Vol. 40 (2020)
No. 1

数 学 杂 志
J. of Math. (PRC)

A CLASS OF NON-MATCHABLE DISTRIBUTIVE LATTICES

WANG Xu, ZHAO Xu-xu, YAO Hai-yuan
(*College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*)

Abstract: In this paper, we consider non-matchable distributive lattices. By introducing the meet-irreducible cell with respect to a perfect matching of a plane elementary bipartite graph and giving its characterizations, we obtain a new class of non-matchable distributive lattices, and extend a result on non-matchable distributive lattices with a cut element.

Keywords: plane (weakly) elementary bipartite graph; Z-transformation directed graph; meet-irreducible cell; non-matchable distributive lattice; planarity

2010 MR Subject Classification: 05C70; 06D50

Document code: A **Article ID:** 0255-7797(2020)01-0029-07

1 Introduction

Zhang et al. [1] introduced a concept of Z-transformation graph (called by some authors resonance graph) on the set of perfect matchings (or 1-factors) of a hexagonal system, then Zhang and Zhang [2] extended the concept to a general plane bipartite graph with a perfect matching, and Zhang [3] surveyed rich theoretical results in several directions. Let G be a plane bipartite graph with a perfect matching. Denote by $\mathcal{M}(G)$ the set of all perfect matchings of G. The Z-transformation directed graph $\vec{Z}(G)$ is an orientation of Z-transformation graph [4].

Lam and Zhang [5] proved that $\mathcal{M}(G)$ equipped with a partial order is a finite distributive lattice and its Hasse diagram is isomorphic to $\vec{Z}(G)$. Recently, Zhang et al. [6] introduced the concept of matchable distributive lattice and got some consequences on matchable distributive lattices, Yao and Zhang [7] obtained some results on non-matchable distributive lattices with a cut element.

In a finite lattice, an element is meet-irreducible if it is covered by exactly one element; from a graphical point of view if and only if it is a vertex of indegree 1 in Hasse diagram. Consider an arc f with its tail M in $\vec{Z}(G)$. We introduce the concept of meet-irreducible cell with respect to M. Furthermore, we have some equivalent characterizations of the concept (i.e., Theorem 3.2). Finally, by Theorem 3.2, we extend a result on non-matchable

---

[*] **Received date:** 2018-09-03 **Accepted date:** 2019-03-05

Foundation item: Supported by National Natural Science Foundation of China (11761064).

Biography: Wang Xu (1988–), male, born at Longnan, Gansu, postgraduate, major in graph theory with applications. E-mail: wangxuac@163.com.

Corresponding author: Yao Haiyuan

distributive lattices obtained by Yao and Zhang [7] (i.e., Theorem 2.3), and obtain a class of non-matchable distributive lattices by Kuratowski's theorem.

2 Preliminaries

A set P equipped with a binary relation $\leq$ satisfying reflexivity, antisymmetry and transitivity is said to be a partially ordered set (poset for short). Given any poset P, the dual P^* of P by defining $x \leq y$ to hold in P^* if and only if $y \leq x$ holds in P. A poset P is a chain if any two elements of P are comparable, and we write $\mathbf{n}$ to denote the chain obtained by giving $\{0, 1, \cdots, n-1\}$ the order in which $0 < 1 < \cdots < n-1$. The subset S of the poset P is called convex if $a, b \in S$, $c \in P$, and $a \leq c \leq b$ imply that $c \in S$. A lattice is nontrivial if it has at least two elements and a finite distributive lattice is irreducible if it cannot be decomposed into a direct product of two nontrivial finite distributive lattices.

