Vol. 40 (2020)
No. 1

数 学 杂 志
J. of Math. (PRC)

BOUNDEDNESS OF THE FRACTIONAL INTEGRAL OPERATOR WITH ROUGH KERNEL AND ITS COMMUTATOR IN VANISHING GENERALIZED VARIABLE EXPONENT MORREY SPACES ON UNBOUNDED SETS

MO Hui-xia, WANG Xiao-juan, HAN Zhe

(*School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China*)

Abstract: In this paper, we study the boundedness of fractional integral operators and their commutators in vanishing generalized Morrey spaces with variable exponent on unbounded sets. Using the properties of variable exponent functions and the pointwise estimates of operators $T_{\Omega,\alpha}$ and their commutators $[b, T_{\Omega,\alpha}]$ in Lebesgue spaces with variable exponent, we obtain the boundedness of fractional integral operators $T_{\Omega,\alpha}$ and their commutators $[b, T_{\Omega,\alpha}]$ in vanishing generalized Morrey spaces with variable exponents on unbounded sets, which extend the previous results.

Keywords: fractional integral operator with rough kernel; commutator; BMO function space; vanishing generalized Morrey space with variable exponents

2010 MR Subject Classification: 42B20; 42B25; 42B35

Document code: A **Article ID: 0255-7797(2020)01-0007-13**

1 Introduction

In recent years, the theory of variable exponent function spaces attracted ever more attention (see [1–8] for example), since Kováčik and Pákosník [1] introduced the variable exponents Lebesgue and Sobolev spaces, which are the fundamental works of variable exponent function spaces. The function spaces with variable exponent were applied widely in the image processing, fluid mechanics and partial differential equations with non-standard growth (see [9–13] for example).

Let $\Omega \in L^s(\mathbb{S}^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, where $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$ and $s \geq 1$. For $0 < \alpha < n$, then the fractional integral with rough kernel is defined by

$$T_{\Omega,\alpha}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy. \tag{1.1}$$

* **Received date:** 2019-04-15 **Accepted date:** 2019-06-17

Foundation item: Supported by National Natural Science Foundation of China (11601035).

Biography: Mo Huixia (1976–), female, born at Shijiazhuang, Hebei, associate professor, major in harmonic analysis. E-mail: huixmo@bupt.edu.cn.

Moreover, let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. Then the commutator generated by $T_{\Omega,\alpha}$ and b can be defined as follows

$$[b,T_{\Omega,\alpha}](f)(x)=\int_{\mathbb{R}^n}[b(x)-b(y)]\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy. \tag{1.2}$$

Recently, Tan and Liu [14] studied the boundedness of $T_{\Omega,\alpha}$ on the variable exponent Lebesgue, Hardy and Herz-type Hardy spaces. Wang [15] et al. obtained the boundedness of $T_{\Omega,\alpha}$ and its commutator $[b,T_{\Omega,\alpha}]$ on Morrey-Herz space with variable exponent. Tan and Zhao [16] established the boundedness of $T_{\Omega,\alpha}$ and its commutator $[b,T_{\Omega,\alpha}]$ in variable exponent Morrey spaces. In 2016, Long and Han [17] established the boundedness of the maximal operator, potential operator and singular operator of Calderón-Zygmund type in the vanishing generalized Morrey spaces with variable exponent.

Inspired by [14–17], in the paper, we consider the boundedness of $T_{\Omega,\alpha}$ and its commutator $[b,T_{\Omega,\alpha}]$ generated by $T_{\Omega,\alpha}$ and BMO functions in vanishing generalized Morrey spaces with variable exponent on unbounded sets.

Notation Throughout this paper, $\mathbb{R}^n$ is the n-dimensional Euclidean space, χ_A is the characteristic function of a set $A\subseteq\mathbb{R}^n$; C is the positive constant, which may have different values even in the same line; $A\lesssim B$ means that $A\le CB$ with some positive constant C independent of appropriate quantities and if $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$.

Now, let us recall some necessary definitions and notations.

Definition 1.1 Let $D\subset\mathbb{R}^n$ be an open set and $p(\cdot):D\to[1,\infty)$ be a measurable function. Then, $L^{p(\cdot)}(D)$ denotes the set of all measurable functions f on D such that for some $\lambda>0$,

$$\int_D\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx<\infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(D)}=\inf\{\lambda>0:\int_D\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx\le 1\}.$$

$L^{p(\cdot)}(D)$ is regarded as variable exponent Lebesgue space. And, if $p(x)=p$ is a positive constant, then $L^{p(\cdot)}(D)$ is exactly the Lebesgue space $L^p(D)$.

Define the sets $\mathcal{P}^0(D)$ and $\mathcal{P}(D)$ as follows $\mathcal{P}^0(D)=\{p(\cdot):D\to[0,\infty),p^->0,p^+<\infty\}$, and $\mathcal{P}(D)=\{p(\cdot):D\to[1,\infty),p^->1,p^+<\infty\}$, where $p^-=\operatorname{ess}\inf_{x\in D}p(x)$ and $p^+=\operatorname{ess}\sup_{x\in D}p(x)<\infty$.

Throughout the paper, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, that is $p'(x)=p(x)/(p(x)-1)$. And note that $s'<p^-$ is equivalent to $s>(p')^+$.

Definition 1.2 Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator is defined by

$$Mf(x)=\sup_{x\in B}\frac{1}{|B|}\int_B|f(y)|dy,$$

where the supremum is taken over all balls containing x.

Let $\mathcal{B}(D)$ be the set of $p(\cdot) \in \mathcal{P}(D)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(D)$.

