Vol. 39 (2019)
No. 4

数 学 杂 志
J. of Math. (PRC)

ASYMPTOTIC BEHAVIOR OF COMPRESSIBLE NAVIER-STOKES FLUID IN POROUS MEDIUM

YUAN Guo-zhi, ZHAO Hong-xing
(*College of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China*)

Abstract: We study the behavior of the solution to the full compressible Navier-Stokes fluid in porous medium. By using standard energy and two-scale convergence, we prove the strong convergence of the density and the temperature with characteristic size of the pores ε in R^n for $n = 2$ or 3 and obtain the homogenized for this model, when $\varepsilon \to 0$, which gives another explanation to the results in references.

Keywords: asymptotic analysis; homogenization; Navier-Stokes flow; Gibbs' equation
2010 MR Subject Classification: 76D05; 76M50; 74Q10
Document code: A **Article ID:** 0255-7797(2019)04-0529-14

1 Introduction

Homogenization is a mathematical tool that allows changing the scale in problems containing several characteristic scales. Typical examples of its utilization are finding effective models for composite materials, in optimal shape design, etc. Another important example, which we are interested in, is the fluid mechanics of the flow through porous medium.

In porous medium, there are at least two length scales: microscopic scale and macroscopic scale. The partial differential equations describing a physical phenomenon are posed at the microscopic level whereas only macroscopic quantities are of interest for the engineers or the physicists. Therefore, effective or homogenized equations should be derived from the microscopic ones by an asymptotic analysis. To this end, it is convenient to assume that the porous medium has a periodic structure.

A number of known laws from the dynamics of fluids in porous media were derived using homogenization. The most well-known example is Darcy's law, being the homogenized equation for one-phase flow through a rigid porous medium. Its formal derivation by two-scale expansion goes back to the classical paper by Sanchez- Palencia [1], Keller [2] and the classical book Bensoussan [3]. It was rigorously derived by using oscillating functions by Tartar [4]. In other cases of periodic porous media, we refer the readers to the papers by Allaire [5–8] and Mikelic [9, 10]. Other works can be seen in [11–13] and the references

* **Received date:** 2017-11-20 **Accepted date:** 2018-05-02
Foundation item: Supported by the research of high schools of Guangzhou (1201420925) and GDNSF (2014A030313526).

Biography: Yuan Guozhi (1992–), male, born at Guangzhou, Guangdong, master, major at partial differential equations. E-mail address: shenggaoxii@163.com.

therein. Besides the Darcy law, Brinkman [14] introduced a new set of equations, which is called the Brinkman law, an intermediate law between the Darcy and Stokes equations. The so-called Brinkman law is obtained from the Stokes equations by adding to the momentum equation a term proportional to the velocity (see [7]).

Inspired by the work from Feireisl [11], we consider the asymptotic behavior of a compressible fluid in a periodic medium. Before stating the system, let us recall the domain we consider. A porous medium is defined as the periodic repetition of an elementary cell of size ε (we assume that $\frac{1}{\varepsilon}$ to be an integral) in a bounded domain Ω of R^n with $n = 2, 3$. The solid part of the porous medium is also taken of size ε. The domain Ω_ε is then defined as the intersection of Ω with the fluid part. We consider the density dependent fluid governed by the full compressible Navier-Stokes equations. So, we have the following equations

$$\begin{cases} \varepsilon^2 \dfrac{\partial \rho_\varepsilon}{\partial t} + \mathrm{div}(\rho_\varepsilon u_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon \times (0,T), \\ \varepsilon^2 \dfrac{\partial(\rho_\varepsilon u_\varepsilon)}{\partial t} + \mathrm{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \mathrm{div}(\mu \nabla u_\varepsilon) + \nabla p_\varepsilon = \rho_\varepsilon f \quad \text{in } \Omega_\varepsilon \times (0,T), \\ \varepsilon^2 \dfrac{\partial}{\partial t}\{\rho_\varepsilon(\dfrac{|u_\varepsilon|^2}{2} + e_\varepsilon)\} + \mathrm{div}\{u_\varepsilon(\rho_\varepsilon \dfrac{|u_\varepsilon|^2}{2} + \rho_\varepsilon e_\varepsilon + p_\varepsilon)\} \\ = \mathrm{div}(\kappa \nabla \mathbb{T}_\varepsilon) + \mathrm{div}(\mu \nabla u_\varepsilon \cdot u_\varepsilon) + \rho_\varepsilon u_\varepsilon f \ \text{ in } \Omega_\varepsilon \times (0,T), \end{cases} \tag{1.1}$$

where $u_\varepsilon, \rho_\varepsilon, \mathbb{T}_\varepsilon$ are the unknown quantities velocity, density and temperature. $p_\varepsilon = p_\varepsilon(\rho_\varepsilon, \mathbb{T}_\varepsilon)$ is the pressure, $e_\varepsilon = e_\varepsilon(\rho_\varepsilon, \mathbb{T}_\varepsilon)$ is the internal energy, f (the exterior force) is given on $\Omega \times (0,T)$. We assume that f is smooth enough. $T \in (0,\infty)$ is fixed; $\kappa = \kappa(\mathbb{T}_\varepsilon), \mu = \mu(\mathbb{T}_\varepsilon)$ are positive for $\mathbb{T}_\varepsilon \geq 0$ and $\kappa, \mu \in W^{1,\infty}([0,\infty))$.

We also assume that u_ε satisfies

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \times (0,T), \tag{1.2}$$

and in order to fix ideas we impose Neumann boundary conditions on $\mathbb{T}_\varepsilon$ namely

$$\frac{\partial \mathbb{T}_\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon \times (0,T), \tag{1.3}$$

where n, as usual, the unit outward normal to $\partial\Omega_\varepsilon$.