The symmetric difference of two finite sets A and B is defined as $A \oplus B := (A \cup B) \setminus (A \cap B)$. If M is a perfect matching of a graph and C is an M-alternating cycle of the graph, then the symmetric difference of M and edge-set $E(C)$ is another perfect matching of the graph, which is simply denoted by $M \oplus C$. Let G be a plane bipartite graph with a matching M, and the vertices of G be colored properly black and white such that the two ends of every edge receive different colors. An M-alternating cycle of G is said to be proper, if every edge of the cycle belonging to M goes from white end-vertex to black end-vertex by the clockwise orientation of the cycle; otherwise improper [8].

For some concepts and notations not explained in the paper, refer to [9, 10] for poset and lattice, [11, 12] for graph theory.

An inner face of a graph is called a cell if its boundary is a cycle, and we will say that the cycle is a cell too. Observe that an M-alternating cell intersecting an improper M-alternating cell must be proper, vice versa. Obviously, we have the following result.

Lemma 2.1 (see [13]) If G is a plane bipartite graph with a matching M, then any two proper (resp. improper) M-alternating cells are disjoint.

Definition 2.1 (see [2]) Let G be a plane bipartite graph. The Z-transformation graph $Z(G)$ is defined on $\mathcal{M}(G)$: $M_1, M_2 \in \mathcal{M}(G)$ are joined by an edge if and only if $M_1 \oplus M_2$ is a cell of G. And Z-transformation digraph $\vec{Z}(G)$ is the orientation of $Z(G)$: an edge M_1M_2 of $Z(G)$ is oriented from M_1 to M_2 if $M_1 \oplus M_2$ form a proper M_1-alternating (thus improper M_2-alternating) cell.

An edge of graph G is allowed if it lies in a perfect matching of G. A graph G is said to be elementary if its allowed edges form a connected subgraph of G. Let G be a bipartite graph. A subgraph H of G is said to be nice if $G - V(H)$ has a perfect matching [14]; from Theorem 4.1.1 in [14], we have that a bipartite graph is elementary if and only if it is connected and every edge of it is allowed.

Definition 2.2 (see [2]) A bipartite graph G is weakly elementary if the subgraph of G consisting of C together with its interior is elementary for every nice cycle C of G.

Let G be a plane bipartite graph with a perfect matching. A binary relation $\leq$ on $\mathcal{M}(G)$

is defined as: for $M_1, M_2 \in \mathcal{M}(G)$, $M_1 \leq M_2$ if and only if $\vec{Z}(G)$ has a directed path from M_2 to M_1 [2]. It is shown that $(\mathcal{M}(G); \leq)$ is a poset [5]. For convenience, we write $\mathcal{M}(G)$ for poset $(\mathcal{M}(G), \leq)$.

Theorem 2.2 (see [5]) If G is a plane (weakly) elementary bipartite graph, then $\mathcal{M}(G)$ is a finite distributive lattice and its Hasse diagram is isomorphic to $\vec{Z}(G)$.

Definition 2.3 (see [6]) A finite distributive lattice L is matchable if there is a plane weakly elementary bipartite graph G such that $L \cong \mathcal{M}(G)$; otherwise non-matchable.

Yao and Zhang [7] obtained the following result on non-matchable distributive lattices with a cut element.

Theorem 2.3 (see [7]) Let L be a finite distributive lattice, and L have cut element covered by m elements and covering n elements. If $m \geq 3$ and $n \geq 3$, then L is non-matchable.

3 Meet-Irreducible Cell

The proof of Lemma 3.7 in [15] implies the following proposition.

Proposition 3.1 If G is a plane elementary bipartite graph with a perfect matching M, then there exists a hypercube in $\vec{Z}(G)$ generated by some pairwise disjoint M-alternating cells. In particular, M is the maximum (resp. minimum) element of the corresponding Boolean lattice in $\mathcal{M}(G)$ if these M-alternating cells are proper (resp. improper).

It is obvious that the dimension of the hypercube is equal to the number of these pairwise disjoint M-alternating cells. In particular, the hypercube is a quadrilateral if and only if it is generated by exactly two disjoint M-alternating cells in G [13, 15, 16].