Definition 1.3 [18] Let $D \subset \mathbb{R}^n$ be an open set $D \subset \mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. $p(\cdot)$ is log-Hölder continuous, if $p(\cdot)$ satisfies the following conditions

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad x, y \in D, \quad |x-y| \le \frac{1}{2},$$

$$|p(x) - p(y)| \le \frac{C}{\log(|x|+e)}, \quad x, y \in D, \quad |y| \ge |x|, \tag{1.3}$$

we denote by $p(\cdot) \in \mathcal{P}^{\log}(D)$. From Theorem 1.1 of [18], we know that if $p(\cdot) \in \mathcal{P}^{\log}(D)$, then $p(\cdot) \in \mathcal{B}(D)$.

If D is an unbounded set, we shall also use the assumption: there exists $p(\infty) =: \lim\limits_{|x|\to\infty} p(x)$. And, we denote the subset of $\mathcal{P}^{\log}(D)$ by $\mathcal{P}^{\log}_{\infty}(D)$ with the exponents satisfying the following decay condition,

$$|p(x) - p(\infty)| \le \frac{C}{\log(e+|x|)}, \quad x \in D. \tag{1.4}$$

Note that if D is an unbounded set and $p(\infty)$ exists, then (1.4) is equivalent to condition (1.3). We would also like to remark that $p(\cdot) \in \mathcal{P}^{\log}_{\infty}(D)$ if and only if $p'(\cdot) \in \mathcal{P}^{\log}_{\infty}(D)$ and $(p(\infty))' = p'(\infty)$ (see [5] for detail).

Let $D \subset \mathbb{R}^n$ be an open set and $\tilde{B}(x,r) = B(x,r) \bigcap D$, where $B(x,r)$ is the ball centered at x and with radius r.

Definition 1.4 [3] For $0 < \lambda(\cdot) < n$ and $1 \le p(\cdot) < \infty$, the Morrey space with variable exponent is defined by $L^{p(\cdot),\lambda(\cdot)}(D) = \left\{ f \in L^{p(\cdot)}_{\rm loc}(D) : \|f\|_{L^{p(\cdot),\lambda(\cdot)}(D)} < \infty \right\}$, where

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(D)} =: \sup_{x\in D, r>0} r^{-\frac{\lambda(x)}{p(x)}} \|f\|_{L^p(\tilde{B}(x,r))}.$$

Definition 1.5 Let $D \subset \mathbb{R}^n$ be an unbounded open set and $\Pi \subseteq D$, $\varphi(x,r)$ belongs to the class $\beth = \beth\,(\Pi \times [0,\infty))$ of non-negative functions on $\Pi \times [0,\infty)$, which are positive on $\Pi \times (0,\infty)$. Then for $1 \le p(x) \le p^+ < \infty$, the generalized Morrey space with variable exponent is defined by $L^{p(\cdot),\varphi}_{\Pi}(D) = \left\{ f \in L^{p(\cdot)}_{\rm loc}(D) : \|f\|_{L^{p(\cdot),\varphi}_{\Pi}} < \infty \right\}$, where

$$\|f\|_{L^{p(\cdot),\varphi}_{\Pi}(D)} =: \sup_{x\in\Pi, r>0} \varphi(x,r)^{-\frac{1}{\theta_p(x,r)}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))}$$

and

$$\frac{1}{\theta_p(x,r)} = \begin{cases} \dfrac{n}{p(x),} & r \le 1, \\ \dfrac{n}{p(\infty),} & r \ge 1. \end{cases}$$

Definition 1.6 Let $D \subset \mathbb{R}^n$ be an unbounded open set. The vanishing generalized Morrey space with variable exponent $VL^{p(\cdot),\varphi}_{\Pi}(D)$ is defined as the space of functions $f \in L^{p(\cdot),\varphi}_{\Pi}(D)$ such that

$$\lim_{r\to 0^+} \sup_{x\in\Pi} \varphi(x,r)^{-\frac{1}{\theta_p(x,r)}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))} = 0.$$

Naturally, it is suitable to impose on $\varphi(x,r)$ with the following conditions

$$\lim_{r\to 0^+}\sup_{x\in\Pi} r^{\frac{n}{\theta_p(x,r)}}\varphi(x,r)^{-\frac{1}{\theta_p(x,r)}}=0 \tag{1.5}$$

and

$$\inf_{r>1}\sup_{x\in\Pi}\varphi(x,r)>0.$$

Noting that, if we replace $\theta_p(x,r)$ with $p(x)$ in Definition 1.5 and Definition 1.6, then $VL_{\Pi}^{p(\cdot),\varphi}(D)$ is the class vanishing generalized Morrey spaces with variable exponent, see [17] for example. Particularly, if $\theta_p(x,r)=p(x)$, $\varphi(x,r)=r^{\lambda(x)}$ and $\Pi=D$, then the generalized Morrey space with variable exponent $L_{\Pi}^{p(\cdot),\varphi}(D)$ is exactly the Morrey space with variable exponent $L^{p(\cdot),\lambda(\cdot)}(D)$.

Definition 1.7 For $b\in L^1_{\rm loc}(\mathbb{R}^n)$, then the space of functions of bounded mean oscillation is defined by $\mathrm{BMO}(\mathbb{R}^n)=\{f\in L^1_{\rm loc}(\mathbb{R}^n):\|b\|_{\mathrm{BMO}}<\infty\}$, where

$$\|b\|_{\mathrm{BMO}}=\sup_B\frac{1}{|B|}\int_B|b(x)-b_B|dx \quad \text{and} \quad b_B=\frac{1}{|B|}\int_B f(y)dy.$$

In the following, let us state the main results of the paper.

