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LEONARD PAIRS CONSTRUCTED FROM THE QUANTUM ALGEBRA $\nu_q(sl_2)$

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Abstract: In this paper, we study the construction of Leonard pairs. By using the finite dimensional irreducible representations of quantum algebra $\nu_q(sl_2)$, we get Leonard pairs, and give the classification of Leonard pairs, which provide more help in studying Leonard triples.

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1 Introduction

Leonard pairs were introduced by Terwilliger in [1], which gave some examples to illustrate how Leonard pairs arise in representation theory, combinatorics, and the theory of orthogonal polynomials. Because these polynomials frequently arise in connection with the finite-dimensional representations of good Lie algebras and quantum groups, it is natural to find Leonard pairs associated with these algebraic objects. Leonard pairs of Krawtchouk type were described in [2] using split basis and normalized semisimple generators of sl_2. Leonard pairs of q-Krawtchouk type were described in [3] using split basis of $U_q(sl_2)$. Recently, Alnajjar and Curtin in [4] gave general construction of Leonard pairs of Racah, Hahn, dual Hahn and Krawtchouk type using equitable basis of sl_2. Alnajjar in [5, 6] gave general construction of Leonard pairs of q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, and affine q-Krawtchouk type using equitable generators of $U_q(sl_2)$. The Leonard pairs and Leonard triples of q-Racah type from the quantum algebra $U_q(sl_2)$ were also discussed by Hou in [7] and [8]. Equivalent presentations for $U_q(sl_2)$ were introduced in [9].

Given a Leonard pair it is often more natural to work with a split basis rather than a standard basis. In this paper, we illustrate this with an example based on the quantum algebra $\nu_q(sl_2)$. Let M denote a finite-dimensional irreducible $\nu_q(sl_2)$-module and assume $\mathbb{A}$ (resp. $\mathbb{B}$) is an arbitrary linear combination of F, K (resp. EH^{-1}, H^{-1}). We give the

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necessary and sufficient conditions on the coefficients for $\mathbb{A}, \mathbb{B}$ to act on M as a Leonard pair.

The rest of this paper are organized as follows. In Section 2, we introduce some facts concerning the Leonard pairs. In Section 3, we recall some facts concerning irreducible finite-dimensional $\nu_q(sl_2)$-modules. In Section 4, we define two linear transformations $\mathbb{A}$ and $\mathbb{B}$ using the elements in $\nu_q(sl_2)$ and describe their properties. At last, we characterize when the pair $\mathbb{A}, \mathbb{B}$ is a Leonard pair.

2 Leonard Pairs

In this section, we recall the definitions and some related facts concerning Leonard pairs, and more details about the Leonard pairs can be found in [3] . Throughout this paper $\mathcal{F}$ denotes an algebraically closed field. Fix a nonzero scalar $q \in \mathcal{F}$ which is not a root of unity. $M_{d+1}(\mathcal{F})$ denote the $\mathcal{F}$-algebra consisting of all $(d+1)$ by $(d+1)$ matrices having rows and columns indexed by $0, 1, 2, \cdots, d$ for a nonnegative integer d.

Let V denote a $\mathcal{F}$-vector space of dimensions $d+1$. Let $\mathrm{End}(V)$ denote the $\mathcal{F}$-algebra consisting of all linear transformations from V to V. Let $\{v_i\}_{i=0}^{d}$ denote a basis for V. For $\mathbb{A} \in \mathrm{End}(V)$ and $X \in M_{d+1}(\mathcal{F})$, we say X represents $\mathbb{A}$ with respect to $\{v_i\}_{i=0}^{d}$ whenever $\mathbb{A}v_j = \sum\limits_{i=0}^{d} X_{ij} v_i$ for $0 \leq j \leq d$, where X_{ij} is the element in the matrix X.

A square matrix is said to be tridiagonal if each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be irreducible if each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair.

Definition 2.1 Let V be a vector space over $\mathcal{F}$ with finite positive dimensions. A Leonard pair on V is an ordered pair of linear transformations $\mathbb{A} : V \to V$ and $\mathbb{A}^* : V \to V$ that satisfy both the conditions below.