Definition 3.1 Let G be a plane (weakly) elementary bipartite graph with a perfect matching M. A proper M-alternating cell f is meet-irreducible with respect to M if $M \oplus f$ is meet-irreducible in $\mathcal{M}(G)$.

In fact, by the definition of meet-irreducible element, it follows that $M \oplus f$ is meet-irreducible in $\mathcal{M}(G)$ whenever f is a meet-irreducible cell with respect to M. The equivalent characterizations of meet-irreducible elements is given as follows, the thought of which is analogous to that in [17]. A part of Theorem 3.2 is published, however, to make the paper self-contained, we would rather include the proof here than refer to [18].

Theorem 3.2 Let G be a plane (weakly) elementary bipartite graph G with a perfect matching M and let f be a proper M-alternating cell.

(1) If G has no improper M-alternating cell (namely, M is the maximum element of $\mathcal{M}(G)$), then every (proper) M-alternating cell is a meet-irreducible cell with respect to M.

(2) If G has some improper M-alternating cells, then the following are equivalent:

(a) the cell f is a meet-irreducible cell with respect to M;

(b) the cell f intersects every improper M-alternating cell;

(c) there is no perfect matching M' in $V(\mathcal{Q}) \backslash \{M\}$ such that f is a proper M'-alternating cell, where $\mathcal{Q}$ is a corresponding Boolean lattice generated by all improper M-alternating cells.

Proof (1) It is trivial by the definition of Z-transformation directed graph.

(2) First suppose that the cell f is a meet-irreducible cell with respect to M, but there is at least one improper M-alternating cell f' such that f and f' are disjoint. Thus $M \oplus f = ((M \oplus f') \oplus f) \oplus f'$, i.e., G has two improper $M \oplus f$-alternating cells, hence $M \oplus f$ is not meet-irreducible, contradicting the supposition that f is a meet-irreducible cell with respect to M.

Next suppose that the cell f intersects every improper M-alternating cell, but there is a perfect matching M' in $V(\mathcal{Q}) \setminus \{M\}$ such that f is a proper M'-alternating cell. In fact, by Proposition 3.1, there is at least one improper M-alternating cell f' that is a proper M'-alternating cell. Hence f and f' are disjoint by Lemma 2.1, a contradiction.

Finally, suppose that there is no perfect matching M' in $V(\mathcal{Q}) \setminus \{M\}$ such that f is a proper M'-alternating cell, but f is not a meet-irreducible cell with respect to M. Thus G has at least one improper $M \oplus f$-alternating cell f' except f, by Lemma 2.1, hence f and f' are disjoint. Therefore f' is an improper M-alternating cell, which implies that f is a proper $M \oplus f'$-alternating cell, i.e., there is a perfect matching $M' = M \oplus f'$ in $V(\mathcal{Q}) \setminus \{M\}$ such that f is a proper M'-alternating cell, a contradiction.

Assume that every proper M-alternating cell is a meet-irreducible cell with respect to M. If G has an improper M-alternating cell, then every proper M-alternating cell intersects every improper M-alternating cell, hence M is a cut element [13]. And M is the maximum element of $\mathcal{M}(G)$ otherwise. Moreover we obtain the following consequence of Theorem 3.2.

Corollary 3.3 (see [4, 13]) If G is a plane elementary bipartite graph with a perfect matching M, and has both proper and improper M-alternating cells, then M is a cut vertex of $Z(G)$ if and only if every proper M-alternating cell intersects every improper M-alternating cell, i.e., every proper M-alternating cell is a meet-irreducible cell with respect to M.

Note that duality of lattice, meet-irreducible cell, Theorem 3.2 and Corollary 3.3 could be treated in dual.

4 Non-Matchable Distributive Lattices

Subdividing an edge e is to delete e, add a new vertex v, and join v to the ends of e. Any graph derived from a graph G by a sequence of edge subdivisions is called a subdivision of G.