2 Main Results

Theorem 2.1 Assume that $D\subseteq\mathbb{R}^n$ is an unbounded open set. Let $0<\alpha<n, T_{\Omega,\alpha}$ be defined as in (1.1), $p(\cdot),q(\cdot)\in\mathcal{P}_\infty^{\log}(D)$, such that $1<p^-\le p^+<\frac{n}{\alpha}$ and $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}, x\in D$. If $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s'<p^-$, then for arbitrary $x_0\in\Pi\subset D$ and $\tilde{B}(x_0,r)=B(x_0,r)\bigcap D$, we have

$$\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}\lesssim r^{\frac{n}{\theta_q(x_0,r)}}\int_{2r}^{\infty}t^{-\frac{n}{\theta_q(x_0,r)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt, \tag{2.1}$$

where $s'=\frac{s}{s-1}$.

Theorem 2.2 Assume that $D\subseteq\mathbb{R}^n$ is an unbounded open set and $\Pi\subset D$. Let $0<\alpha<n, T_{\Omega,\alpha}$ be defined as in (1.1), $p(\cdot),q(\cdot)\in\mathcal{P}_\infty^{\log}(D)$, such that $1<p^-\le p^+<\frac{n}{\alpha}$ and $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}, x\in D$. Let $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s'<p^-$ and $\varphi,\psi\in\beth=\beth(\Pi\times[0,\infty))$. Then the operator $T_{\Omega,\alpha}$ is bounded from $VL_{\Pi}^{p(\cdot),\varphi}(D)$ to $VL_{\Pi}^{q(\cdot),\psi}(D)$, if

$$c_\delta=:\int_\delta^\infty\frac{\sup_{x\in\Pi}\varphi^{\frac{1}{\theta_p(x,t)}}(x,t)dt}{t^{1+\frac{n}{\theta_q(x,t)}}}<\infty \tag{2.2}$$

for each $\delta>0$, and

$$\int_r^\infty\frac{\varphi^{\frac{1}{\theta_p(x,t)}}(x,t)dt}{t^{1+\frac{n}{\theta_q(x,t)}}}\le c_0\frac{\psi^{\frac{1}{\theta_q(x,r)}}(x,r)}{r^{\frac{n}{\theta_q(x,r)}}}, \tag{2.3}$$

where c_0 does not depend on $x\in\Pi$ and $r>0$.

Theorem 2.3 Assume that $D\subseteq\mathbb{R}^n$ is an unbounded open set. Let $0<\alpha<n, [b,T_{\Omega,\alpha}]$ be defined as in (1.2) and $p(\cdot),q(\cdot)\in\mathcal{P}_\infty^{\log}(D)$, such that $1<p^-\le p^+<\frac{n}{\alpha}$ and

$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}, x\in D$. If $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s'<p^-$, then for arbitrary $x_0\in\Pi\subset D$, $\tilde{B}(x_0,r)=B(x_0,r)\bigcap D$ and $b\in \mathrm{BMO}(\mathbb{R}^n)$, we have

$$\|[b,T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}\lesssim\|b\|_{\mathrm{BMO}}r^{\frac{n}{\theta_q(x_0,r)}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta_q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt. \tag{2.4}$$

Theorem 2.4 Assume that $D\subseteq\mathbb{R}^n$ is an unbounded open set and $\Pi\subset D$. Let $0<\alpha<n, [b,T_{\Omega,\alpha}]$ be defined as (1.2), $p(\cdot),q(\cdot)\in\mathcal{P}_\infty^{\log}(D)$, such that $1<p^-\le p^+<\frac{n}{\alpha}$ and $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}, x\in D$. Let $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s'<p^-, b\in\mathrm{BMO}(\mathbb{R}^n)$ and $\varphi,\psi\in\daleth=\daleth\left(\Pi\times[0,\infty)\right)$. Then the commutator $[b,T_{\Omega,\alpha}]$ is bounded from $VL_{\Pi}^{p(\cdot),\varphi}(D)$ to $VL_{\Pi}^{q(\cdot),\psi}(D)$, if

$$c_\delta=:\int_\delta^\infty(1+\ln\frac{r}{t})\frac{\sup\limits_{x\in\Pi}\varphi^{\frac{1}{\theta_p(x,t)}}(x,t)dt}{t^{1+\frac{n}{\theta_q(x,t)}}}<\infty \tag{2.5}$$

for each $\delta>0$ and

$$\int_r^\infty(1+\ln\frac{r}{t})\frac{\varphi^{\frac{1}{\theta_p(x,t)}}(x,t)dt}{t^{1+\frac{n}{\theta_q(x,t)}}}\le c_0\frac{\psi^{\frac{1}{\theta_q(x,r)}}(x,r)}{r^{\frac{n}{\theta_q(x,r)}}}, \tag{2.6}$$

where c_0 does not depend on $x\in\Pi$ and $r>0$.

3 Some Lemmas

In this part, we give some requisite lemmas.

Lemma 3.1 (see [19, 20]) (Generalized Hölder's inequality) Let $D\subseteq\mathbb{R}^n$, $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p(x)}+\frac{1}{q(x)}\equiv 1$. If $f\in L^{p(\cdot)}(D)$ and $g\in L^{q(\cdot)}(D)$, then

$$\int_D|f(x)g(x)|dx\le C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{q(\cdot)}},$$

with $C=\sup\limits_{x\in D}\frac{1}{p(x)}+\sup\limits_{x\in D}\frac{1}{q(x)}$

In general, if $p_1(\cdot),p_2(\cdot)\dots p_m(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that $\sum\limits_{k=1}^m\frac{1}{p_k(x)}\equiv 1,\ x\in D$. Then for $f_i\in L^{p_i(\cdot)}(D), i=1,2,\dots,m$, we have

$$\int_D|f_1(x)\cdots f_m(x)|dx\le C\|f_1\|_{L^{p_1(\cdot)}}\cdots\|f_m\|_{L^{p_m(\cdot)}},$$

where $C=\sum\limits_{k=1}^m\sup\limits_{x\in D}\frac{1}{p_k(x)}$.