In this paper, we assume that the initial conditions

$$\rho_\varepsilon|_{t=0} = \rho_{0,\varepsilon}, \quad u_\varepsilon|_{t=0} = u_{0,\varepsilon}, \ \ \mathbb{T}_\varepsilon|_{t=0} = \mathbb{T}_{0,\varepsilon}, \ \ e_\varepsilon|_{t=0} = e_{0,\varepsilon} \tag{1.4}$$

are bounded in $L^\infty(\Omega_\varepsilon)$.

In this paper, we also assume that the transport coefficients $\mu(\mathbb{T}_\varepsilon)$ and $\kappa(\mathbb{T}_\varepsilon)$ satisfying the following conditions

$$\begin{aligned} &\kappa, \ \mu \in W^{1,\infty}[0,+\infty), \ \ 0 < \underline{\mu}(1+\mathbb{T}_\varepsilon) \leq \mu(\mathbb{T}_\varepsilon), \\ &0 < \underline{\kappa}(1+\mathbb{T}_\varepsilon^3) \leq \kappa(\mathbb{T}_\varepsilon) \leq \overline{\kappa}(1+\mathbb{T}_\varepsilon^3) \end{aligned} \tag{1.5}$$

for all $\mathbb{T}_\varepsilon \geq 0$. $\underline{\mu}, \underline{\kappa}, \overline{\kappa}$ are positive constants.

Let us recall that the equation with temperature in (1.1), it is equivalent (at least formally) to

$$\varepsilon^2\frac{\partial}{\partial t}(\rho_\varepsilon e_\varepsilon)+\mathrm{div}(\rho_\varepsilon u_\varepsilon e_\varepsilon)-\mathrm{div}(\kappa\nabla\mathbb{T}_\varepsilon)-\mu|\nabla u_\varepsilon|^2+p_\varepsilon\mathrm{div}u_\varepsilon=0. \tag{1.6}$$

For simplicity, in this paper, we consider the models in astrophysics and the state equation for the pressure p_ε and the internal energy e_ε satisfying the Joule's law (see [15])

$$p_\varepsilon=a\rho_\varepsilon^\gamma+b\rho_\varepsilon^\beta\mathbb{T}_\varepsilon,\qquad e_\varepsilon=\frac{a}{\gamma-1}\rho_\varepsilon^{\gamma-1}+c\mathbb{T}_\varepsilon^{\frac{3}{2}},$$

where $\gamma\geq n$ for $n=2,3$, $\frac{3}{2}<\beta<\gamma$, a,b,c are positive constants. Then the specific entropy reads,

$$s_\varepsilon(\rho_\varepsilon,\mathbb{T}_\varepsilon)=3c\sqrt{\mathbb{T}_\varepsilon}-\frac{b}{\beta-1}\rho_\varepsilon^{\beta-1}.$$

We assume that the initial condition

$$\rho_\varepsilon s_\epsilon^+(\rho_\varepsilon,\mathbb{T}_\varepsilon)\in C([0,+\infty)^2),\;\; \rho_\varepsilon(s_\varepsilon)^-(x,0)\;\text{ is bounded in }\; L^1(\Omega_\varepsilon).$$

Let us also recall that, at least formally, the following identity holds

$$\varepsilon^2\frac{\partial}{\partial t}(\rho_\varepsilon s_\varepsilon)+\mathrm{div}(\rho_\varepsilon u_\varepsilon s_\varepsilon)-\mathrm{div}(\kappa\frac{\nabla\mathbb{T}_\varepsilon}{\mathbb{T}_\varepsilon})=\mu(\mathbb{T}_\varepsilon)\frac{|\nabla u_\varepsilon|^2}{\mathbb{T}_\varepsilon}+\kappa(\mathbb{T}_\varepsilon)\frac{|\nabla\mathbb{T}_\varepsilon|^2}{\mathbb{T}_\varepsilon^2}. \tag{1.7}$$

Our aim here is to investigate the asymptotic behaviors of ρ_ε, u_ε and $\mathbb{T}_\varepsilon$ as $\varepsilon\to 0$ under the assumptions mentioned above. The main difficulty in this paper is how to pass the limit in the momentum and energy equations. To overcome this obstacle, we have to regularize the system both in time and in space before we can pass the limit. In this paper, we exert the conditions on the entropy to get the estimates. Moreover, we rigorously proved that the low boundary of γ would be n when passing the limit to the convection term. At the limit process, we fall back on the two-scale convergence method to obtain the homogenized model. Those are quite different from [11].

1.1 The Domain

Let Ω be an open bounded subset of R^n with $n=2$ or 3 and defined $\mathcal{Y}=[0,1]^n$ to be the unit open cube of R^n. Let $\mathcal{Y}_s$ be a closed smooth subset of $\mathcal{Y}$ with a strictly positive measure. The fluid part is then defined by $\mathcal{Y}_f=\mathcal{Y}-\mathcal{Y}_s$. Let $\theta=|\mathcal{Y}_f|$. The constant θ is called the porosity of the porous medium. We assume that $0<\theta<1$.