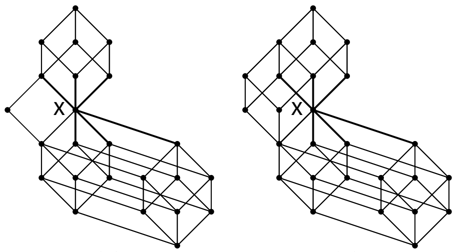
Given a plane graph G, its (geometric) dual G^* is constructed as follows: place a vertex in each face of G (including the exterior face) and, if two faces have an edge e in common, join the corresponding vertices by an edge e^* crossing only e [12]. It is easy to see that the dual G^* of a plane graph G is itself a plane graph [11].

Theorem 4.1 (Kuratowski's theorem) A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.

Similar to the proof of Lemma 4.2 in [7] and Theorem 3.2, we have Theorem 4.2.

Theorem 4.2 Let L be a finite distributive lattice and $x \in L$. If x is covered by at least three elements and covers at least three meet-irreducible elements, then L is non-matchable.

Proof Suppose to the contrary that L is matchable. Then there is a plane (weakly) elementary bipartite graph G such that $\mathcal{M}(G) \cong L$ [6], which implies that a perfect matching M_x of G corresponds to $x \in L$. According to the premise, G has at least three improper M_x-alternating cells, and at least three proper M_x-alternating cells f_1, f_2 and f_3 that are meet-irreducible. By Theorem 3.2, such three proper M_x-alternating cells intersect all improper M_x-alternating cells. This shows that the dual G^* of G contains a $K_{3,3}$ as subgraph. But it is impossible by Kuratowski's theorem.



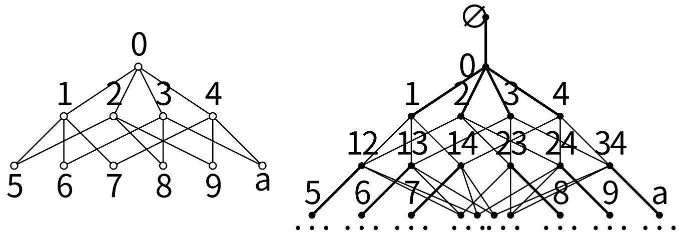
(a) (b)

Figure 1: two non-matchable distributive lattices

If x is a cut element, Theorem 4.2 implies Theorem 2.3 (i.e., Theorem 4.3 in [7]); on the other hand, Theorem 4.2 can determine some non-matchable distributive lattice that can not be determined only by Theorem 2.3. For instance, it is easy to see that each distributive lattice in Figure 1 is non-matchable by Theorem 4.2, but not determined only by Theorem 2.3.

Obviously, a dual version of Theorem 4.2 could be obtained easily.

Corollary 4.3 If L is a matchable distributive lattice, then for every element of L, it either is covered by at most two elements or covers at most two meet-irreducible elements in both L and L^*.



(a) (b)

Figure 2: (a) the poset Δ and (b) a part of $\mathcal{F}(\Delta)$

Let P be a poset and $F \subseteq P$. The subposet F is a filter if, whenever $x \in F$, $y \in P$ and $x \leq y$, we have $y \in F$ [9]. The set of all filters of a poset P is denoted by $\mathcal{F}(P)$, and carries the usual anti-inclusion order; and the filter lattice $\mathcal{F}(P)$ is a distributive lattice [9, 10]. Figure 2 (a) shows the Hasse diagram of a poset Δ; and Figure 2 (b) is the highest five layers of Hasse diagram of $\mathcal{F}(\Delta)$, where every filter is labeled by its minimal element(s).

Combining Theorem 3.2 and Kuratowski's theorem, we have another class of non-matchable distributive lattices.

Theorem 4.4 The distributive lattice $\mathcal{F}(\Delta)$ is non-matchable, where Δ is the poset as shown in Figure 2(a).