Lemma 3.2 (see [21]) Let $p(\cdot),\tilde{q}(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If $p^+<q<\infty$ and $\frac{1}{p(x)}=\frac{1}{\tilde{q}(x)}+\frac{1}{q}, x\in\mathbb{R}^n$, then we have $\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$ for all measurable functions f and g.

Lemma 3.3 (see [5], Corollary 4.5.9) If D is an unbounded set and $p\in\mathcal{P}_\infty^{\log}(D)$, then

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(D)}\le Cr^{\frac{n}{\theta_p(x,r)}},\quad x\in D.$$

Lemma 3.4 (see [22]) Let $b \in \mathrm{BMO}(\mathbb{R}^n), 1 < p < \infty$ and $0 < r_1, r_2 < \infty$. Then

$$(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}|b(y)-b_B(x,r_2)|^p dy)^{\frac{1}{p}} \leq C(1+\ln\frac{r_2}{r_1})\|b\|_{\mathrm{BMO}}.$$

Lemma 3.5 Let $D \subseteq \mathbb{R}^n$ be an unbounded open set, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P}_\infty^{\log}(D)$. Then for any $x \in D, 0 < r_1, r_2 < \infty$ and $\tilde{B}(x,r_i) = B(x,r_i)\bigcap D, i = 1,2$, we have

$$\frac{1}{\|\chi_{\tilde{B}(x,r_1)}\|_{L^{p(\cdot)}(D)}}\|(b(\cdot)-b_{\tilde{B}(x,r_2)})\chi_{\tilde{B}(x,r_1)}\|_{L^{p(\cdot)}(D)} \leq C(1+\ln\frac{r_2}{r_1})\|b\|_{\mathrm{BMO}}.$$

Proof Let $p^+ < p_1 < \infty, p_2(x) \in \mathcal{P}_\infty^{\log}(D)$, such that $\frac{1}{p(x)} = \frac{1}{p_1} + \frac{1}{p_2(x)}, x \in D$. Then by Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned}
&\frac{1}{\|\chi_{\tilde{B}(x,r_1)}\|_{L^{p(\cdot)}(D)}}\|(b(\cdot)-b_{\tilde{B}(x,r_2)})\chi_{\tilde{B}(x,r_1)}\|_{L^{p(\cdot)}(D)}\\
\leq&\frac{C}{\|\chi_{\tilde{B}(x,r_1)}\|_{L^{p(\cdot)}(D)}}\|(b(\cdot)-b_{\tilde{B}(x,r_2)})\chi_{\tilde{B}(x,r_1)}\|_{L^{p_1}(D)}\|\chi_{\tilde{B}(x,r_1)}\|_{L^{p_2(\cdot)}(D)}\\
=&C(\frac{1}{|\tilde{B}(x,r_1)|}\int_{\tilde{B}(x,r_1)}|b(y)-b_{\tilde{B}(x,r_2)}|^{p_1}dy)^{\frac{1}{p_1}}\frac{|\tilde{B}(x,r_1)|^{\frac{1}{p_1}}\|\chi_{\tilde{B}(x,r_1)}\|_{L^{p_2(\cdot)}(D)}}{\|\chi_{\tilde{B}(x,r_1)}\|_{L^{p(\cdot)}(D)}}\\
\leq&C(1+\ln\frac{r_2}{r_1})\|b\|_{BMO}\frac{r_1^{\frac{n}{\theta p_2(x,r_1)}}r_1^{\frac{n}{p_1}}}{r_1^{\frac{n}{\theta p(x,r_1)}}} = C(1+\ln\frac{r_2}{r_1})\|b\|_{\mathrm{BMO}}.
\end{aligned}$$

Lemma 3.6 (see [23]) Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n), 0 < \alpha < n, 1 < p^- \leq p^+ < \frac{n}{\alpha}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ for any $x \in \mathbb{R}^n$. If $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}(\mathbb{R}^n), \Omega \in L^s(\mathbb{S}^{n-1})$ with $1 \leq s' < p^-$, then there exists $C > 0$ such that

$$\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3.7 (see [23]) Let $b \in \mathrm{BMO}(\mathbb{R}^n), p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n), 0 < \alpha < n, 1 < p^- \leq p^+ < \frac{n}{\alpha}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ for any $x \in \mathbb{R}^n$. If $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}(\mathbb{R}^n), \Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s' < p^-$, then there exists a constant $C > 0$ such that

$$\|[b, T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{BMO}}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

4 Proof of Main Results

In this section, we will give the proofs of the main results.

Proof of Theorem 2.1 Let $f_1 = f\chi_{\tilde{B}(x_0,2r)}$ and $f_2 = f\chi_{D\setminus\tilde{B}(x_0,2r)}$, then we have $f = f_1 + f_2$. By the sublinearity of the operator $T_{\Omega,\alpha}$, we obtain that

$$\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} \leq \|T_{\Omega,\alpha}(f_1)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} + \|T_{\Omega,\alpha}(f_2)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}. \tag{4.1}$$

Noting that $\frac{q(\cdot)(n-\alpha)}{n} \in \mathcal{B}(D)$, then from Lemma 3.6, we have

$$\|T_{\Omega,\alpha}(f_1)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} \lesssim \|f_1\|_{L^{p(\cdot)}(D)} \lesssim \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,2r))}. \tag{4.2}$$

For the other part, let us estimate $|T_{\Omega,\alpha}(f_2)(x)|$ for $x \in \tilde{B}(x_0, r)$, first.

