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# THE GROUND STATE OF THE CHERN-SIMONS-SCHRÖDINGER SYSTEM

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**Abstract:** We study the nonlinear Chern-Simons-Schrödinger system with superlinear nonlinearities. By the concentration compactness principle combined with the Nehari manifold, we prove the existence of positive ground state to this problem. Moreover, we obtain that the ground state has exponential decay at infinity.

**Keywords:** the ground state; the Chern-Simons-Schrödinger system; the variational method; the Nehari manifold; the concentration compactness principle

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## 1 Introduction and Main Result

We are interested in the existence of ground state $u \in H^1(\mathbb{R}^2)$ to the following Chern-Simons-Schrödinger system (**CSS** system)

$$\begin{cases} -\Delta u + u + A_0 u + \sum\limits_{j=1}^{2} A_j^2 u = |u|^{p-2}u, \\ \partial_1 A_0 = A_2|u|^2, \quad \partial_2 A_0 = -A_1|u|^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0, \end{cases} \tag{1.1}$$

where $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, and $p > 6$.

The CSS system comes from the study of the standing wave of Chern-Simons-Schrödinger system which describes the dynamics of large number of particles in a electromagnetic field and the physical background of the high-temperature superconductor, fractional quantum Hall effect and Aharovnov-Bohm scattering. For the more physical background of CSS system, we refer to the references that we will mention below and references therein. Recently, many scholars paid much attention to the CSS system proposed in [1], [2], and [3]. Berge, De Bouard, Saut [4] studied the blowing up time-dependent solution and Liu, Smith, Tataru [5] considered the local wellposedness. Byeon, Huh, Seok [6, 7], Huh [8, 9] investigated the

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existence and non-existence of standing wave solutions by variational methods. Authors in [10–13] obtained a series of existence results of solitary waves. Wan and Tan proved the existence, non-existence, and multiplicity of standing waves to the nonlinear **CSS** system with an external potential $V(x)$ without the Ambrosetti-Rabinowitz condition in [14], the existence of nontrivial solutions to Chern-Simons-Schrödinger systems (1.1) with $V(x)$ be a constant and the argument of global compactness with $p > 4$, $V(x) \in C(\mathbb{R}^2)$ and $0 < V_0 < V(x) < V_\infty$ in [15], and the concentration of standing waves to Chern-Simons-Schrödinger systems with $p > 6$, $V(x)$ satisfying the same condition as [15] in [16].

Inspired by [6, 14–15, 17–18], the purpose of the present paper is to study the existence of ground state for system (1.1) where $p > 6$. We consider this problem on the standard Nehari manifold, which is different from the idea in [15] where they use a constraint manifold of Pohozaev-Nehari type and [16] where $V(x)$ cannot be constants. The components $A_j$, $j = 0, 1, 2$ of the gauge field yield $p > 6$. The main characteristic of system (1.1) is that the non-local term $A_j$, $j = 0, 1, 2$ depend on $u$ and there is a lack of compactness in $\mathbb{R}^2$. By using the variational method joined with the Nehari manifold and concentration compactness principle [17], we can obtain the following result.

**Theorem 1.1** If $p > 6$, then (1.1) has a positive ground state which has exponential decay at infinity.

The paper is organized as follows. In Section 2, we introduce the framework as well as show some important propositions of $A_j$, $j = 0, 1, 2$ and some technical lemmas. In Section 3, we prove the compactness result and Theorem 1.1.

## 2 Mathematical Framework

Let $H^1(\mathbb{R}^2)$ denote the usual Sobolev space with

$$\|u\| = \Big( \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 \, dx \Big)^{1/2}.$$

Define the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \Big( |\nabla u|^2 + |u|^2 + A_1^2 |u|^2 + A_2^2 |u|^2 \Big) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \tag{2.1}$$

From (1.1), integrating by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} A_0 |u|^2 \, dx = -2 \int_{\mathbb{R}^2} A_0 (\partial_1 A_2 - \partial_2 A_1) \, dx \\ =&2 \int_{\mathbb{R}^2} (A_2 \partial_1 A_0 - A_1 \partial_2 A_0) \, dx = 2 \int_{\mathbb{R}^2} (A_1^2 + A_2^2) |u|^2 \, dx. \end{aligned} \tag{2.2}$$

We obtain the derivative of $J$ in $H^1(\mathbb{R}^2)$ as follow

$$\begin{aligned} &\langle J'(u), \eta \rangle \\ =& \int_{\mathbb{R}^2} \Big( \nabla u \nabla \eta + u\eta + (A_1^2(u) + A_2^2(u)) u\eta + A_0 u\eta - |u|^{p-2} u\eta \Big) \, dx \end{aligned} \tag{2.3}$$

for all $\eta \in C_0^\infty(\mathbb{R}^2)$. Specially, by (2.2), we have that

$$\langle J'(u),\, u\rangle = \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + |u|^2 + 3\big(A_1^2(u) + A_2^2(u)\big)|u|^2 - |u|^p\Big)\, dx. \tag{2.4}$$

