle \nabla g, \nu \rangle \nu \rangle \Big|_{\partial M} \\ &= \langle (\nabla f)^\top, (\nabla g)^\top \rangle \Big|_{\partial M} + \frac{\partial f}{\partial \nu} h = \frac{\partial f}{\partial \nu} h \end{aligned}$$

and

$$\Delta g \Big|_{\partial M} = (n-1)Hh.$$

So, we infer from above two equalities that

$$\Delta_f g \Big|_{\partial M} = (\Delta g - \langle \nabla f, \nabla g \rangle) \Big|_{\partial M} = (n-1)Hh - \frac{\partial f}{\partial \nu} h = (n-1)H^f h. \quad (2.6)$$

Combining (2.5) and (2.6), we have

$$\Gamma_1 = \frac{\int_M (\Delta_f g)^2 d\mu - \int_{\partial M} (n-1)H^f h^2 d\mu}{\int_M g^2 d\mu}. \quad (2.7)$$

Taking g into (2.3), we have

$$\begin{aligned} \int_M \frac{m-1}{m} (\Delta_f g)^2 d\mu &\geq \int_M \text{Ric}^m(\nabla g, \nabla g) d\mu + \int_{\partial M} (n-1)H^f h^2 d\mu \\ &\geq (m-1)\kappa \int_M |\nabla g|^2 d\mu + \int_{\partial M} (n-1)H^f h^2 d\mu, \end{aligned}$$

which implies

$$- \int_{\partial M} (n-1)H^f h^2 d\mu \geq - \int_M \frac{m-1}{m} (\Delta_f g)^2 d\mu + (m-1)\kappa \int_M |\nabla g|^2 d\mu. \quad (2.8)$$

Combining (2.7) and (2.8), we have

$$\Gamma_1 \geq \frac{\frac{1}{m} \int_M (\Delta_f g)^2 d\mu + (m-1)\kappa \int_M |\nabla g|^2 d\mu}{\int_M g^2 d\mu}. \quad (2.9)$$

For any nonzero function which vanishes on ∂M , it is well known from the Poincaré inequality and the Schwarz inequality that

$$\lambda_1 \int_M g^2 d\mu \leq - \int_M g \Delta_f g d\mu \leq \left\{ \int_M g^2 d\mu \int_M (\Delta_f g)^2 d\mu \right\}^{\frac{1}{2}},$$

which implies that

$$\lambda_1^2 \int_M g^2 d\mu \leq \lambda_1 \int_M |\nabla g|^2 d\mu \leq \int_M (\Delta_f g)^2 d\mu. \quad (2.10)$$

The equality holds in above inequality if and only if g is the a first Dirichlet eigenfunction of the drifting Laplacian of M .

Combining (2.9) and (2.10), we have

$$\Lambda_1 \geq \lambda_1 \left(\frac{1}{m} \lambda_1 + (m-1)\kappa \right). \quad (2.11)$$

If $\Lambda_1 = \lambda_1 \left(\frac{1}{m} \lambda_1 + (m-1)\kappa \right)$, we know that equality holds in (2.10), which implies that g is the a first Dirichlet eigenfunction of the drifting Laplacian of M , then we have

$$\Delta_f g = -\lambda_1 g \quad \text{and} \quad \Delta_f^2 g = \lambda_1^2 g \quad \text{in } M. \quad (2.12)$$

Then, we infer from $\Delta_f^2 g = \Gamma_1 g$ in M that $\lambda_1 = m\kappa > 0$.

When $m = n$, then f is a constant and $\Delta_f = \Delta$, from [24, Theorem 1.1], we know that $\Gamma_1 \geq \lambda_1 \left(\frac{1}{n} \lambda_1 + (n-1)\kappa \right)$ with equality holding if and only if M is isometric to an n -dimensional Euclidean unit semi-sphere.