When $x \in \tilde{B}(x_0, r)$ and $y \in D\backslash\tilde{B}(x_0, 2r))$, it is easy to see that $|x_0 - y| \approx |x - y|$. Let $u(x) \in \mathcal{P}^{\log}_{\infty}(D)$, such that $\frac{1}{p(x)} + \frac{1}{s} + \frac{1}{u(x)} = 1, x \in D$. Then by the generalized Hölder's inequality and Lemma 3.3, we obtain

$$\begin{aligned} |T_{\Omega,\alpha}(f_2)(x)| &\leq \int_{D\backslash\tilde{B}(x_0,2r)} \frac{|f(y)||\Omega(x-y)|}{|x-y|^{n-\alpha}} dy \\ &\lesssim \int_{D\backslash\tilde{B}(x_0,2r)} |f(y)||\Omega(x-y)| \int_{|x-y|}^{\infty} t^{\alpha-n-1} dt dy \\ &\lesssim \int_{2r}^{\infty} t^{\alpha-n-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} \|\Omega(x-\cdot)\|_{L^s(\tilde{B}(x_0,t))} \|\chi_{D\bigcap\tilde{B}(x_0,t)}\|_{L^{u(\cdot)}(D)} dt \\ &\lesssim \int_{2r}^{\infty} t^{\alpha-n-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} \|\Omega(x-\cdot)\|_{L^s(\tilde{B}(x_0,t))} |\tilde{B}(x_0,t)|^{\frac{1}{\theta_u(x_0,t)}} dt. \end{aligned} \tag{4.3}$$

When $x \in \tilde{B}(x_0, t), 2r < t < \infty$, it's easy to see that

$$\begin{aligned} \|\Omega(x-\cdot)\|_{L^s(\tilde{B}(x_0,t))} &= \Big(\int_{\tilde{B}(x-x_0,t)} |\Omega(z)|^s dz\Big)^{\frac{1}{s}} \\ &\leq \Big(\int_{\tilde{B}(0,t+|x-x_0|)} |\Omega(z)|^s dz\Big)^{\frac{1}{s}} \leq \Big(\int_{\tilde{B}(0,2t)} |\Omega(z)|^s dz\Big)^{\frac{1}{s}} \\ &\leq \Big(\int_{\mathbb{S}^{n-1}} \int_0^{2t} |\Omega(z')|^s d\sigma(z') r^{n-1} dr\Big)^{\frac{1}{s}} \lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |\tilde{B}(x_0,2t)|^{\frac{1}{s}}. \end{aligned} \tag{4.4}$$

Since $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ and $\frac{1}{p(x)} + \frac{1}{s} + \frac{1}{u(x)} = 1$, then for $x \in \tilde{B}(x_0, r)$, from (4.3) and (4.4), we get

$$\begin{aligned} |T_{\Omega,\alpha}(f_2)(x)| &\lesssim \int_{2r}^{\infty} t^{\alpha-n-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |\tilde{B}(x_0,2t)|^{\frac{1}{s}} |\tilde{B}(x_0,t)|^{\frac{1}{\theta_u(x_0,t)}} dt \\ &\lesssim \int_{2r}^{\infty} t^{-\frac{n}{\theta_q(x_0,t)}-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} dt. \end{aligned} \tag{4.5}$$

Thus,

$$\begin{aligned} \|T_{\Omega,\alpha}(f_2)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} &\lesssim \int_{2r}^{\infty} t^{-\frac{n}{\theta_q(x_0,t)}-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} dt \|\chi_{\tilde{B}(x_0,r)}\|_{L^{q(\cdot)}(D)} \\ &\lesssim r^{\frac{n}{\theta_q(x_0,r)}} \int_{2r}^{\infty} t^{-\frac{n}{\theta_q(x_0,t)}-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} dt. \end{aligned} \tag{4.6}$$

Combining the estimates of (4.1), (4.2) and (4.6) we have

$$\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} \lesssim r^{\frac{n}{\theta_q(x_0,r)}} \int_{2r}^{\infty} t^{-\frac{n}{\theta_q(x_0,t)}-1} \|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))} dt,$$

which completes the proof of Theorem 2.1.

Proof of Theorem 2.2 For every $f \in VL^{p(\cdot),\varphi}_{\Pi}(D)$ we need to prove that

$$\|T_{\Omega,\alpha}(f)\|_{VL^{q(\cdot),\psi}_{\Pi}(D)} \lesssim \|f\|_{VL^{p(\cdot),\varphi}_{\Pi}(D)}$$

and

$$\lim_{r\to 0^+}\sup_{x\in\Pi}\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}=0.$$

In fact, from Theorem 2.1 and (2.3), it follows that

$$\begin{aligned}\|T_{\Omega,\alpha}(f)\|_{VL_\Pi^{q(\cdot),\psi}(D)}&=\sup_{x\in\Pi,r>0}\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}\\&\lesssim\sup_{x\in\Pi,r>0}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\int_{2r}^{\infty}t^{-\frac{n}{\theta q(x,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,t))}dt\\&=\sup_{x\in\Pi,r>0}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\int_{2r}^{\infty}\frac{\varphi^{\frac{1}{\theta p(x,t)}}(x,t)}{t^{1+\frac{n}{\theta q(x,t)}}}\frac{\|f\|_{L^{p(\cdot)}(\tilde{B}(x,t))}}{\varphi^{\frac{1}{\theta p(x,t)}}(x,t)}dt\\&\lesssim\|f\|_{VL_\Pi^{p(\cdot),\varphi}(D)}\sup_{x\in\Pi,r>0}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\int_{r}^{\infty}\frac{\varphi^{\frac{1}{\theta p(x,t)}}(x,t)}{t^{1+\frac{n}{\theta q(x,t)}}}dt\lesssim\|f\|_{VL_\Pi^{p(\cdot),\varphi}(D)}.\end{aligned}$$

Now, let us show that

$$\lim_{r\to 0^+}\sup_{x\in\Pi}\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}=0.$$