Notice that

$$\Delta A_1 = \partial_2(\frac{|u|^2}{2}), \quad \Delta A_2 = -\partial_1(\frac{|u|^2}{2}),$$

which give

$$A_1 = A_1(u) = K_2 * (\frac{|u|^2}{2}) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x-y|^2}\frac{|u|^2(y)}{2}\, dy, \tag{2.5}$$

$$A_2 = A_2(u) = -K_1 * (\frac{|u|^2}{2}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x-y|^2}\frac{|u|^2(y)}{2}\, dy, \tag{2.6}$$

where $K_j = \frac{-x_j}{2\pi|x|^2}$ for $j = 1, 2$ and $*$ denotes the convolution. The identity $\Delta A_0 = \partial_1(A_2|u|^2) - \partial_2(A_1|u|^2)$ provides

$$A_0 = A_0(u) = K_1 * (A_1|u|^2) - K_2 * (A_2|u|^2). \tag{2.7}$$

As [15], we have that $J$ is well defined in $H^1(\mathbb{R}^2)$, $J \in C^1(H^1(\mathbb{R}^2))$, and the weak solution of (1.1) is the critical point of the functional $J$ by the following two properties.

**Proposition 2.1** (see [15]) Let $1 < s < 2$ and $\frac{1}{s} - \frac{1}{q} = \frac{1}{2}$.

(i) Then there is a constant $C$ depending only on $s$ and $q$ such that

$$\Big(\int_{\mathbb{R}^2} \big|Tu(x)\big|^q\, dx\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^2} |u(x)|^s\, dx\Big)^{\frac{1}{s}},$$

where the integral operator $T$ is given by

$$Tu(x) := \int_{\mathbb{R}^2} \frac{u(y)}{|x-y|}\, dy.$$

(ii) If $u \in H^1(\mathbb{R}^2)$, then we have that for $j = 1, 2$,

$$\|A_j^2(u)\|_{L^q(\mathbb{R}^2)} \le C\|u\|^2_{L^{2s}(\mathbb{R}^2)}$$

and

$$\|A_0(u)\|_{L^q(\mathbb{R}^2)} \le C\|u\|^2_{L^{2s}(\mathbb{R}^2)}\|u\|^2_{L^4(\mathbb{R}^2)}.$$

(iii) For $q' = \frac{q}{q-1}$, $j = 1, 2$,

$$\|A_j(u)u\|_{L^2(\mathbb{R}^2)} \le \||A_j(u)|^2\|_{L^q(\mathbb{R}^2)}\|u\|^2_{L^{2q'}(\mathbb{R}^2)}.$$

**Proposition 2.2** (see [15]) Suppose that $u_n$ converges to $u$ a.e. in $\mathbb{R}^2$ and $u_n$ converges weakly to $u$ in $H^1(\mathbb{R}^2)$. Let $A_{j,n} := A_j(u_n(x))$, $j = 0, 1, 2$. Then

(i) $A_{j,n}$ converges to $A_j(u(x))$ a.e. in $\mathbb{R}^2$.

(ii) $\int_{\mathbb{R}^2} A_{i,n}^2 u_n u\,dx$, $\int_{\mathbb{R}^2} A_{i,n}^2|u|^2\,dx$, and $\int_{\mathbb{R}^2} A_{i,n}^2|u_n|^2\,dx$ converge to $\int_{\mathbb{R}^2} A_i^2|u|^2\,dx$ for $i=1,2$; $\int_{\mathbb{R}^2} A_{0,n}u_n u\,dx$ and $\int_{\mathbb{R}^2} A_{0,n}|u_n|^2\,dx$ converge to $\int_{\mathbb{R}^2} A_0|u|^2\,dx$.

(iii) $\int_{\mathbb{R}^2} |A_i(u_n-u)|^2|u_n-u|^2\,dx = \int_{\mathbb{R}^2} |A_i(u_n)|^2|u_n|^2\,dx - \int_{\mathbb{R}^2} |A_i(u)|^2|u|^2\,dx + o_n(1)$ for $i=1,2$.