When $m > n$, if $\Gamma_1 = \lambda_1 \left(\frac{1}{m} \lambda_1 + (m-1)\kappa \right)$, we know that equality holds in (2.3), which means

$$0 = \Delta_f g + \frac{m}{m-n} \langle \nabla f, \nabla g \rangle = \Delta g + \frac{n}{m-n} \langle \nabla f, \nabla g \rangle$$

holds everywhere on M . Multiplying in above inequality with g and integrating on M with respect to $e^{\frac{n}{m-n}f} d\nu$ give that

$$0 = \int_M g \left(\Delta g + \frac{n}{m-n} \langle \nabla f, \nabla g \rangle \right) e^{\frac{n}{m-n}f} d\nu = - \int_M |\nabla g|^2 e^{\frac{n}{m-n}f} d\nu. \quad (2.13)$$

From above equality, we know that g is a constant function on M , which is a contradiction since g is the first eigenfunction of drifting Laplacian and cannot be a constant. Therefore, we know that $\Lambda_1 \geq \lambda_1 \left(\frac{1}{m} \lambda_1 + (m-1)\kappa \right)$ with equality holding if and only if M is isometric to an n -dimensional Euclidean unit semi-sphere and f is a constant. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 Let ϕ be an eigenfunction of problem (1.4) corresponding to the first eigenvalue Λ_1 . That is

$$\Delta_f^2 \phi = -\Lambda_1 \Delta_f \phi \quad \text{in } M, \quad \phi = \frac{\partial^2 \phi}{\partial \nu^2} = 0 \quad \text{on } \partial M. \quad (2.14)$$

Multiplying (2.14) by ϕ and integrating on M , we infer from the divergence theorem that

$$\Lambda_1 \int_M |\nabla \phi|^2 d\mu = \int_M (\Delta_f \phi)^2 d\mu - \int_{\partial M} h \Delta_f \phi d\mu, \quad (2.15)$$

where $\psi = \frac{\partial \phi}{\partial \nu} \Big|_{\partial M}$. Since $\phi \Big|_{\partial M} = \frac{\partial^2 \phi}{\partial \nu^2} \Big|_{\partial M} = 0$, we infer from (2.7) that

$$\Delta_f \phi \Big|_{\partial M} = (\Delta \phi - \langle \nabla f, \nabla \phi \rangle) \Big|_{\partial M} = (n-1)H\psi - \frac{\partial f}{\partial \nu} \psi = (n-1)H^f \psi. \quad (2.16)$$

Combining (2.15) and (2.16), we have

$$\Lambda_1 = \frac{\int_M (\Delta_f \phi)^2 d\mu - \int_{\partial M} (n-1)H^f \psi^2 d\mu}{\int_M |\nabla \phi|^2 d\mu}. \quad (2.17)$$

Taking ϕ into (2.3), we have

$$-\int_{\partial M} (n-1)H^f \psi^2 d\mu \geq -\int_M \frac{m-1}{m} (\Delta_f \phi)^2 d\mu + (m-1)\kappa \int_M |\nabla \phi|^2 d\mu. \quad (2.18)$$

Combining (2.17)–(2.18), we have

$$\Lambda_1 \geq \frac{\frac{1}{m} \int_M (\Delta_f \phi)^2 d\mu + (m-1)\kappa \int_M |\nabla \phi|^2 d\mu}{\int_M |\nabla \phi|^2 d\mu}. \quad (2.19)$$

From the Poincaré inequality, we have

$$\lambda_1 \int_M |\nabla \phi|^2 d\mu \leq \int_M (\Delta_f \phi)^2 d\mu. \quad (2.20)$$

Taking above inequality into (2.19), we have $\Lambda_1 \geq \frac{1}{m}\lambda_1 + (m-1)\kappa$, the equality holds in above inequality if and only if ϕ is the first Dirichlet eigenfunction of the drifting Laplacian of M .

When $m = n$, then f is a constant and $\Delta_f = \Delta$, from [24, Theorem 1.2], we know that $\Lambda_1 \geq \frac{1}{n}\lambda_1 + (n-1)\kappa$ with equality holding if and only if M is isometric to an n -dimensional Euclidean unit semi-sphere.

When $m > n$, if $\Lambda_1 = \frac{1}{m}\lambda_1 + (m-1)\kappa$, we know that equality holds in (2.3), then by similar discussion in the proof of Theorem 1.1, we know that u is a constant function on M , which is a contradiction since ϕ is the first eigenfunction of drifting Laplacian and cannot be a constant. Therefore, we have $\Lambda_1 \geq \frac{1}{m}\lambda_1 + (m-1)\kappa$. This completes the proof of Theorem 1.2.