Repeating the domain $\mathcal{Y}_f$ by $\mathcal{Y}$-periodicity, we get the whole fluid domain D_f, we can write it as

$$D_f=\{x\in R^n|\exists k\in Z^n\text{ such that }x-k\in\mathcal{Y}_f\}.$$

Then the solid part is defined by $D_s=R^n-D_f$. It is easy to see that D_f is a connected domain, while D_s is formed by separated smooth subsets. In the sequel, we denote for all $k\in Z^n$, $\mathcal{Y}^k=\mathcal{Y}+k$ and then $\mathcal{Y}_f^k=\mathcal{Y}_f+k$. For all ε, we define the domain Ω_ε as the intersection of Ω with the fluid domain scaled by ε, namely, $\Omega_\varepsilon = \Omega \cap \varepsilon D_f$. To get a smooth connected domain, we will not remove the solid part of the cells which intersect with the boundary of Ω. Now, the fluid domain can be also defined by

$$\Omega_\varepsilon = \Omega - \cup\{\varepsilon\mathcal{Y}_s^k, k \in Z^n, \varepsilon\mathcal{Y}^k \subset \Omega\}.$$

1.2 Some Notations and Preliminaries

Throughout this paper, we denote $L^p(0,T;L^q(X))$, the time-space Lebesgue spaces, where X would be Ω or Ω_ε; $W^{s,p}(X)$ is the classical Sobolev space with all functions, whose all derivatives up to order s belong to L^p and $H^s(X) = W^{s,2}(X)$; $W_0^{1,p}(X)$ is the subset of $W^{1,p}(X)$ with trace 0 on X. We also denote $W^{-s,p'}(X)$, the dual space of $W_0^{s,p}(X)$, where p' is the conjugate exponent of p; C is a constant that may differ from one place to another. Throughout this paper, we use $||\cdot||_X$ to denote the modules for all vectors and matrixes if there is no confusion.

Due to the presence of the holes, the domain Ω_ε depends on ε and hence to study the convergence of $\{u_\varepsilon, \rho_\varepsilon, p_\varepsilon\}$, we have to extend the functions defined in Ω_ε to the whole domain. This can be done in two different possible ways.

Definition 1.1 For any fixed $\varepsilon \in L^1(\Omega_\varepsilon)$, we define

$$\widetilde{\varphi} = \begin{cases} \varphi & \text{in } \Omega, \\ 0 & \text{in } \Omega - \Omega_\varepsilon \end{cases}$$

the null extension and

$$\widehat{\varphi} = \begin{cases} \varphi & \text{in } \Omega_\varepsilon, \\ \dfrac{1}{|\varepsilon\mathcal{Y}_f^k|}\displaystyle\int_{\varepsilon\mathcal{Y}_f^k} \varphi(x)dx & \text{in } \Omega \cap \varepsilon\mathcal{Y}_s^k \end{cases}$$

the mean value extension.

The relation between the weak limits of both types of extensions is given by the following lemma (see [13]).

Lemma 1.1 For all $\omega_\varepsilon \in L^1(\Omega_\varepsilon)$, the following two assertions are equivalent

1. $\widehat{\omega}_\varepsilon \rightharpoonup \omega$ in $L^1(\Omega)$; 2. $\widetilde{\omega}_\varepsilon \rightharpoonup \theta\omega$ in $L^1(\Omega)$.

A very important property of the porous media is a variant of the Poincare's inequality. Due to the presence of the holes in Ω_ε, the Poincare's inequality reads in [12].

Lemma 1.2 Let $1 \le p, q < \infty$ and $u \in W_0^{1,p}(\Omega_\varepsilon)$, then

$$||u||_{L^q(\Omega_\varepsilon)} \le C\varepsilon^{1+n(\frac{1}{q}-\frac{1}{p})}||\nabla u||_{L^p(\Omega_\varepsilon)},$$

where C depends only on $\mathcal{Y}_f$ and p, q satisfies (1) $1 \le p < n, p \le q \le p^* = \frac{np}{n-p}$; (2) $p = n, p \le q < \infty$; (3) $p > n, p \le q \le \infty$.

We also need the restriction operator constructed by Tartar (see [4]).

Lemma 1.3 There exists an operator $\mathcal{R}_\varepsilon$ with the following properties

1. $\mathcal{R}_\varepsilon$ is a bounded linear operator on $W_0^{1,p}(\Omega)$ ranging in $W_0^{1,p}(\Omega_\varepsilon)$, $p \geq 2$;
2. $\mathcal{R}_\varepsilon[\varphi] = \varphi|_{\Omega_\varepsilon}$ provide $\varphi = 0$ in $\Omega - \Omega_\varepsilon$;
3. $\mathrm{div}_x \varphi = 0$ in Ω implies $\mathrm{div}_x \mathcal{R}_\varepsilon[\varphi] = 0$ in Ω_ε;
4. $||\mathcal{R}_\varepsilon[\varphi]||_{L^p(\Omega_\varepsilon)} + \varepsilon||\nabla \mathcal{R}_\varepsilon[\varphi]||_{L^p(\Omega_\varepsilon)} \leq C(||\varphi||_{L^p(\Omega)} + \varepsilon||\nabla\varphi||_{L^p(\Omega)})$.

In addition, we can find the restriction operator $\mathcal{R}_\varepsilon$ satisfies a compatibility relation with the extension operator introduces in Definition 1.1, namely,

$$\langle \nabla\widehat{\omega}, \varphi\rangle = -\int_\Omega \widehat{\omega}\ \mathrm{div}\varphi dx = -\int_{\Omega_\varepsilon} \omega\ \mathrm{div} R_\varepsilon[\varphi] dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Lemma 1.4 (Bogovskii operator) There exists a linear operator $\mathcal{B}_\varepsilon$ with the properties: if $f \in L^p(\Omega_\varepsilon)$, then $\phi = \mathcal{B}_\varepsilon[f]$ such that

$$\phi \in W_0^{1,p}(\Omega_\varepsilon), \quad \mathrm{div}\phi = f - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} f dx.$$

Moreover, the following estimates is satisfied

$$||\mathcal{B}_\varepsilon[f]||_{W_0^{1,p}(\Omega_\varepsilon)} \leq C\varepsilon^{-1}||f||_{L^p(\Omega_\varepsilon)}, \quad 1 < p < \infty.$$

In addition, if $f = \mathrm{div} g$ with $g \in L^q(\Omega_\varepsilon)$, $g \cdot n|_{\partial\Omega_\varepsilon} = 0$, then

$$||\mathcal{B}_\varepsilon[f]||_{L^q(\Omega_\varepsilon)} \leq C||g||_{L^q(\Omega_\varepsilon)}, \quad 1 < q < \infty.$$

There are many ways to construct $\mathcal{B}_\varepsilon$. An explicit formula was proposed by Bogovskii [16] on Lipschitz domains. Some properties of $\mathcal{B}_\varepsilon$ were discussed by Galdi [17]. In the domain with porous medium, the relevant estimates were obtained by Masmoudi [13].