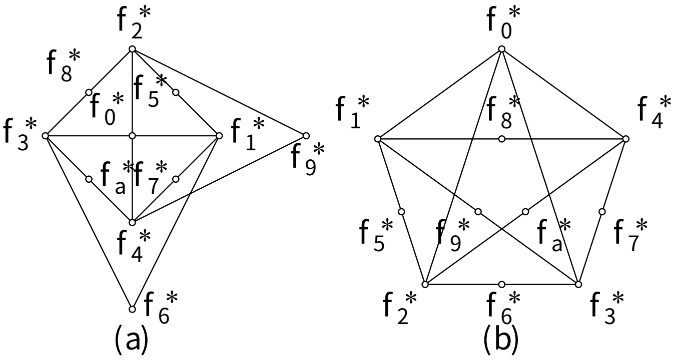


Figure 3: proof of Theorem 4.4

Proof Recall that $\mathcal{F}(\Delta)$ is a finite distributive lattice. Suppose that $\mathcal{F}(\Delta)$ is matchable. Since $\mathcal{F}(\Delta)$ has a cut element labeled by 0 (see Figure 2), and is irreducible, there exists a plane elementary bipartite graph G such that $\vec{Z}(G) \cong \mathcal{F}(\Delta)$ [6].

Consider a part of $\mathcal{F}(\Delta)$ as drawn in Figure 2 (b). The vertices $\emptyset$, 0, 1, $\cdots$, a correspond to the perfect matchings $M_\emptyset$, M_0, M_1, $\cdots$, M_a of G, respectively. Let $f_0 = M_\emptyset \oplus M_0$, $f_1 = M_0 \oplus M_1$, $f_5 = M_{12} \oplus M_5$, $f_6 = M_{13} \oplus M_6$, $\cdots$, and $f_a = M_{34} \oplus M_a$. By definition of Z-transformation graph, then f_0 is a nice cell, so are $f_1, \cdots, f_a$. Since the cells $f_0, f_1, \cdots, f_a$ are meet-irreducible cells, by Theorem 3.2 (2), the cell f_0 intersects f_1, f_2, f_3 and f_4; the cell f_5 intersects f_1 and f_2, but it does not intersect f_3 or f_4, because f_5, f_3 and f_4 are proper M_{12}-alternating cells. Thus f_0 and f_5 are distinct; analogously, f_0 and f_i $(i \in \{6, 7, 8, 9, a\})$ are distinct too.

Next, consider the dual G^* of G, as drawn in Figure 3 (a), vertex f_0^* is adjacent with f_1^*, f_2^*, f_3^* and f_4^*, and f_5^* is adjacent with f_1^* and f_2^*, etc. Therefore, let $V' = \{f_0^*, \cdots, f_a^*\}$, thus G^* contains a subgraph $S^* := G^*[V']$. Clearly S^* (see Figure 3 (b)) is a subdivision of K_5. By Kuratowski's theorem, hence S^* is non-planar, contradicting the planarity of G.

If a poset P contains Δ as a convex subposet, then there are 11 elements in P cover relations of which are identical in Δ. Similar to proof of Theorem 4.4, we prove the following result.

Corollary 4.5 Let P be a poset containing Δ as a convex subposet. Then distributive lattice $\mathcal{F}(P)$ is non-matchable. In addition, for any finite distributive lattice L, the Cartesian product, linear sum and vertical sum [9] of $\mathcal{F}(P)$ and L are non-matchable.

Note that Δ is a filter of $\mathbf{2}^4$, the following corollary is immediate.

Corollary 4.6 The distributive lattice $\mathcal{F}(\mathbf{2}^4)$ is non-matchable. In addition, the distributive lattice $\mathcal{F}\left(\prod_{j=1}^{k} \mathbf{n}_j\right)$ is non-matchable, where $k \geq 4$, $\mathbf{n}_j$ is a chain of length n_j and $n_j \geq 2$ for every $j = 1, 2, \cdots, k$.