For $0<r<\delta_0<\infty$, by estimation (2.1), it follows that

$$\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}\le C[A_{\delta_0}(x,r)+B_{\delta_0}(x,r)],\tag{4.7}$$

where

$$A_{\delta_0}(x,r):=\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\left(\int_r^{\delta_0}\frac{\varphi^{\frac{1}{\theta p(x,t)}}(x,t)}{t^{1+\frac{n}{\theta q(x,t)}}}\sup_{0<r<t}\frac{1}{\varphi^{\frac{1}{\theta p(x,t)}}(x,r)}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))}dt\right)$$

and

$$B_{\delta_0}(x,r):=\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\left(\int_{\delta_0}^{\infty}\frac{\varphi^{\frac{1}{\theta p(x,t)}}(x,t)}{t^{1+\frac{n}{\theta q(x,t)}}}\sup_{0<r<t}\frac{1}{\varphi^{\frac{1}{\theta p(x,t)}}(x,r)}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))}dt\right).$$

For any $\varepsilon>0$, now we choose a fixed δ_0 such that whenever $0<r<\delta_0$,

$$\sup_{x\in\Pi}\sup_{0<r<t}\frac{1}{\varphi^{\frac{1}{\theta p(x,r)}}(x,r)}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))}<\frac{\varepsilon}{2Cc_0},$$

where c_0 and C are constants from (2.3) and (4.7), which is possible since $f\in VL_\Pi^{p(\cdot),\varphi}(D)$.

This allows us to estimate the first term uniformly for $0<r<\delta_0$,

$$\sup_{x\in\Pi}CA_{\delta_0}(x,r)<\frac{\varepsilon}{2}.\tag{4.8}$$

By choosing r small enough, we obtain the estimate of the second term. Indeed, by (2.2), we get

$$B_{\delta_0}(x,r)\le c_{\delta_0}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|f\|_{VL_\Pi^{p(\cdot),\varphi}(D)},$$

where c_{δ_0} is the constant from (2.2). Since ψ satisfies condition (1.6), we can choose r small enough such that

$$\sup_{x\in\Pi}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}<\frac{\varepsilon}{2Cc_{\delta_0}\|f\|_{VL_{\Pi}^{p(\cdot),\varphi}(D)}}.$$

Thus

$$\sup_{x\in\Pi}CB_{\delta_0}(x,r)<\frac{\varepsilon}{2}. \tag{4.9}$$

Combining the estimates of (4.8) and (4.9), we obtain $\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}<\varepsilon$.

Therefore, $\lim\limits_{r\to0^+}\sup\limits_{x\in\Pi}\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|T_{\Omega,\alpha}(f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}=0$. The proof of Theorem 2.2 is completed.

Proof of Theorem 2.3 Let $f_1=f\chi_{\tilde{B}(x_0,2r)}$ and $f_2=f\chi_{D\backslash\tilde{B}(x_0,2r)}$, then we have $f=f_1+f_2$. Thus, it follows that

$$\|[b,T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}\leq\|[b,T_{\Omega,\alpha}](f_1)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}+\|[b,T_{\Omega,\alpha}](f_2)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} \tag{4.10}$$

Noting that $\frac{q(\cdot)(n-\alpha)}{n}\in\mathcal{B}(D)$, then from Lemma 3.7, we know that the commutator $[b,T_{\Omega,\alpha}]$ is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, so

$$\|[b,T_{\Omega,\alpha}](f_1)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}\lesssim\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,2r))}. \tag{4.11}$$

For the second part of (4.10), let us estimate $|[b,T_{\Omega,\alpha}](f_2)(x)|$ for $x\in\tilde{B}(x_0,r)$, first. It is easy to see that

$$\begin{aligned}
&|[b,T_{\Omega,\alpha}](f_2)(x)|\\
\lesssim&\int_{D\backslash\tilde{B}(x_0,2r)}\frac{|b(x)-b_{\tilde{B}(x_0,r)}||\Omega(x-y)||f(y)|}{|x-y|^{n-\alpha}}dy+\int_{D\backslash\tilde{B}(x_0,2r)}\frac{|b_{\tilde{B}(x_0,r)}-b(y)||\Omega(x-y)||f(y)|}{|x-y|^{n-\alpha}}dy\\
\lesssim&|b(x)-b_{\tilde{B}(x_0,r)}|\int_{D\backslash\tilde{B}(x_0,2r)}\frac{|\Omega(x-y)||f(y)|}{|x-y|^{n-\alpha}}dy+\int_{D\backslash\tilde{B}(x_0,2r)}\frac{|b_{\tilde{B}(x_0,r)}-b(y)||\Omega(x-y)||f(y)|}{|x-y|^{n-\alpha}}dy\\
=:&I_1+I_2.
\end{aligned}$$

From (4.5), it is easy to see that

$$I_1\lesssim|b(x)-b_{\tilde{B}(x_0,r)}|\int_{2r}^{\infty}t^{-\frac{n}{\theta q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt.$$

Thus, according to Lemma 3.3 and Lemma 3.5, we deduce that

$$\begin{aligned}
&\|I_1\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}\\
\lesssim&\|(b(x)-b_{\tilde{B}(x_0,r)})\chi_{\tilde{B}(x_0,r)}\|_{L^{q(\cdot)}(D)}\int_{2r}^{\infty}t^{-\frac{n}{\theta q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt\\
\lesssim&\|b\|_{\mathrm{BMO}}\|\chi_{\tilde{B}(x_0,r)}\|_{L^{q(\cdot)}(D)}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt\\
\lesssim&\|b\|_{\mathrm{BMO}}r^{\frac{n}{\theta q(x_0,r)}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt.
\end{aligned} \tag{4.12}$$