Finally, we define the permeability matrix $\mathcal{A}$. For $1 \leq i \leq n$, let $(\omega_i, \pi_i) \in H^1(\mathcal{Y}_f) \times L^2(\mathcal{Y}_f)/R$ be the unique solution of the following system

$$\begin{cases} -\triangle\omega_i + \nabla\pi_i = e_i \quad \text{in } \mathcal{Y}_f, \\ \mathrm{div}\omega_i = 0 \quad \text{in } \mathcal{Y}_f, \\ \omega_i = 0 \ \text{ on } \partial\mathcal{Y}_s, \ \ \omega_i, \pi_i \ \text{ are } \mathcal{Y}\text{-periodic}, \end{cases}$$

where e_i is the standard basis of R^n. Set $\omega_i^\varepsilon = \omega_i(\frac{x}{\varepsilon})$, $\pi_i^\varepsilon = \pi_i(\frac{x}{\varepsilon})$. Then we get the cell problem

$$\begin{cases} -\varepsilon^2\triangle\omega_i^\varepsilon + \varepsilon\nabla\pi_i^\varepsilon = e_i \quad \text{in } \varepsilon\mathcal{Y}_f, \\ \mathrm{div}\omega_i^\varepsilon = 0 \quad \text{in } \varepsilon\mathcal{Y}_f, \\ \omega_i^\varepsilon = 0 \ \text{ on } \partial(\varepsilon\mathcal{Y}_s), \ \ \omega_i^\varepsilon, \pi_i^\varepsilon \ \text{ are } \varepsilon\mathcal{Y}\text{-periodic}. \end{cases}$$

Lemma 1.5 Let $\omega_i^\varepsilon, \pi_i^\varepsilon$ be the solution to the cell problem and be extended to zero outside Ω_ε. Then the following estimates hold

$$||\omega_i^\varepsilon||_{[L^q(\Omega_\varepsilon)]^n} \leq C, \ \ ||\pi_i^\varepsilon||_{L^q(\Omega_\varepsilon)/R} \leq C, \ \ \varepsilon||\nabla\omega_i^\varepsilon||_{[L^q(\Omega_\varepsilon)]^n} \leq C$$

for any $1 \le q \le +\infty$, C only depends on q and $\mathcal{Y}_f$.

The construction of ω_i^ε and the properties of ω_i^ε, π_i^ε stated in Lemma 1.5 can be found in Tartar [4] or Masmoudi [13]. Let us define

$$\mathcal{A} = (\mathcal{A}_{i,j})_{i,j=1}^n, \quad \mathcal{A}_{i,j} = \int_{\mathcal{Y}_f} (\omega_i)_j dx.$$

It is easy to see that $\mathcal{A}$ is a symmetric positive defined matrix. The form of the permeability matrix has different form if $\mathcal{Y}_s$ has different forms. For more information about $\mathcal{A}$, we refer the interested readers to Allaire [7] for detail.

1.3 The Main Results

Now we introduce the definition of weak solution to the systems (1.1)–(1.4)

Definition 1.2 We shall say that a trio $\{u_\varepsilon, \rho_\varepsilon, p_\varepsilon\}$ is a weak solution of (1.1), supplemented with the boundary and initial conditions (1.2)–(1.4) if only if

1. $\rho_\varepsilon \in L^\infty(0,T;L^\gamma(\Omega_\varepsilon))$, $\gamma \ge n$ and $u_\varepsilon \in L^2(0,T;H_0^1(\Omega_\varepsilon))$, and the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} (\varepsilon^2 \rho_\varepsilon \varphi_t + \rho_\varepsilon u_\varepsilon \cdot \nabla\varphi) dxdt = -\int_{\Omega_\varepsilon} \varepsilon^2 \rho_{0,\varepsilon} \varphi(x,0) dx$$

holds for any test function $\varphi \in C_0^\infty([0,T) \times \overline{\Omega}_\varepsilon)$.

2. $p_\varepsilon \in L^q(\Omega_\varepsilon \times (0,T))$ for some $q > 1$, and

$$\begin{aligned}&\int_0^T \int_{\Omega_\varepsilon} (\varepsilon^2 \rho_\varepsilon u_\varepsilon \varphi_t + \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla\varphi + p_\varepsilon \mathrm{div}\varphi) dxdt \\ =&\int_0^T \int_{\Omega_\varepsilon} \mu \nabla u_\varepsilon : \nabla\varphi dxdt - \int_{\Omega_\varepsilon} \varepsilon^2 m_{0,\varepsilon} \varphi(x,0) dx - \int_0^T \int_{\Omega_\varepsilon} \rho_\varepsilon f \varphi dxdt.\end{aligned}$$

3. $\mathbb{T}_\varepsilon \in L^2(0,T;L^6(\Omega_\varepsilon)) \cap L^2(0,T;H^1(\Omega_\varepsilon))$, $\mathbb{T}_\varepsilon > 0$ a.a in $\Omega_\varepsilon \times (0,T)$ and the integral identity

$$\begin{aligned}&\int_0^T \int_{\Omega_\varepsilon} (\varepsilon^2 \rho_\varepsilon s_\varepsilon \varphi_t + \rho_\varepsilon u_\varepsilon s_\varepsilon \cdot \nabla\varphi - \kappa(\mathbb{T}_\varepsilon) \frac{\nabla \mathbb{T}_\varepsilon}{\mathbb{T}_\varepsilon} \nabla\varphi) dxdt \\ =&-\int_0^T \int_{\Omega_\varepsilon} (\mu(\mathbb{T}_\varepsilon) \frac{|\nabla u_\varepsilon|^2}{\mathbb{T}_\varepsilon} + \kappa(\mathbb{T}_\varepsilon) \frac{|\nabla \mathbb{T}_\varepsilon|^2}{\mathbb{T}_\varepsilon^2}) \varphi dxdt + \int_{\Omega_\varepsilon} \varepsilon^2 \rho_{\varepsilon 0} s_\varepsilon(x,0) \varphi(x,0) dx\end{aligned}$$

holds for any $\varphi \in C_0^\infty([0,T) \times \bar{\Omega}_\varepsilon)$.