(1) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(2) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

There are so many examples of Leonard pairs which arise in representation theory, combinatorics, and the theory of orthogonal polynomials, for details can be found in [3].

Given a Leonard pair $\mathbb{A}, \mathbb{A}^*$, it is natural to represent one of $\mathbb{A}, \mathbb{A}^*$ by an irreducible tridiagonal matrix and the other by a diagonal matrix. In order to distinguish the two representations, Terwilliger introduced the standard basis and split basis for this pair in [3]. A square matrix is said to be lower bidiagonal whenever every nonzero entry lies on either the diagonal or the subdiagonal. A lower bidiagonal is said to be irreducible lower bidiagonal whenever each entry on the subdiagonal is nonzero. A matrix is upper bidiagonal (resp. irreducible upper bidiagonal) whenever its transpose is lower bidiagonal (resp. irreducible lower bidiagonal).

Definition 2.2 Let V denote a vector space over $\mathcal{F}$ with finite positive dimension and let $\mathbb{A}, \mathbb{A}^*$ denote a Leonard pair on V. A standard basis for this pair is a basis for V with respect to which the matrix representing $\mathbb{A}$ is irreducible tridiagonal and the matrix representing $\mathbb{A}^*$ is diagonal.

Definition 2.3 Let V denote a vector space over $\mathcal{F}$ with finite positive dimension and let $\mathbb{A}, \mathbb{A}^*$ denote a Leonard pair on V. A split basis for this pair is a basis for V with respect to which the matrix representing $\mathbb{A}$ is irreducible lower bidiagonal and the matrix representing $\mathbb{A}^*$ is irreducible upper bidiagonal.

The following theorem in [3] provides a way to recognize a Leonard pair.

Theorem 2.4 (see [3]) Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. Let $\mathbb{A} : V \longrightarrow V$ and $\mathbb{B} : V \longrightarrow V$ denote linear transformations. Let us assume there exists a basis for V with respect to which the matrices representing $\mathbb{A}$ and $\mathbb{B}$ have the following form

$$A : \begin{pmatrix} \theta_0 & & & & \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}, \qquad B : \begin{pmatrix} \theta_0^* & \varphi_1 & & & \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_3 \\ & & & & \theta_d^* \end{pmatrix}. \tag{2.1}$$

Then the pair $\mathbb{A}, \mathbb{B}$ is a Leonard pair on V if and only if there exist scalars ϕ_i $(1 \le i \le d)$ in $\mathcal{F}$ such that conditions (i)–(v) hold below

(i) $\varphi_i \neq 0, \phi_i \neq 0$, $1 \le i \le d$;

(ii) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$, $0 \le i, j \le d$;

(iii) $\varphi_i = \phi_1 \sum\limits_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$, $1 \le i \le d$;

(iv) $\phi_i = \varphi_1 \sum\limits_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0)$, $1 \le i \le d$;

(v) The expressions $\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}$, $\frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$ are equal and independent of i for $2 \le i \le d-1$.

In the rest of this paper, we use the theorem to get Leonard pairs.

3 $\nu_q(sl_2)$-Modules

In this section, we recall some facts concerning irreducible finite-dimensional $\nu_q(sl_2)$-modules in [10].

Definition 3.1 The quantum algebra $\nu_q(sl_2)$ is defined as the associative algebra (with 1 and over $\mathcal{F}$) with the generators E, F, K, K^{-1}, H, H^{-1} and the following relations

$$\begin{aligned} &KK^{-1} = HH^{-1} = 1 = K^{-1}K = H^{-1}H, KH = HK, \\ &KEK^{-1} = q^2E, \quad HEH^{-1} = q^2E, \\ &KFK^{-1} = q^{-2}F, \quad HFH^{-1} = q^{-2}F, EF - q^{-2}FE = q^{-2}(KH - 1). \end{aligned}$$

Lemma 3.2 Let $t \geq 1$ be an integer. Then we have the following formulas in $\nu_q(sl_2)$,

$$\begin{aligned} EF^t &= q^{-2t}F^tE + F^{t-1}[q^{-4(t-1)}(t)_qHK - (t)_{q^{-1}}]q^{-2}, \\ FE^t &= q^{2t}E^tF + E^{t-1}[(t)_q - q^{4(t-1)}(t)_{q^{-1}}HK], \end{aligned}$$

where $(t)_q = 1 + q^2 + \cdots + q^{2(t-1)}$.