Next, we define the Nehari manifold related to the functionals above and discuss the property of the least energy of the critical points. Set $X := H^1(\mathbb{R}^2)$ and

$$\Sigma = \{u \in X \setminus \{0\} : \langle J'(u), u\rangle = 0\}.$$

**Lemma 2.3** If $p \geq 6$, then $\Sigma$ is a smooth manifold.

**Proof** Set

$$\omega(u) = \langle J'(u), u\rangle, \qquad u \in \Sigma.$$

Then

$$\langle \omega'(u), u\rangle = 2\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 + 9A_1^2u^2 + 9A_2^2u^2\right) dx - p\int_{\mathbb{R}^2} |u|^p\,dx.$$

By $u \in \Sigma$, we have

$$\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 + 3A_1^2u^2 + 3A_2^2u^2\right) dx = \int_{\mathbb{R}^2} |u|^p\,dx.$$

Therefore, if $p \geq 6$ we have

$$\langle \omega'(u), u\rangle = 2\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 + 9A_1^2u^2 + 9A_2^2u^2\right) dx - p\int_{\mathbb{R}^2} |u|^p\,dx < 0.$$

By the implicit function theorem, $\Sigma$ is a smooth manifold.

Now we can define critical values of the functional on the Nehari manifold. Let

$$c = \inf_{u\in\Sigma} J(u),\ c^* = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),\ c^{**} = \inf_{u\in X\setminus\{0\}}\max_{t\geq 0} J(tu),$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ J(\gamma(1)) < 0\}$. These critical values have the following property which is similar to Lemma 2.4 in [16]. For the reader's convenience, we show its proof.

**Lemma 2.4**

$$c = c^* = c^{**},\ \text{ if } p > 6.$$

**Proof** First, we show $c = c^{**}$. Indeed, this will follow if we can prove that for any $u \in X \setminus \{0\}$, the ray $R_t = \{tu :\ t \geq 0\}$ intersects the solution manifold $\Sigma$ once and only once at $\theta u$ ($\theta > 0$), where $J(\theta u)$, $\theta \geq 0$, achieves its maximum.

$$\begin{aligned}\langle J'(tu),\, tu\rangle = &\int_{\mathbb{R}^2} \Big(t^2|\nabla u|^2 + t^2u^2 + 3t^6(K_2 * \frac{1}{2}u^2)u^2 \\ &+ 3t^6(-K_1 * \frac{1}{2}u^2)u^2 - t^p|u|^p\Big)\,dx \\ =& t^2\Big(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx + 3t^4\int_{\mathbb{R}^2} \left(A_1^2u^2 + A_2^2u^2\right) dx - t^{p-2}\int_{\mathbb{R}^2} |u|^p\,dx\Big).\end{aligned}$$

Let

$$h(t) = A + t^4 B - t^{p-2} D, \quad t \in [0, +\infty],$$

where

$$A = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx,\ B = 3\int_{\mathbb{R}^2} \left(A_1^2 u^2 + A_2^2 u^2\right) dx,\ D = \int_{\mathbb{R}^2} |u|^p \, dx.$$

We claim that there exists a unique $t_0 \in (0, +\infty)$ such that $h(t_0) = 0$. In fact, by computing, we have that

$$\begin{cases} h'' > 0, & t < t_1 := \left(\frac{12B}{(p-2)(p-3)D}\right)^{\frac{1}{p-6}}, \\ h'' < 0, & t > t_1 := \left(\frac{12B}{(p-2)(p-3)D}\right)^{\frac{1}{p-6}}. \end{cases}$$

Also, there exist $t_2 = 0$, $t_3 = \left(\frac{4B}{(p-2)D}\right)^{\frac{1}{p-6}}$ satisfying $t_2 < t_1 < t_3$, such that $h'(t) = 0$ and $h(t)$ is strictly decreasing for $t \geq t_3$ as well as strictly increasing for $t \leq t_3$. Since $h(t_2) = A > 0$ and $h(t) \to -\infty$ as $t \to +\infty$, there exists a unique $t_0 > t_3$ such that $h(t_0) = 0$. Therefore, the ray $R_t$ intersects $\Sigma$ only once. We obtain that $c = c^{**}$.

Next, we show $c^* = c^{**}$. It is easy to see that $c^{**} \geq c^*$. Let us prove $c^{**} \leq c^*$. For $u \in X \setminus \{0\}$ fixed, let $\bar{t}$ be the unique point such that $\bar{t}u \in \Sigma$. Then, we can write

$$c^{**} = \inf_{u \in K} J(u)$$

with

$$K = \{\bar{u} = \bar{t}u : \ u \in X, u \neq 0, \bar{t} < \infty\}.$$

Let $\gamma \in \Gamma$ be a path. If for all $\gamma \in \Gamma$, $\gamma \cap K \neq \emptyset$, then the inequality is obtained. If there exists $\gamma \in \Gamma$ such that $\gamma(t) \notin K$ for all $t \in [0, 1]$, then we have

$$\int_{\mathbb{R}^2} \left(|\nabla\gamma|^2 + \gamma^2 + 3A_1^2(\gamma)\gamma^2 + 3A_2^2(\gamma)\gamma^2\right) dx > \int_{\mathbb{R}^2} |\gamma|^p \, dx.$$