With all the preparation above, we are now in the position to state our main result in this paper.

Theorem 1.1 Let $\{u_\varepsilon, \rho_\varepsilon, \mathbb{T}_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to system (1.1). We assume that $\gamma \ge n$ for $n = 2,3$ and

$$\widehat{\rho_\varepsilon(x,0)} \to \rho_0(x), \;\; e_\varepsilon(x,0) \to e_0(x) \;\text{ strongly in }\; L^1(\Omega), \;\text{ respectively.}$$

Then, there exist three functions u, ρ, Ξ such that

$$\begin{cases} \widehat{\rho_\varepsilon} \to \rho \quad \text{in } L^p(0,T;L^q(\Omega)),\ 1<p<\infty,\ 1<q<\gamma+1, \\ \widehat{\mathbb{T}_\varepsilon} \to \Xi \quad \text{in } L^2(\Omega\times(0,T)), \\ \dfrac{u_\varepsilon}{\varepsilon^2} \rightharpoonup u \quad \text{weakly in } L^2(\Omega\times(0,T)), \end{cases}$$

where $\Xi=\Xi(t)$ is a spatially homogeneous function, $\rho_(x,0)=\rho_0(x)$. Moveover, $\{u,\rho,\Xi\}$ satisfies the following homogenized system

$$\begin{cases} |\theta|\partial_t\rho + \operatorname{div}(\rho u) = 0 \ \text{ in } \Omega\times(0,T), \\ \mu(\Xi)u = \mathcal{A}(-\nabla p(\rho,\Xi)+\rho f) \quad \text{on the set } \{\rho>0\}\ \Omega\times(0,T), \\ \dfrac{\partial(\rho e)}{\partial t} = \rho u\cdot f, \end{cases}$$

where p, e are given by

$$p = a\rho^\gamma + b\rho^\beta\Xi,\ \ e = \frac{a}{\gamma-1}\rho^{\gamma-1} + c\Xi^{\frac{3}{2}},$$

and $\mathcal{A}$ is the so-called permeability matrix. The specific homogenized entropy s related to the homogenized pressure p and the homogenized inner energy e through Gibbs' equation

$$\Xi Ds(\rho,\Xi) = De(\rho,\Xi) + p(\rho,\Xi)D(\frac{1}{\rho}) \quad \text{on the set } \{\rho>0\},$$

where s is given by $s = 3c\sqrt{\Xi} - \frac{b}{\beta-1}\rho^{\beta-1}$.

2 Uniform Bounds

In this section, we collect all available bounds on the family $\{u_\varepsilon, \rho_\varepsilon, \mathbb{T}_\varepsilon\}$. Let us begin with the basic estimates

2.1 Basic a Priori Estimates

In this subsection, we obtain some estimates for the solutions to system (1.1) which are independent of ε. First, from the conservation of mass, we have $\rho_\varepsilon \in L^\infty(0,T;L^1(\Omega_\varepsilon))$. We set

$$\int_{\Omega_\varepsilon} \rho_\varepsilon dx = \int_{\Omega_\varepsilon} \rho_{0,\varepsilon} dx = M. \tag{2.1}$$

Next, integrating (1.7) over $\Omega_\varepsilon\times(0,t)$ for any $t\in[0,T]$, we have

$$\varepsilon^{-1}||\nabla\mathbb{T}_\varepsilon||_{L^2(\Omega_\varepsilon\times(0,T))} \le C,\ \ \varepsilon^{-1}||\nabla u_\varepsilon||_{L^2(\Omega_\varepsilon\times(0,T))} \le C. \tag{2.2}$$

We then deduce the uniform bound of $\rho_\varepsilon|s_\varepsilon|$ in $L^\infty(0,T;L^1(\Omega_\varepsilon))$ and ρ_ε is uniformly bounded in $L^\infty(0,T;L^\gamma(\Omega_\varepsilon))$. By Lemma 1.2 and the special s_ε, e_ε, we also have

$$\begin{aligned} &\varepsilon^{-2}||u_\varepsilon||_{L^2(\Omega_\varepsilon\times(0,T))} \le C; \\ &\rho_\varepsilon|u_\varepsilon|^2,\ \rho_\varepsilon e_\varepsilon,\ \rho_\varepsilon\mathbb{T}_\varepsilon^{\frac{3}{2}} \text{ are uniformly bounded in } L^\infty(0,T;L^\gamma(\Omega_\varepsilon)). \end{aligned} \tag{2.3}$$

2.2 Refined Temperature Estimatess

In this subsection, we want to deduce the uniform bounds on $\mathbb{T}_\varepsilon$. Note that we only have the bounds of $\nabla\mathbb{T}_\varepsilon$ and $\rho_\varepsilon\mathbb{T}_\varepsilon^{\frac{3}{2}}$. The estimate on $\mathbb{T}_\varepsilon$ itself isn't included. To fill this gap, we fall back on the Nečas' lemma [18] and the Sobolev embedding theorem in the porous medium

Lemma 2.1 Let Ω be a bounded Lipschitz domain in R^2 or R^3. Let M, K be two positive real numbers and ρ a non-negative function such that

$$0 < M \leq M_\rho = \int_\Omega \rho dx, \quad \int_\Omega \rho^\gamma dx \leq K \quad \text{for a certain } \gamma \geq 2.$$

Then there exists a constant $C = C(M, K)$ such that

$$||\omega - \frac{1}{M_\rho}\int_\Omega \omega\rho dx||_{L^2(\Omega)} \leq C(M, K)||\nabla\omega||_{H^{-1}(\Omega)}$$

for any $\omega \in L^2(\Omega)$.