When $x\in B(x_0,r)$ and $y\in D\backslash\tilde{B}(x_0,2r)$, it is easy to see that $|x_0-y|\approx|x-y|$. Let $h(x)\in\mathcal{P}_\infty^{\log}(D)$ such that $\frac{1}{p(x)}+\frac{1}{s}+\frac{1}{h(x)}=1, x\in D$. Then for $x\in\tilde{B}(x_0,r)$, by Lemma 3.1, we obtain

$$\begin{aligned} I_2 &\lesssim \int_{D\backslash\tilde{B}(x_0,2r)}|b_{\tilde{B}(x_0,r)}-b(y)||\Omega(x-y)||f(y)|\int_{|x-y|}^{\infty}\frac{dt}{t^{n-\alpha+1}}dy\\ &\le \int_{2r}^{\infty}\int_{y\in D:2r<|x-y|<t}|b_{\tilde{B}(x_0,r)}-b(y)||\Omega(x-y)||f(y)|dy\frac{dt}{t^{n-\alpha+1}} \\ &\lesssim \int_{2r}^{\infty}\|b_{\tilde{B}(x_0,r)}-b(\cdot)\|_{L^{h(\cdot)}(\tilde{B}(x_0,t))}\|\Omega(x-\cdot)\|_{L^s(\tilde{B}(x_0,t))}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}\frac{dt}{t^{n-\alpha+1}}. \end{aligned} \tag{4.13}$$

From (4.4) and Lemma 3.5, we have

$$\|\Omega(x-\cdot)\|_{L^s(\tilde{B}(x_0,t))}\lesssim\|\Omega\|_{L^s(\mathbb{S}^{n-1})}|\tilde{B}(x_0,2t)|^{\frac{1}{s}} \tag{4.14}$$

and

$$\|b_{\tilde{B}(x_0,r)}-b(\cdot)\|_{L^{h(\cdot)}(\tilde{B}(x_0,t))}\lesssim(1+\ln\frac{r}{t})|\tilde{B}(x_0,t)|^{\frac{1}{\theta_h(x_0,t)}}. \tag{4.15}$$

Therefore, for $x\in\tilde{B}(x_0,r)$, from (4.13), (4.14) and (4.15), we obtain

$$\begin{aligned} I_2 &\lesssim \|b\|_{\mathrm{BMO}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}|\tilde{B}(x_0,t)|^{\frac{1}{\theta_h(x_0,t)}+\frac{1}{s}}t^{-n+\alpha-1}dt\\ &\lesssim \|b\|_{\mathrm{BMO}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta_q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt. \end{aligned}$$

Thus

$$\begin{aligned} \|I_2\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))} &\lesssim \|b\|_{\mathrm{BMO}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta_q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt\|\chi_{\tilde{B}(x_0,r)}\|_{L^{q(\cdot)}(D)}\\ &\lesssim \|b\|_{\mathrm{BMO}}r^{\frac{n}{\theta_q(x_0,r)}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta_q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt. \end{aligned} \tag{4.16}$$

From (4.12) and (4.16), we get

$$\|[b,T_{\Omega,\alpha}](f_2)\|_{L^{q(\cdot)}(\tilde{B}(x_0,r))}\lesssim\|b\|_{\mathrm{BMO}}r^{\frac{n}{\theta_q(x_0,r)}}\int_{2r}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta_q(x_0,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x_0,t))}dt. \tag{4.17}$$

Combining the estimates of (4.10), (4.11) and (4.17), we completed the proof of Theorem 2.3.

Proof of Theorem 2.4 The proof is similar to that of Theorem 2.2. From Theorem 2.3 and (2.6), it is easy to see that $\|[b,T_{\Omega,\alpha}](f)\|_{VL_\Pi^{q(\cdot),\psi}(D)}\lesssim\|b\|_{\mathrm{BMO}}\|f\|_{VL_\Pi^{p(\cdot),\varphi}(D)}$. So, we just have to show that

$$\lim_{r\to0^+}\sup_{x\in\Pi}\frac{1}{\psi^{\frac{1}{\theta_q(x,r)}}(x,r)}\|[b,T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))}=0.$$

For $0 < r < \delta_0 < \infty$. Then by (2.4), it follows that

$$\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|[b,T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))} \le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{4.18}$$

where

$$I_{\delta_0}(x,r) =: \|b\|_{\mathrm{BMO}}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\int_r^{\delta_0}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta q(x,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,t))}dt$$

and

$$J_{\delta_0}(x,r) =: \|b\|_{\mathrm{BMO}}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\int_{\delta_0}^{\infty}(1+\ln\frac{r}{t})t^{-\frac{n}{\theta q(x,t)}-1}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,t))}dt.$$

For any $\varepsilon > 0$, now we choose a fixed $\delta_0 > 0$ such that whenever $0 < r < \delta_0$,

$$\sup_{x\in\Pi}\sup_{0<r<t}\frac{1}{\varphi^{\frac{1}{\theta p(x,r)}}(x,r)}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))} < \frac{\varepsilon}{2Cc_0\|b\|_{\mathrm{BMO}}},$$

where c_0 and C are constants from (2.6) and (4.18), respectively. It is possible since $f \in VL_{\Pi}^{p(\cdot),\varphi}(D)$. This allows us to estimate the first term uniformly for $0 < r < \delta_0$,

$$\sup_{x\in\Pi} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}. \tag{4.19}$$

For the second part, from (2.5), it follows that

$$J_{\delta_0}(x,r) \le c_{\delta_0}\|b\|_{\mathrm{BMO}}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|f\|_{VL_{\Pi}^{p(\cdot),\varphi}(D)},$$

where c_{δ_0} is the constant from (2.5). Since ψ satisfies condition (1.6), it is possible to choose r small enough such that

$$\sup_{x\in\Pi}\frac{r^{\frac{n}{\theta q(x,r)}}}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)} \le \frac{\varepsilon}{2Cc_{\delta_0}\|b\|_{\mathrm{BMO}}\|f\|_{VL_{\Pi}^{p(\cdot),\varphi}(D)}}.$$

Thus

$$\sup_{x\in\Pi} CJ_{\delta_0(x,r)} < \frac{\varepsilon}{2}. \tag{4.20}$$

So, from the estimates of (4.19) and (4.20), it follows that

$$\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|[b,T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))} < \varepsilon,$$

which means that

$$\lim_{r\to 0^+}\sup_{x\in\Pi}\frac{1}{\psi^{\frac{1}{\theta q(x,r)}}(x,r)}\|[b,T_{\Omega,\alpha}](f)\|_{L^{q(\cdot)}(\tilde{B}(x,r))} = 0.$$

Therefore, the proof of Theorem 2.4 is completed.