3 Proofs of Theorem 1.4–Theorem 1.5

In this section, we give the proof of Theorem 1.4–Theorem 1.5 listed in Section 1.

Proof of Theorem 1.4 Let w be an eigenfunction corresponding to first eigenvalue q_1 of problem (1.7), that is

$$\Delta_f^2 w = 0 \text{ in } M, \quad w = \frac{\partial^2 w}{\partial \nu^2} - q_1 \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial M. \quad (3.1)$$

We know that w is not a constant since $w \Big|_{\partial M} = 0$. Set $\eta = \frac{\partial w}{\partial \nu} \Big|_{\partial M}$, then $\eta \neq 0$. Otherwise, if $\eta = 0$, we have

$$w = \nabla w = \frac{\partial^2 w}{\partial \nu^2} = 0 \text{ on } \partial M,$$

combining $\Delta_f^2 w = 0$ in M , we can deduce from the maximum principle that $\Delta_f w = 0$ on M , which in turn implies that $w = 0$. This is a contradiction.

Since $w|_{\partial M} = 0$, we can infer from the divergence theorem that

$$\int_M \langle \nabla w, \nabla(\Delta_f w) \rangle d\mu = - \int_M w \Delta_f^2 w d\mu = 0,$$

which implies that

$$\int_{\partial M} \Delta_f w \frac{\partial w}{\partial \nu} d\mu = \int_M \langle \nabla w, \nabla(\Delta_f w) \rangle d\mu + \int_M (\Delta_f w)^2 d\mu = \int_M (\Delta_f w)^2 d\mu. \quad (3.2)$$

On the other hand, since

$$\begin{aligned} \Delta_f w|_{\partial M} &= (\Delta w - \langle \nabla f, \nabla w \rangle)|_{\partial M} = \frac{\partial^2 w}{\partial \nu^2} + (n-1)H \frac{\partial w}{\partial \nu} - \frac{\partial f}{\partial \nu} \frac{\partial w}{\partial \nu} \\ &= \frac{\partial^2 w}{\partial \nu^2} + (n-1)H^f \frac{\partial w}{\partial \nu} \\ &= q_1 \frac{\partial w}{\partial \nu} + (n-1)H^f \frac{\partial w}{\partial \nu}, \end{aligned} \quad (3.3)$$

we can get

$$q_1 = \frac{\int_M (\Delta_f w)^2 d\mu - (n-1) \int_{\partial M} H^f \eta^2 d\mu}{\int_{\partial M} \eta^2 d\mu}. \quad (3.4)$$

Substituting w into (2.3) and noticing $w|_{\partial M} = 0$, $\text{Ric}^m(\nabla w, \nabla w) \geq 0$, then we have

$$\begin{aligned} \int_M \frac{m-1}{m} (\Delta_f w)^2 d\mu &\geq \int_M \text{Ric}^m(\nabla w, \nabla w) d\mu + \int_{\partial M} (n-1)H^f \eta^2 d\mu \\ &\geq \int_{\partial M} (n-1)H^f \eta^2 d\mu. \end{aligned} \quad (3.5)$$

Combining (3.4) and (3.5), we have

$$q_1 \geq \frac{1}{m} \frac{\int_M (\Delta_f w)^2 d\mu}{\int_{\partial M} \eta^2 d\mu}. \quad (3.6)$$

Since p_1 is given by

$$p_1 = \min_{w|_{\partial M}=0, w \neq \text{const.}} \frac{\int_M (\Delta_f w)^2 d\mu}{\int_{\partial M} \left(\frac{\partial w}{\partial \nu}\right)^2 d\mu}, \quad (3.7)$$

we have from the variational characterization that

$$\frac{\int_M (\Delta_f w)^2 d\mu}{\int_{\partial M} \eta^2 d\mu} \geq p_1. \quad (3.8)$$

Combining (3.6) and (3.8), we have $q_1 \geq \frac{1}{m} p_1$.