Proof see [11].

Following the idea in [11] and [12], we prove the Sobolev embedding theorem in the porous medium has the form.

Lemma 2.2 Let $v \in W^{1,p}(\Omega_\varepsilon)$. Then we have

$$||v||_{L^q(\Omega_\varepsilon)} \leq C(|\int_{\Omega_\varepsilon} vdx| + \varepsilon^{n(\frac{1}{q}-\frac{1}{p})}||\nabla v||_{L^p(\Omega_\varepsilon)})$$

with C independent of ε, $1 < p \leq q \leq \frac{np}{n-p}$ for $n \geq 3$ and $1 < p \leq q < \infty$ for $n = 2$.

Proof Obviously, it is enough to show

$$||v||_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{n(\frac{1}{q}-\frac{1}{p})}||\nabla v||_{L^p(\Omega_\varepsilon)}, \quad v \in W^{1,p}(\Omega_\varepsilon), \int_{\Omega_\varepsilon} vdx = 0.$$

By the definition of the module of L^p and Lemma 1.2, Lemma 1.3, we have

$$\begin{aligned}||v||_{L^q(\Omega_\varepsilon)} =& \sup_{||\varphi||_{L^{q'}} \leq 1, \int_{\Omega_\varepsilon}\varphi dx = 0} \int_{\Omega_\varepsilon} \nabla v \mathcal{B}_\varepsilon[\varphi]dx \\ \leq& \sup_{||\varphi||_{L^{q'}} \leq 1, \int_{\Omega_\varepsilon}\varphi dx = 0} ||\nabla v||_{L^p(\Omega_\varepsilon)}\varepsilon^{1+n(\frac{1}{p'}-\frac{1}{q'})}||\nabla\mathcal{B}_\varepsilon[\varphi]||_{L^{q'}(\Omega_\varepsilon)} \\ \leq& C\varepsilon^{n(\frac{1}{q}-\frac{1}{p})}||\nabla v||_{L^p(\Omega_\varepsilon)},\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$.

By Lemma 1.3 and (2.2), we obtain

$$|\langle\nabla\widehat{\mathbb{T}_\varepsilon}, \varphi\rangle| \leq C||\nabla\mathbb{T}_\varepsilon||_{L^2(\Omega_\varepsilon)}(||\varphi||_{L^2(\Omega_\varepsilon)} + \varepsilon||\nabla\varphi||_{L^2(\Omega_\varepsilon)}) \leq C\varepsilon||\varphi||_{H_0^1(\Omega_\varepsilon)},$$

which implies that

$$||\nabla\widehat{\mathbb{T}_\varepsilon}||_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} \leq C\varepsilon. \tag{2.4}$$

Next, we write

$$\mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon - \frac{1}{M_\varepsilon}\int_{\Omega_\varepsilon} \rho_\varepsilon \mathbb{T}_\varepsilon dx + \frac{1}{M_\varepsilon}\int_{\Omega_\varepsilon} \rho_\varepsilon \mathbb{T}_\varepsilon dx.$$

In accordance with Lemma 2.1 and (2.2)–(2.4), we have

$$||\mathbb{T}_\varepsilon - \frac{1}{M_\varepsilon}\int_{\Omega_\varepsilon} \rho_\varepsilon \mathbb{T}_\varepsilon dx||_{L^2(\Omega_\varepsilon)} \le C\varepsilon,\ \Xi_\varepsilon = \frac{1}{M_\varepsilon}\int_{\Omega_\varepsilon} \rho_\varepsilon \mathbb{T}_\varepsilon dx \text{ is bounded in } L^\infty(0,T).$$

We deduce that $||\mathbb{T}_\varepsilon||_{L^2(\Omega_\varepsilon\times(0,T))} \le C$. By Lemma 2.2, we obtain

$$||\mathbb{T}_\varepsilon||_{L^2(0,T;L^6(\Omega_\varepsilon))} \le C \tag{2.5}$$

with C independent of ε.

2.3 Refined Density Estimates

We get the uniform bound on ρ_ε in $L^\infty(0,T;L^\gamma(\Omega_\varepsilon))$. This is not enough when we consider the strong convergence of the density. In this part, we want to obtain some more delicate estimates. To this end, we choose the test function $v_\varepsilon = \psi(t)\mathcal{B}_\varepsilon(\rho_\varepsilon - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\rho_\varepsilon dx)$, where $\psi(t) \in C_0^\infty(0,T)$, $\psi(t) > 0$, $\mathcal{B}_\varepsilon$ denotes Bogovskii's operator introduced in Lemma 1.3.