If $p > 6$, then

$$\begin{aligned} J(\gamma) =& \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla\gamma|^2 + \gamma^2 + A_1^2(\gamma)\gamma^2 + A_2^2(\gamma)\gamma^2\Big) dx - \frac{1}{p}\int_{\mathbb{R}^2} |\gamma|^p \, dx \\ >& \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla\gamma|^2 + \gamma^2 + A_1^2(\gamma)\gamma^2 + A_2^2(\gamma)\gamma^2\Big) dx \\ &- \frac{1}{p}\int_{\mathbb{R}^2} \Big(|\nabla\gamma|^2 + \gamma^2 + 3A_1^2(\gamma)\gamma^2 + 3A_2^2(\gamma)\gamma^2\Big) dx \\ >& 0, \end{aligned}$$

which contradicts $J(\gamma(1)) < 0$. Hence $c^* = c^{**}$.

## 3 The Proof of Theorem 1.1

First, we should obtain the following compactness result to prove Theorem 1.1.

**Lemma 3.1** Let $\{\bar{u}_n\}$ be a minimizing sequence of $c$. If $p>6$, then

(i) there exists $\{u_n\}\subset\Sigma$ such that $J(u_n)\to c,\quad J'(u_n)\to 0,$ and $\|u_n-\bar{u}_n\|_X\to 0$ as $n\to\infty$;

(ii) there exists $\{\xi_n\}\subset\mathbb{R}^2$ such that $\{v_n\}$ is precompact, where $v_n(\cdot):=u_n(\cdot+\xi_n)$.

**Proof** (i) It is a direct consequence of the Ekeland's variational principle (see [18]).

(ii) We are going to use the concentration compactness principle given in [17]. Since $J(u_n)\to c$ as $n\to\infty$, $u_n\in\Sigma$, and $p>6$ for $n$ large, we have

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^2}\left(|\nabla u_n|^2+u_n^2\right)dx+\left(\frac{1}{2}-\frac{3}{p}\right)\int_{\mathbb{R}^2}\left(A_{1,n}^2u_n^2+A_{2,n}^2u_n^2\right)dx=c+o(1),$$

where $A_{1,n}:=A_1(u_n)$ and $A_{2,n}:=A_2(u_n)$. Then, $\{u_n\}$ is bounded in $X$. For any $n\in\mathbb{N}$, we consider the following measure

$$\mu_n(\Omega)=\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega}\left(|\nabla u_n|^2+u_n^2\right)dx+\left(\frac{1}{2}-\frac{3}{p}\right)\int_{\Omega}\left(A_{1,n}^2u_n^2+A_{2,n}^2u_n^2\right)dx.$$

From the concentration compactness lemma in [19], there is a subsequence of $\{\mu_n\}$, which we will always denote by $\{\mu_n\}$, satisfying one of the three following possibilities.

**Vanishing** Assume that there exists a subsequence of $\{\mu_n\}$, such that for all $\rho>0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_\rho(y)}d\mu_n=0.$$

**Dichotomy** Suppose there exist a constant $\bar{c}$ with $0<\bar{c}<c$, sequences $\{\xi_n\}\subset\mathbb{R}^2$, $\{\rho_n\}$ such that $|\xi_n|,\{\rho_n\}\to\infty$ and two nonnegative measures $\mu_n^1$ and $\mu_n^2$ satisfying the following

$$\begin{aligned}&0\le\mu_n^1+\mu_n^2\le\mu_n,\\&\operatorname{supp}(\mu_n^1)\subset B_{\rho_n}(\xi_n),\operatorname{supp}(\mu_n^2)\subset B_{2\rho_n}^c(\xi_n),\\&\mu_n^1(\mathbb{R}^2)\to\bar{c},\quad\mu_n^2(\mathbb{R}^2)\to c-\bar{c}\quad\text{as}\ \ n\to\infty.\end{aligned}$$

**Compactness** There is a sequence $\{\xi_n\}\subset\mathbb{R}^N$ such that for any $\delta>0$ there exists a radius $\rho>0$ such that

$$\int_{B_\rho(\xi_n)}d\mu_n\ge c-\delta\ \text{ for all } n. \tag{3.1}$$

**Step 1** We show vanishing is impossible.

If $\{\mu_n\}$ is vanishing, then $\{u_n\}$ is also vanishing. Namely, there is a subsequence of $\{u_n\}$ such that for all $\rho>0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_\rho(y)}\left(|\nabla u_n|^2+u_n^2\right)dx=0.$$