References

[1] Zhang Fuji, Guo Xiaofeng, Chen Rongsi. Z-transformation graphs of perfect matchings of hexagonal systems [J]. Discrete Math., 1988, 72(1): 405–415.

[2] Zhang Heping, Zhang Fuji. Total Z-transformation graphs of perfect matching of plane bipartite graphs [J]. Electron Notes Discrete Math., 2000, 5: 317–320.

[3] Zhang Heping. Z-transformation graphs of perfect matchings of plane bipartite graphs: a survey [J]. Match. Commun. Math. Comput. Chem., 2006, 56(3): 457–476.

[4] Zhang Heping, Zhang Fuji. Block graphs of Z-transformation graphs of perfect matchings of plane elementary bipartite graphs [J]. Ars. Combin., 1999, 53: 309–314.

[5] Lam Che Bor P, Zhang Heping. A distributive lattice on the set of perfect matchings of a plane bipartite graph[J]. Order, 2003, 20: 13–29.

[6] Zhang Heping, Yang Dewu, Yao Haiyuan. Decomposition theorem on matchable distributive lattices [J]. Discrete Appl. Math., 2014, 166: 239–248.

[7] Yao Haiyuan, Zhang Heping. Non-matchable distributive lattices [J]. Discrete Math., 2015, 338(3): 122–132.

[8] Zhang Heping, Zhang Fuji. The rotation graphs of perfect matchings of plane bipartite graphs [J]. Discrete Appl. Math., 1997, 73(1): 5–12.

[9] Davey B A, Priestley H A. Introduction to lattices and order (2nd ed.) [M]. Cambridge: Cambridge University Press, 2002.

[10] Stanley R P. Cambridge studies in advanced mathematics: volume 49. Enumerative combinatorics: volume 1 (2nd ed.) [M]. Cambridge: Cambridge University Press, 2011.

[11] Bondy J A, Murty U S R. Graduate texts in mathematics[M]. Volume 244, Graph Theory, London: Springer-Verlag, 2008.

[12] Harary F. Graph theory [M]. New Delhi: Narosa Publishing House Reading, 1988.

[13] Zhang Heping, Zha Rijun, Yao Haiyuan. Z-transformation graphs of maximum matchings of plane bipartite graphs [J]. Discrete Appl. Math., 2004, 134(1–3): 339–350.

[14] Lovász L, Plummer M D. Matching theory [M]. Amsterdam: North-Holland, 1986.

[15] Zhang Heping, Yao Haiyuan, Yang Dewu. A min-max result on outerplane bipartite graphs [J]. Appl. Math. Lett., 2007, 20(2): 199–205.

[16] Zhang Heping, Ou Lifeng, Yao Haiyuan. Fibonacci-like cubes as Z-transformation graphs [J]. Discrete Math., 2009, 309: 1284–1293.

[17] Qi Zhongbin, Zhang Heping. The relation between the R-rotation graph and the $\bar{R}$-rotation graph of a coronoid system [J]. Acta Math. Appl. Sinica, 2010, 33(2): 269–280 (Chinese).

[18] Yao Haiyuan, Wang Xu, Zhao Xuxu. Several classes of non-matchable distributive lattices [J]. J. Northwest Norm. Univ. (Natur. Sci.), 2019, 55(1): 35–38 (Chinese).

一类非匹配型分配格

王 旭, 赵姁姁, 姚海元

(西北师范大学数学与统计学院, 甘肃 兰州 730070)

摘要: 本文研究了非匹配型分配格. 通过引入基于平面基本二部图某个完美匹配的交不可约内环并给出其等价刻画, 得到了一类新的非匹配型分配格, 推广了一个有割元非匹配型分配格的结论.

关键词: 平面(弱)基本二部图; Z 变换图; 交不可约内环; 非匹配型分配格; 平面性

MR(2010)**主题分类号:** 05C70; 06D50 **中图分类号:** O157.5; O153.1