For any $t \in (0,T)$, we have

$$\begin{aligned} ||v||_{L^2(\Omega_\varepsilon)} &\le C\varepsilon||\nabla v||_{L^2(\Omega_\varepsilon)} \le C||\rho_\varepsilon||_{L^2(\Omega_\varepsilon)}, \\ ||v||_{L^\gamma(\Omega_\varepsilon)} &\le C\varepsilon||\nabla v||_{L^\gamma(\Omega_\varepsilon)} \le C||\rho_\varepsilon||_{L^\gamma(\Omega_\varepsilon)}. \end{aligned} \tag{2.6}$$

Multiplying the second equation of (1.1) by v_ε, we get (we drop the $dxdt$)

$$\begin{aligned} \int_0^T \psi(t)\int_{\Omega_\varepsilon} a\rho_\varepsilon^{\gamma+1} &= \int_0^T\int_{\Omega_\varepsilon} \varepsilon^2\rho_\varepsilon u_\varepsilon\cdot\partial_t v_\varepsilon + \frac{a}{|\Omega_\varepsilon|}\int_0^T \psi(t)\int_{\Omega_\varepsilon}\rho_\varepsilon^\gamma\int_{\Omega_\varepsilon}\rho_\varepsilon \\ &\quad + \int_0^T\int_{\Omega_\varepsilon}\rho_\varepsilon f\cdot v_\varepsilon + \int_0^T\int_{\Omega_\varepsilon}\mu\nabla u_\varepsilon : \nabla v_\varepsilon - \int_0^T\psi(t)\int_{\Omega_\varepsilon} b\rho_\varepsilon^{\beta+1}\mathbb{T}_\varepsilon \\ &\quad + \frac{b}{|\Omega_\varepsilon|}\int_0^T\psi(t)\int_{\Omega_\varepsilon} b\rho_\varepsilon^\beta\mathbb{T}_\varepsilon\int_{\Omega_\varepsilon}\rho_\varepsilon - \int_0^T\int_{\Omega_\varepsilon}\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : \nabla v_\varepsilon \\ &= \sum_{i=1}^7 I_i. \end{aligned} \tag{2.7}$$

The first term is the most technical and requires some spatial regularization of v_ε (see [13]). Let us explain how the difficulty related to I_1 can be solved. The estimation of rest terms are the same as [13]. Set $\chi(x) \in C_0^\infty(R^n)$ such that $\chi(x) \ge 0$, $\int_{R^n}\chi(x)dx = 0$. For all $\delta \in (0,1)$, we denote $\chi_\delta(x) = \frac{1}{\delta^n}\chi(\frac{x}{\delta})$. Next, we denote $\widetilde{\rho}_{\varepsilon,\delta} = \widetilde{\rho}_\varepsilon * \chi_\delta(x)$, where $\widetilde{\rho}_\varepsilon$ is defined in Definition 1.1. Then we have the following relation

$$\varepsilon^2\frac{\partial\widetilde{\rho}_{\varepsilon,\delta}}{\partial t} + \operatorname{div}(\widetilde{\rho}_{\varepsilon,\delta}\widetilde{u}_\varepsilon) = r_{\varepsilon,\delta}, \tag{2.8}$$

where $r_{\varepsilon,\delta}$ is nothing but a commutator, $r_{\varepsilon,\delta}\to 0$ in $L^2(0,T;L^{\frac{2\gamma}{\gamma+2}}(\Omega))$ as $\delta\to 0$. Now we take $v_{\varepsilon,\delta}=\psi(t)\mathcal{B}_\varepsilon(\widetilde{\rho}_{\varepsilon,\delta}-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\widetilde{\rho}_{\varepsilon,\delta}dx)$ in stead of v_ε. Taking the time derivative of t of $v_{\varepsilon,\delta}$, we have

$$\begin{aligned}\frac{\partial v_{\varepsilon,\delta}}{\partial t}=&\psi(t)\mathcal{B}(\frac{\partial\widetilde{\rho}_{\varepsilon,\delta}}{\partial t}-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\frac{\partial\widetilde{\rho}_{\varepsilon,\delta}}{\partial t})+\psi'(t)\mathcal{B}_\varepsilon(\widetilde{\rho}_{\varepsilon,\delta}-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\widetilde{\rho}_{\varepsilon,\delta}dx)\\=&\psi(t)I_{11}+\psi'(t)I_{12}.\end{aligned}$$

It is easy to check that

$$|\int_0^T\int_{\Omega_\varepsilon}\varepsilon^2\rho_\varepsilon u_\varepsilon\cdot\psi'(t)I_{12}|\le C\varepsilon||\rho_\varepsilon||^2_{L^\infty(0,T;L^\gamma(\Omega_\varepsilon))}||u_\varepsilon||_{L^2(0,T;L^p(\Omega_\varepsilon))}\le C\varepsilon^{3+n(\frac{1}{p}-\frac{1}{2})},$$

where $3+n(\frac{1}{p}-\frac{1}{2})\ge 2$ for $n=3$ and $3+n(\frac{1}{p}-\frac{1}{2})>2$ for $n=2$.

Next, after a straightforward manipulation we have

$$I_{11}=-\varepsilon^{-2}\mathcal{B}(\mathrm{div}(\widetilde{\rho}_{\varepsilon,\delta}\widetilde{u}_\varepsilon))+\mathcal{B}_\varepsilon(r_{\varepsilon,\delta}-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}r_{\varepsilon,\delta}dx)=I_{111}+I_{112}.$$

By Lemma 1.2 and Lemma 1.3, we deduce that

$$\left|\int_0^T\int_{\Omega_\varepsilon}\varepsilon^2\rho_\varepsilon u_\varepsilon\cdot\psi(t)I_{111}\right|\le C||\rho_\varepsilon||^2_{L^\infty(0,T;L^\gamma(\Omega_\varepsilon))}||u_\varepsilon||^2_{L^2(0,T;L^p(\Omega_\varepsilon))}\le C\varepsilon^{4+\frac{2n}{p}-n}.$$

To I_{112}, we have

$$\left|\int_0^T\int_{\Omega_\varepsilon}\varepsilon^2\rho_\varepsilon u_\varepsilon\cdot\psi(t)I_{112}\right|\le C\varepsilon^{\frac{2\gamma-n}{\gamma}}||\rho_\varepsilon u_\varepsilon||_{L^2_T(\Omega_\varepsilon)}||r_{\varepsilon,\delta}||_{L^2(0,T;L^{\frac{2\gamma}{\gamma+2}}(\Omega_\varepsilon))},$$

where $L^2_T(\Omega_\varepsilon)=L^2(0,T;L^2(\Omega_\varepsilon))$. Then we let $\delta\to 0$ and deduce that $\int_0^T\int_{\Omega_\varepsilon}\varepsilon^2\rho_\varepsilon u_\varepsilon\cdot\psi(t)I_{112}$ also tends to 0.