By the Lion's lemma [17], $u_n\to 0$, in $L^s(\mathbb{R}^2)$, $s>2$. Because

$$\begin{aligned}\langle J'(u_n),\,u_n\rangle&=\int_{\mathbb{R}^2}\left(|\nabla u_n|^2+u_n^2+3A_{1_n}^2u_n^2+3A_{2_n}^2u_n^2-|u_n|^p\right)dx\\&\to 0\ \text{ as } n\to\infty,\end{aligned}$$

and $\int_{\mathbb{R}^2} |u_n|^p \, dx \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \Big( |\nabla u_n|^2 + u_n^2 + 3A_{1_n}^2 u_n^2 + 3A_{2_n}^2 u_n^2 \Big) \, dx = 0.$$

Consequently

$$\begin{aligned} 0 = &\lim_{n\to\infty} \Big( (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^2} \big( |\nabla u_n|^2 + u_n^2 \big) \, dx \\ &+ (\frac{1}{2} - \frac{3}{p}) \int_{\mathbb{R}^2} \big( A_{1_n}^2 u_n^2 + A_{2_n}^2 u_n^2 \big) \, dx \Big) \\ =& c > 0, \end{aligned}$$

which is impossible.

**Step 2** We prove dichotomy is impossible.

Take a cut-off function $\phi_n \in C_0^1(\mathbb{R}^2)$ such that $\phi_n \equiv 1$ in $B_{\rho_n}(\xi_n)$, $\phi_n \equiv 0$ in $B_{2\rho_n}^c(\xi_n)$, $0 \le \phi_n \le 1$, and $|\nabla \phi_n| \le 2/\rho_n$. Write $u_n := u_{1,n} + u_{2,n}$, where

$$u_{1,n} := \phi_n u_n, \quad u_{2,n} := (1 - \phi_n) u_n.$$

Define

$$I(u) = (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^2} \big( |\nabla u|^2 + u^2 \big) \, dx + (\frac{1}{2} - \frac{3}{p}) \int_{\mathbb{R}^2} \big( A_1^2 u^2 + A_2^2 u^2 \big) \, dx.$$

We know that $u_{2,n}$ converges to 0 a.e. in $\mathbb{R}^2$, and $A_j(u_{2,n}) \to 0$ a.e. in $\mathbb{R}^2$.

If $\|(1 - \phi_n) u_n\|$ is bounded and $\mathrm{supp}((1 - \phi_n) u_n) \subset B_{\rho_n}^c$, then the Hölder inequality yields

$$\begin{aligned} |A_j((1 - \phi_n) u_n)| &\le C \|u_n^2\|_{L^{\frac{4}{3}}(B_{\rho n}^c(x))} \Big( \int_{B_{\rho n}^c(x)} \frac{dy}{|x - y|^4} \, dy \Big)^{\frac{1}{4}} \\ &\le C \frac{1}{\rho_n^{1/2}} \xrightarrow{n} 0 \end{aligned}$$

and

$$\begin{aligned} &\Big| \int_{\mathbb{R}^2} K_j(x - y)(1 - \phi_n)\phi_n |u_n(y)|^2 \, dy \Big| \\ \le &\|u_n^2\|_{L^{\frac{4}{3}}(\Omega_n)} \Big( \int_{\Omega_n} \frac{dy}{|x - y|^4} \, dy \Big)^{\frac{1}{4}} \le C \frac{1}{\rho_n^{1/2}} \xrightarrow{n} 0, \end{aligned}$$

where

$$\Omega_n := B_{2\rho_n}(\xi_n) \backslash B_{\rho_n}(\xi_n). \tag{3.2}$$

By $\|u_n\| \le C$, we have

$$\lim_{n\to\infty} A_j((1 - \phi_n) u_n) = 0, \tag{3.3}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} A_1(u_{1,n}) A_1(u_{2,n}) |u_{j,n}|^2 \, dx = 0. \tag{3.4}$$

For $q'=\frac{q}{q-1}$, $\frac{1}{s}-\frac{1}{q}=\frac{1}{2}$, we get

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} A_j^2(u_{2,n})u_{1,n}^2(x)\,dx\Big| &\le \|A_j^2(u_{2,n})\|_{L^q(\mathbb{R}^2)}\|u_{1,n}\|^2_{L^{2q'}(\mathbb{R}^2)}\\ &\le C\|u_{2,n}\|^2_{L^{2s}(\mathbb{R}^2)}\|u_{1,n}\|^2_{L^{2q'}(\mathbb{R}^2)}.\end{aligned}$$

Then

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}|A_j(u_{2,n})|^2|u_{1,n}|^2\,dx=0. \tag{3.5}$$

It is clear that $\liminf\limits_{n\to\infty} I(u_{1,n})\ge \bar{c}$ and $\liminf\limits_{n\to\infty} I(u_{2,n})\ge c-\bar{c}$. Furthermore, we have

$$\mu_n(\Omega_n)\to 0 \text{ as } n\to\infty,$$

namely,

$$\begin{aligned}\int_{\Omega_n}\left(|\nabla u_n|^2+u_n^2\right)dx\to 0 \ \text{ as } n\to\infty,\\ \int_{\Omega_n}\left(A_{1,n}^2u_n^2+A_{2,n}^2u_n^2\right)dx\to 0 \ \text{ as } n\to\infty.\end{aligned} \tag{3.6}$$