Lemma 3.3 Given an nonnegtative integer n and $a, b \in \mathcal{F}$ with $ab = q^{2n}$. Let M be a $n+1$-dimensional vector space with basis $\{m_0, m_1, \cdots, m_n\}$. We define the $\nu_q(sl_2)$-action on M as follows

$$\begin{aligned} Km_i &= aq^{-2i}m_i, \quad Hm_i = bq^{-2i}m_i \quad \text{for} \quad 0 \leq i \leq n, \\ Em_i &= \begin{cases} [q^{-4(i-1)}(i)_qab - (i)_{q^{-1}}]q^{-2}m_{i-1}, & \text{if} \quad 0 < i \leq n, \\ 0, & \text{if} \quad i = 0, \end{cases} \\ Fm_i &= \begin{cases} m_{i+1}, & \text{if} \quad 0 \leq i < n, \\ 0, & \text{if} \quad i = n. \end{cases} \end{aligned}$$

Then M becomes a $\nu_q(sl_2)$-module, we denote by $M(n, a, b)$.

Theorem 3.4 Suppose that V is a finite dimensional irreducible $\nu_q(sl_2)$-module with dimension $n+1$, then V is isomorphic to $M(n, a, b)$ for some $a, b \in \mathcal{F}$ with $ab = q^{2n}$.

We describe the construction of Leonard pairs from $\nu_q(sl_2)$-modules by using generators of $\nu_q(sl_2)$ in the next section.

4 Leonard Pairs From $\nu_q(sl_2)$

In this section, we define two linear transformations $\mathbb{A}$ and $\mathbb{B}$ of elements in $\nu_q(sl_2)$ and characterize when the pair $\mathbb{A}, \mathbb{B}$ is a Leonard pair.

Definition 4.1 Referring to Definition 3.1 and Lemma 3.3 , let α, β denote nonzero scalars in $\mathcal{F}$. Then define two linear transformations $\mathbb{A}, \mathbb{B}$ as follows

$$\mathbb{A} = \alpha F + \frac{a^{-1}K}{q^2-1}, \qquad \mathbb{B} = \frac{b}{q^2-1}\beta EH^{-1} + \frac{bH^{-1}}{q^2-1}. \tag{4.1}$$

Now we give the main result in this paper.

Theorem 4.2 Let n be an nonnegative integer and $a, b \in \mathcal{F}$ with $ab = q^{2n}$. Then the pair $\mathbb{A}, \mathbb{B}$ defined in (4.1) acts on $M(n, a, b)$ as a Leonard pair provided $\alpha\beta$ is not among $q^{-2}, q^{-4} \cdots, q^{-2n}$.

To prove the above theorem, we apply Theorem 2.4. Before do this, we first give some lemmas.

Lemma 4.3 There exists a basis for $M(n, a, b)$ with respect to which the matrices representing $\mathbb{A}$, $\mathbb{B}$ have the form of (2.1).

Proof We can obtain this basis by modifying the basis $\{m_0, m_1, \cdots, m_n\}$ given in Lemma 3.3. For $0 \leq i \leq n$, we define $u_i = \alpha^i m_i$. We observe $\{u_0, u_1, \cdots, u_n\}$ is a basis for

$M(n,a,b)$. The elements E, F, K, H act on this basis as follows

$$\begin{aligned} Ku_i &= aq^{-2i}u_i, \quad Hu_i = bq^{-2i}u_i \quad \text{for } 0 \le i \le n, \\ Eu_i &= \begin{cases} \alpha[q^{-4(i-1)}(i)_q ab - (i)_{q^{-1}}]q^{-2}u_{i-1}, & \text{if } 0 < i \le n, \\ 0, & \text{if } i = 0, \end{cases} \\ Fu_i &= \begin{cases} \alpha^{-1}u_{i+1}, & \text{if } 0 \le i < n, \\ 0, & \text{if } i = n. \end{cases} \end{aligned} \tag{4.2}$$