When $m = n$, then f is constant, so $\Delta_f = \Delta$, then by Theorem 1.3 in [24], we know that equality holds in (1.8) if and only if M is isometric to an Euclidean ball.

When $m > n$, if the equality holds in (1.8), the equality holds in (2.3), then $\Delta_f w = -\frac{m}{m-n} \langle \nabla f, \nabla w \rangle$, multiplying with f and integrating on M with respect to $e^{\frac{n}{m-n}f} d\nu$, we can get

$$\int_M w \Delta_f w e^{\frac{n}{m-n}f} d\nu = \int_M -w \frac{m}{m-n} \langle \nabla f, \nabla w \rangle e^{\frac{n}{m-n}f} d\nu,$$

which implies $\int_M |\nabla w|^2 e^{\frac{n}{m-n}f} d\nu = 0$. So w is constant, which is a contradiction. Thus, equality holds in (1.8) if and only if M is isometric to an Euclidean ball of radius $\frac{1}{c}$ and f is constant. This completes the proof of Theorem 1.3 .

Proof of Theorem 1.5 Let φ be an eigenfunction corresponding to first eigenvalue p_1 of problem (1.6), that is

$$\Delta_f^2 \varphi = 0 \text{ in } M, \quad \varphi = \Delta_f \varphi - q_1 \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial M. \quad (3.9)$$

Set $\psi = \frac{\partial \varphi}{\partial \nu} |_{\partial M}$, then

$$q_1 = \frac{\int_M (\Delta_f \varphi)^2 d\mu}{\int_{\partial M} \psi^2 d\mu}. \quad (3.10)$$

Substituting φ into (2.3) and noticing $\psi = 0$, $\text{Ric}^m(\nabla \varphi, \nabla \varphi) \geq -(m-1)\kappa$ and $(n-1)H^\varphi \geq (m-1)c$, then we infer from that $-\int_M |\nabla \varphi|^2 d\mu \geq \frac{1}{\lambda_1} \int_M (\Delta_f \varphi)^2 d\mu$ that

$$\begin{aligned} \int_M \frac{m-1}{m} (\Delta_f \varphi)^2 d\mu &\geq \int_M \text{Ric}^m(\nabla \varphi, \nabla \varphi) d\mu + \int_{\partial M} (n-1)H^\varphi \psi^2 d\mu \\ &\geq -(m-1)\kappa \int_M |\nabla \varphi|^2 d\mu + (m-1)c \int_{\partial M} \psi^2 d\mu \\ &\geq -\frac{(m-1)\kappa}{\lambda_1} \int_M (\Delta_f \varphi)^2 d\mu + (m-1)c \int_{\partial M} \psi^2 d\mu. \end{aligned} \quad (3.11)$$

Combining (3.10) and (3.11), we have $q_1 \geq \frac{mc\lambda_1}{\lambda_1 + m\kappa}$.

When $m = n$, then f is constant, so $\Delta_f = \Delta$, then by Theorem 1.4 in [24], we know that equality holds in (1.8) if and only if M is isometric to an Euclidean ball.

When $m > n$, if the equality holds in (1.8), the equality holds in (2.3), then by the similar discussion, we know that φ is constant, which is a contradiction. Thus, the equality holds in (1.8) if and only if M is isometric to an Euclidean ball of radius $1/c$ and f is constant.

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双漂移拉普拉斯特征值的最优估计

李艳丽^{1,2}, 杜 锋^{2,3}

(1.荆楚理工学院电子信息工程学院, 湖北 荆门 448000)

(2.湖北大学数学与统计学学院, 湖北 武汉 430062)

(3.荆楚理工学院数理学院, 湖北 荆门 448000)

摘要: 本文研究了四类双漂移拉普拉斯算子的特征值问题. 利用带权Reilly公式, 当 m -权重Ricci曲率满足一定条件时, 得到了紧致带边光滑度量测度空间上四类双漂移拉普拉斯算子的第一非零特征值的最优估计. 推广了双调和算子特征值的相应结果.

关键词: 特征值; 漂移拉普拉斯; 光滑度量测度空间; m -权重Ricci曲率; Steklov 问题.

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