Then we obtain

$$\begin{aligned}\int_{\Omega_n}\left(|\nabla u_{1,n}|^2+u_{1,n}^2\right)dx\to 0 \ \text{ as } n\to\infty,\\ \int_{\Omega_n}\left(|\nabla u_{2,n}|^2+u_{2,n}^2\right)dx\to 0 \ \text{ as } n\to\infty.\end{aligned}$$

Thus we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left(|\nabla u_n|^2+u_n^2\right)dx=&\int_{\mathbb{R}^2}\left(|\nabla u_{1,n}|^2+u_{1,n}^2\right)dx\\ &+\int_{\mathbb{R}^2}\left(|\nabla u_{2,n}|^2+u_{2,n}^2\right)dx+o_n(1),\end{aligned} \tag{3.7}$$

$$\int_{\mathbb{R}^2}|u_n|^p\,dx=\int_{\mathbb{R}^2}|u_{1,n}|^p\,dx+\int_{\mathbb{R}^2}|u_{2,n}|^p\,dx+o_n(1). \tag{3.8}$$

We note that

$$\begin{aligned}A_{1,n}=&-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_2-y_2}{|x-y|^2}\frac{1}{2}|u_{1,n}+u_{2,n}|^2\,dy\\ =&A_1(u_{1,n})+A_1(u_{2,n})-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_2-y_2}{|x-y|^2}u_{1,n}u_{2,n}\,dy\\ =&A_1(u_{1,n})+A_1(u_{2,n})+o_n(1),\end{aligned}$$

then

$$\int_{\mathbb{R}^2} A_1^2(u_n)|u_n|^2\,dx = \int_{\mathbb{R}^2} \big(A_1(u_{1,n}) + A_1(u_{2,n}) + o_n(1)\big)^2 |u_{1,n} + u_{2,n}|^2\,dx$$

$$= \int_{\mathbb{R}^2} \big[A_1^2(u_{1,n})|u_{1,n}|^2 + A_1^2(u_{2,n})|u_{2,n}|^2 + 2A_1(u_{1,n})A_1(u_{2,n})\big(|u_{1,n}|^2 + |u_{2,n}|^2\big)$$

$$+ A_1^2(u_{1,n})|u_{2,n}|^2 + A_1^2(u_{2,n})|u_{1,n}|^2 + 2\big(A_1^2(u_{1,n}) + A_1^2(u_{2,n})\big)u_{1,n}u_{2,n}$$

$$+ 4A_1(u_{1,n})A_1(u_{2,n})u_{1,n}u_{2,n}\big]\,dx + o_n(1).$$

By (3.3), (3.4), (3.5), and (3.6), we have

$$\int_{\mathbb{R}^2} A_1^2(u_n)|u_n|^2\,dx = \int_{\mathbb{R}^2} A_1^2(u_{1,n})|u_{1,n}|^2\,dx + \int_{\mathbb{R}^2} A_1^2(u_{2,n})|u_{2,n}|^2\,dx + o_n(1). \tag{3.9}$$

Similarly, we get

$$\int_{\mathbb{R}^2} A_2^2(u_n)|u_n|^2\,dx = \int_{\mathbb{R}^2} A_2^2(u_{1,n})|u_{1,n}|^2\,dx + \int_{\mathbb{R}^2} A_2^2(u_{2,n})|u_{2,n}|^2\,dx + o_n(1). \tag{3.10}$$

Thus, by (3.7), (3.9), and (3.10), we have

$$I(u_n) = I(u_{1,n}) + I(u_{2,n}) + o_n(1).$$

Then

$$c = \lim_{n\to\infty} I(u_n) \geq \liminf_{n\to\infty} I(u_{1,n}) + \liminf_{n\to\infty} I(u_{2,n}) \geq \bar{c} + (c - \bar{c}) = c.$$

Hence

$$\lim_{n\to\infty} I(u_{1,n}) = \bar{c},\ \ \lim_{n\to\infty} I(u_{2,n}) = c - \bar{c}. \tag{3.11}$$

Let

$$g(u) := \langle J'(u),\, u\rangle = \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + u^2 + 3A_1^2u^2 + 3A_2^2u^2 - |u|^p\Big)\,dx.$$