Take $ab = q^{2n}$ into (4.2), we can get the coefficient of u_{i-1} as below

$$\begin{aligned} \alpha[q^{-4(i-1)}(i)_q ab - (i)_{q^{-1}}]q^{-2} &= \alpha[q^{-4(i-1)}(i)_q q^{2n} - (i)_{q^{-1}}]q^{-2} \\ &= \alpha q^{-2i}[q^{2n-2i+2}(i)_q - (i)_{q^{-1}}q^{2i-2}] \\ &= \alpha q^{-2i}(i)_q(n-i+1)_q(q^2-1). \end{aligned}$$

Using these comments we can get

$$\mathbb{A}u_i = u_{i+1} + \frac{q^{-2i}}{q^2-1}u_i, \qquad \mathbb{B}u_i = \alpha\beta(i)_q(n-i+1)_q u_{i-1} + \frac{a^{-1}q^{2i}}{q^2-1}u_i,$$

where $u_{-1} = u_{n+1} = 0$. Thus, with respect to the basis $\{u_0, u_1, \cdots, u_n\}$ the matrices representing $\mathbb{A}$, $\mathbb{B}$ are given in (2.1), where

$$\begin{aligned} \theta_i &= \frac{q^{-2i}}{q^2-1}, \quad \theta_i^* = \frac{q^{2i}}{q^2-1} \quad (0 \le i \le n), \\ \varphi_i &= \alpha\beta(i)_q(n-i+1)_q \quad (0 \le i \le n). \end{aligned}$$

Lemma 4.4 Referring to Lemma 4.3, the following two equations hold

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = q^2 + q^{-2} + 1, \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = q^2 + q^{-2} + 1.$$

Proof Immediate from Lemma 4.3 and a simple calculation.

Lemma 4.5 Referring to Lemma 4.3, the scalars θ_i also satisfy the following equation

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{n-h}}{\theta_0 - \theta_n} = \frac{(i)_q(n-i+1)_q}{(n)_q}. \tag{4.3}$$

Proof Using the sum of the geometric progression, we have

$$(q^2-1)(t)_q = q^{2t} - 1. \tag{4.4}$$

Then from (4.4), equation (4.3) holds.

Proof of Theorem 4.1 Define $\phi_i = (i)_q(n-i+1)_q(\alpha\beta - q^{-2(n-i+1)}), 1 \le i \le n$. Let us assume $\alpha\beta$ is not among $q^{-2}, q^{-4} \cdots, q^{-2n}$. Then the above scalars θ_i, θ_i^*, φ_i, ϕ_i satisfy conditions (i)–(v) of Theorem 2.4 by Lemmas 4.3, 4.4 and 4.5.

Remark 1 Applying Theorem 2.4 we find the pair $\mathbb{A}$, $\mathbb{B}$ acts on $M(n,a,b)$ as a Leonard pair. With respect to the basis $\{u_0, u_1, \cdots, u_n\}$, the matrix representing $\mathbb{A}$ (resp. $\mathbb{B}$) is irreducible lower bidiagonal (resp. irreducible upper bidiagonal). Therefore this basis is a split basis for $\mathbb{A}$, $\mathbb{B}$ in view of Definition 2.3.

Remark 2 By the classification of Leonard pairs in [11], those with $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i} = q^2+q^{-2}+1$ are the families q-Racah, q-Hahn, dual q-Hahn, quantum q-Krawtchouk, affine q-Krawtchouk, q-Krawtchouk, or dual q-Krawtchouk, and since the pair $\mathbb{A}, \mathbb{B}$ has this property (see Lemma 4.4), it's easy to show that this pair is of quantum q-Krawtchouk type.

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勒纳德对与量子代数$\nu_q(sl_2)$

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摘要: 本文研究了勒纳德对的构造问题. 利用量子代数$\nu_q(sl_2)$的有限维既约表示, 获得了一系列的勒纳德对, 并讨论了它们的分类. 为进一步研究勒纳德三元组提供了帮助.

关键词: 勒纳德对; 量子代数; 既约表示; 分裂基

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