References

[1] Kováčik O, Rákosník J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$[J]. Czech. Math. J., 1991, 41(4): 592–618.

[2] Fan X, Zhao D. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$[J]. J. Math. Anal. Appl., 2001, 263(2): 424–446.

[3] Almeida A, Hasanov J, Samko S. Maximal and potential operators in variable exponent Morrey spaces[J]. Georgian. Math. J., 2008, 15(2): 195–208.

[4] Almeida A, Hästö P. Besov spaces with variable smoothness and integrability[J]. J. Funct. Anal., 2010, 258(5): 1628–1655.

[5] Diening L, Harjulehto P, Hästö P, Ruzicka M. Lebesgue and Sobolev spaces with variable exponents[M]. Springer, 2011.

[6] Cruz-Uribe D V, Fiorenza A. Variable Lebesgue spaces: foundations and harmonic analysis[M]. Springer Science & Business Media, 2013.

[7] Yang D, Zhuo C, Yuan W. Triebel-Lizorkin type spaces with variable exponents[J]. Banach. J. Math. Anal., 2015, 9(4): 146–202.

[8] Yang D, Zhuo C, Yuan W. Besov-type spaces with variable smoothness and integrability[J]. J. Funct. Anal., 2015, 269(6): 1840–1898.

[9] Ruzicka M. Electrorheological fluids: modeling and mathematical theory[M]. Springer Science & Business Media, 2000.

[10] Chen Y, Levine S, Rao M. Variable exponent, linear growth functionals in image restoration[J]. Siam J. Appl. Math., 2006, 66(4): 1383–1406.

[11] Li F, Li Z, Pi L. Variable exponent functionals in image restoration[J]. Appl. Math. Comput., 2010, 216(3): 870–882.

[12] Harjulehto P, Hästö P, Út V Lê, Nuortio M. Overview of differential equations with non-standard growth[J]. Nonlinear Anal. Theor., 2010, 72(12): 4551–4574.

[13] Harjulehto P, Hästö P, Latvala V, Toivanen O. Critical variable exponent functionals in image restoration[J]. Appl. Math. Lett., 2013, 26(1): 56–60.

[14] Tan J, Liu Z G. Some boundedness of homogeneous fractional integrals on variable exponent function spaces[J]. Acta. Math. Sci. (Chinese Series), 2015, 58(2): 309–320.

[15] Wang H, Wang J, Fu Z. Morrey meets Herz with variable exponent and applications to commutators of homogeneous fractional integrals with rough kernels[J]. J. Funct. Space., 2017, Article ID: 1908794.

[16] Tan J, Zhao J. Rough fractional integrals and its commutators on variable Morrey spaces[J]. C. R. Math., 2015, 353(12): 1117–1122.

[17] Long P, Han H. Characterizations of some operators on the vanishing generalized Morrey spaces with variable exponent[J]. J. Math. Anal. Appl., 2016, 437(1): 419–430.

[18] Cruz-Uribe D, Fiorenza A, Martell J M, Pérez C. The boundedness of classical operators on variable L^p spaces[J]. Ann. Acad. Sci. Fenn. Math., 2006, 31(1): 239–264.

[19] Samko S G. Differentiation and integration of variable order and the spaces $L^{p(x)}$[J]. Operator Theory for Complex and Hypercomplex Analysis (Mexico City, 1994), 1994, 203–219.

[20] Sharapudinov I I. Topology of the space $L^{p(t)}([0,1])$[J]. Mathematical Notes, 1979, 26(4): 796–806.

[21] Nakai E, Sawano Y. Hardy spaces with variable exponents and generalized Campanato spaces[J]. J. Funct. Anal., 2012, 262(9): 3665–3748.

[22] Lin Y, Lu S. Strongly singular Calderón-Zygmund operators and their commutators[J]. Jordan Journal of Mathematics and Statistics, 2008, 1(1): 31–49.

[23] Ho K P. Atomic decomposition of Hardy spaces and characterization of BMO via banach function spaces[J]. Anal. Math., 2012, 38(3): 173–185.

无界集上带粗糙核的分数次积分算子及其交换子在消失广义变指标Morrey空间的有界性

默会霞, 王晓娟, 韩 哲

(北京邮电大学理学院, 北京 100876)

摘要: 本文研究了无界集上带粗糙核的分数次积分算子及其交换子在消失广义变指标Morrey的空间有界性. 利用变指标函数的性质和算子$T_{\Omega,\alpha}$及其交换子$[b, T_{\Omega,\alpha}]$在变指标Lebesgue空间的逐点估计, 获得了无界集上带粗糙核的分数次积分算子$T_{\Omega,\alpha}$及其交换子$[b, T_{\Omega,\alpha}]$在消失广义变指标Morrey空间的有界性, 推广了已有的结果.

关键词: 带粗糙核的分数次积分算子; 交换子; BMO函数空间; 消失广义变指标Morrey空间

MR(2010)**主题分类号:** 42B20; 42B25; 42B35 **中图分类号:** O174.2