By (3.7)–(3.10), and $u_n \in \Sigma$, we obtain

$$0 = g(u_n) \geq g(u_{1,n}) + g(u_{2,n}) + o_n(1). \tag{3.12}$$

By Lemma 2.4, for any $n \geq 1$, $\exists \theta_n > 0$ such that $\theta_n u_{1,n} \in \Sigma$, and then

$$\begin{aligned} &\theta_n^2 \int_{\mathbb{R}^2} \big(|\nabla u_{1,n}|^2 + V(x)u_{1,n}^2\big)\,dx + \theta_n^6 \int_{\mathbb{R}^2} 3\big[A_1^2(u_{1,n})u_{1,n}^2 + A_2^2(u_{1,n})u_{1,n}^2\big]\,dx \\ = \ &\theta_n^p \int_{\mathbb{R}^2} |u_{1,n}|^p\,dx. \end{aligned} \tag{3.13}$$

**Case 1** Up to a subsequence, $g(u_{1,n}) \leq 0$. By (3.13) and $p > 6$, we obtain

$$\begin{aligned} &\big(\theta_n^p - \theta_n^2\big) \int_{\mathbb{R}^2} \big(|\nabla u_{1,n}|^2 + u_{1,n}^2\big)\,dx \\ &+\big(\theta_n^p - \theta_n^6\big) \int_{\mathbb{R}^2} 3\big[A_1^2(u_{1,n})u_{1,n}^2 + A_2^2(u_{1,n})u_{1,n}^2\big]\,dx \leq 0, \end{aligned}$$

which yields that $\theta_n \le 1$. Hence, for all $n \ge 1$,

$$c \le J(\theta_n u_{1,n}) = I(\theta_n u_{1,n}) \le I(u_{1,n}) \to \bar{c} < c \text{ as } n \to \infty,$$

which is impossible.

**Case 2** Up to a subsequence, $g(u_{2,n}) \le 0$. We can have the argument as in Case 1.

**Case 3** Up to a subsequence, $g(u_{1,n}) > 0$ and $g(u_{2,n}) > 0$. By (3.12), we have that $g(u_{1,n}) = o_n(1)$ and $g(u_{2,n}) = o_n(1)$. If $\theta_n \le 1 + o_n(1)$, we can repeat the arguments of Case 1. Assume that $\lim\limits_{n\to\infty} \theta_n = \theta_0 > 1$, we get

$$\begin{aligned} o_n(1) =& g(u_{1,n}) \\ =& \int_{\mathbb{R}^2} \Big(|\nabla u_{1,n}|^2 + u_{1,n}^2 + 3A_1^2(u_{1,n})u_{1,n}^2 + 3A_2^2(u_{1,n})u_{1,n}^2 - |u_{1,n}|^p\Big)\, dx \\ =& \big(1 - \frac{1}{\theta_n^{p-2}}\big) \int_{\mathbb{R}^2} \big(|\nabla u_{1,n}|^2 + V(x)u_{1,n}^2\big)\, dx \\ &+ 3\big(1 - \frac{1}{\theta_n^{p-6}}\big) \int_{\mathbb{R}^2} \big(A_1^2(u_{1,n})u_{1,n}^2 + A_2^2(u_{1,n})u_{1,n}^2\big)\, dx. \end{aligned}$$

Hence, $u_{1,n} \to 0$ as $n \to \infty$ in $X$. Then, we obtain a contradiction with (3.11).

**Step 3** We show the strong convergence. By the proof above, we have that there exists a subsequence of $\{\mu_n\}$ such that it is compact, that is, there is a sequence $\{\xi_n\} \subset \mathbb{R}^N$ such that for any $\delta > 0$, there exists a radius $\rho > 0$ such that

$$\int\limits_{B_\rho(\xi_n)} d\mu_n \ge c - \delta \text{ for all } n.$$

We define the new sequence of functions $v_n(\cdot) = u_n(\cdot - \xi_n) \in X$. We have that $A_j(v_n(\cdot)) = A_j(u_n(\cdot - \xi_n))$, $j = 1, 2$ and thus $v_n \in \Sigma$. Furthermore, from (3.1), we have that for any $\delta > 0$, there is a radius $\rho > 0$ such that

$$\|v_n\|_{H^1(B_\rho^c)} < \delta \text{ uniformly for } n \ge 1. \tag{3.14}$$

By $\{u_n\}$ is bounded in $X$, $\{v_n\}$ is also bounded in $X$. Then, there exist a subsequence of $\{v_n\}$ and $u \in X$ such that

$$v_n \rightharpoonup u \text{ weakly in X;} \tag{3.15}$$

$$v_n \to u \text{ in } L^s_{\text{loc}}(\mathbb{R}^2) \text{ for } 1 \le s < +\infty \tag{3.16}$$

as $n \to \infty$. According to (3.14), (3.15), and (3.16), we obtain that, taken $s \in (2, +\infty)$, for any $\delta > 0$, there exists $\rho > 0$ such that for any $n \ge 1$ sufficiently large

$$\begin{aligned} \|v_n - u\|_{L^s(\mathbb{R}^2)} \le& \|v_n - u\|_{L^s(B_\rho)} + \|v_n - u\|_{L^s(B_\rho^c)} \\ \le& \delta + C\big(\|v_n\|_{H^1(B_\rho^c)} + \|u\|_{H^1(B_\rho^c)}\big) \\ \le& (1 + 2C)\delta, \end{aligned}$$

where $C>0$ is the constant of the embedding $H^1(B_\rho^c)\subset L^s(B_\rho^c)$. We have

$$v_n\to u \ \text{ in } L^s(\mathbb{R}^2) \text{ for any } s\in(2,+\infty). \tag{3.17}$$

Thus

$$v_n\to u \ \text{ a.e in } \mathbb{R}^2. \tag{3.18}$$

We have

$$\langle J'(v_n), v_n-u\rangle\to 0 \ \text{ as } n\to\infty.$$

From (3.15), (3.17), (3.18), and Proposition 2.2, we have $\|v_n\|\to\|u\|$ as $n\to\infty$, which yields $v_n$ strongly converges to $u$ in $X$.

**The Proof of Theorem 1.1** From the proof of (ii) in Lemma 3.1, we get $u\in\Sigma$ and $J(u)=c$. Thus, $u$ is a ground state solution of (1.1). Next, we show that $u\in H^1(\mathbb{R}^2)$ does not change sign. Let $u_+=\max\{u,0\}$ and $u_-=\max\{-u,0\}$, then $u=u_+-u_-$. We have that

$$J(u)=\left(\frac12-\frac16\right)\int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)dx+\left(\frac16-\frac1p\right)\int_{\mathbb{R}^2}|u|^p\,dx.$$

We obtain

$$J(u_+), J(u_-)<J(u_+-u_-),$$

which implies $u_-\equiv 0$ or $u_+\equiv 0$. We suppose that $u\ge 0$. Combining the Sobolev theorem with the Moser iteration to weak solution $u\in H^1(\mathbb{R}^2)$ of (1.1), we know that $u$ is bounded in $L^\infty(\mathbb{R}^2)$. Hence, for all $q\in[2,\infty)$, there exists $C_1$ such that $\|u\|_{W^{1,q}(\mathbb{R}^2)}\le C_1$. Furthermore, we obtain that $u\in C^\alpha(\mathbb{R}^2)$ for some $\alpha\in(0,1)$. Then, we have $u\in\bigcap_{q=2}^\infty W^{2,q}(\mathbb{R}^2)$ by the standard bootstrap argument. Consequently, we obtain $u\in C^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha\in(0,1)$ by the classical elliptic estimate. From the maximum principle, we achieve that $u\ge 0$. Last, we show the ground state $u$ has exponential decay at infinity. Let $\Psi(x)=Me^{-\theta(|x|-L)}$, where $M=\max\{|u(x)|: |x|=L\}$ and $0<\theta<1$. Then we have $\Delta\Psi=(\theta^2-\frac{\theta}{|x|})\Psi$. Define

$$\phi_R=\begin{cases} 0, & x\in B_R^o,\\ b_1u-\Psi, & x\in\mathbb{R}^2\backslash B_R^o\end{cases}$$

with $b_1>0$. From (2.3), choosing $\eta=\phi_R$, we obtain

$$\int_{\mathbb{R}^2}\left(|\nabla\phi_R|^2+|\phi_R|^2\right)dx\le\int_{\mathbb{R}^2}\left((\theta^2-\frac{\theta}{|x|})-1\right)\Psi\phi_R\,dx+\int_{\mathbb{R}^2}b_1|u|^{p-2}u\phi_R\,dx+o_R(1).$$

Let $R>0$ such that $|u|^{p-2}\le 1-\theta^2$ for $|x|>R$. Then

$$\begin{aligned}\int_{|x|>R}\phi_R^2\,dx&\le\int_{|x|>R}\left(|\nabla\phi_R|^2+|\phi_R|^2\right)dx\le\int_{|x|>R}(b_1u-\Psi)(1-\theta^2)\phi_R\,dx+o_R(1)\\ &=(1-\theta^2)\int_{|x|>R}\phi_R^2\,dx+o_R(1).\end{aligned}$$

Consequently, $\phi_R\equiv 0$. It implies exponential decay at infinity.

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# Chern-Simons-Schrödinger方程组的基态解

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**摘要**: 本文研究了带超线性非线性项的陈-西蒙斯-薛定谔方程组. 利用集中紧致原理和Nehari流形,证明了该方程组基态解的存在性, 得到了该基态解在无穷远处是指数衰减的.

**关键词**: 基态解; 陈-西蒙斯-薛定谔方程组; 变分法;Nehari流形; 集中紧致原理

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