From above, we get the estimate

$$\rho_\varepsilon \text{ is bounded uniformly in } L^{\gamma+1}(\Omega_\varepsilon\times(0,T)). \tag{2.9}$$

With those estimations, we can also obtain $\widehat{p_\varepsilon}$ converges strongly to p in $L^2(0,T;L^s(\Omega))$ for some $s>1$, see [12].

3 Proof of Theorem 1.2

We divide three steps to finish the proof.

Step 1 Passing the limit in the continuum equation. As in [13], we can prove that

$$\frac{\partial\widetilde{\rho}_\varepsilon}{\partial t}+\varepsilon^{-2}\mathrm{div}(\widehat{\rho}_\varepsilon\widetilde{u}_\varepsilon)=0. \tag{3.1}$$

Multiplying above equation by $\varphi \in C_0^\infty(\Omega \times (0,T])$ and integrating over $\Omega \times (0,T)$, we have

$$-\int_0^T \int_\Omega \widetilde{\rho}_\varepsilon \partial_t \varphi dxdt - \int_0^T \int_\Omega \widehat{\rho}_\varepsilon \frac{\widetilde{u}_\varepsilon}{\varepsilon^2} \cdot \nabla \varphi dxdt = \int_\Omega \widetilde{\rho}_\varepsilon(x,0)\varphi(x,0)dx.$$

Passing the limit and using the strong convergence of $\widehat{\rho}_\varepsilon$ and $\widehat{\rho}_{0,\varepsilon}$, we obtain

$$-|\theta| \int_0^T \int_\Omega \rho \partial_t \varphi dxdt - \int_0^T \int_\Omega \rho u \cdot \nabla \varphi dxdt = |\theta| \int_\Omega \rho_0(x)\varphi(x,0)dx.$$

We then recover the homogenized equation and the initial condition

$$|\theta|\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad \rho(x,0) = \rho_0(x). \tag{3.2}$$

Step 2 Passing the limit in the momentum equations.

To pass the limit, we have to regularize the second equation of (1.1) both x and t. To this end, we set $\chi_1(x) \in C_0^\infty(R^n)$ such that $\chi_1(x) \geq 0$, $\int_{R^n} \chi_1(x)dx = 1$. Let $\eta_1 \in (0,1)$ and set $\chi_{1\eta_1}(x) = \frac{1}{\eta_1^n}\chi_1(\frac{x}{\eta_1})$. We also set $\chi_2(t) \in C_0^\infty(R^+)$ such that $\chi_2(t) \geq 0$, $\int_{R^+} \chi_2(t)dt = 1$. Let $\eta_2 \in (0,1)$ and set $\chi_{2\eta_2}(t) = \frac{1}{\eta_2}\chi_2(\frac{t}{\eta_2})$. Take $T_m(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}\omega_n^\varepsilon \varphi$ as the test function, where T_m is the truncated function by integer m and $T_m(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}$ is prolonged by zero outside $\Omega \times (0,T)$, ω_n^ε be the solution of the cell problem and $\varphi \in C_0^\infty(\Omega \times (0,T))$. We have

$$\begin{aligned}
&\int_0^T \int_\Omega \mu(\mathbb{T}_\varepsilon)\nabla u_\varepsilon : \nabla(\mathbb{T}_\varepsilon(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}\omega_n^\varepsilon \varphi)dxdt \\
=&\int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla(\mathbb{T}_\varepsilon(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}\omega_n^\varepsilon \varphi)dxdt \\
&+ \varepsilon^2 \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \omega_n^\varepsilon \, \partial_t(\mathbb{T}_\varepsilon(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}\varphi)dxdt \\
&+ \int_0^T \int_\Omega \rho_\varepsilon f \cdot \mathbb{T}_\varepsilon(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}\omega_n^\varepsilon \varphi dxdt \\
&+ \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon, \mathbb{T}_\varepsilon)\operatorname{div}(\mathbb{T}_\varepsilon(\widehat{\rho}_\varepsilon) * \chi_{1\eta_1} * \chi_{2\eta_2}\omega_n^\varepsilon \varphi)dxdt \\
=&I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.3}$$

To pass the limit in I_1 and I_2, we can easily deduce that $I_1 \to 0$, $I_2 \to 0$ as $\varepsilon \to 0$. Now, we can pass the limit in I_3 and I_4 since $\widehat{\rho}_\varepsilon$ converges strongly,

$$I_3 \to \mathcal{A}_n \int_0^T \int_\Omega \rho f \cdot \mathbb{T}_\varepsilon(\rho) * \chi_{1\eta_1} * \chi_{2\eta_2}\varphi dxdt \tag{3.4}$$

and

$$\begin{aligned}
I_4 \to&\mathcal{A}_n \int_0^T \int_\Omega p(\rho, \mathbb{T})\operatorname{div}(\mathbb{T}_\varepsilon(\rho) * \chi_{1\eta_1} * \chi_{2\eta_2}\varphi)dxdt \\
=&-\mathcal{A}_n \int_0^T \int_\Omega \nabla p(\rho, \mathbb{T})\mathbb{T}_\varepsilon(\rho) * \chi_{1\eta_1} * \chi_{2\eta_2}\varphi dxdt.
\end{aligned} \tag{3.5}$$

Finally, we consider the limit on the left-hand side. We write it as

$$\begin{aligned}
&\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)\nabla u_\varepsilon : \nabla(\mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2}\omega_n^\varepsilon\varphi)dxdt\\
=&\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)\nabla u_\varepsilon : \nabla\omega_n^\varepsilon\ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2}\ \varphi dxdt\\
&+\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)\nabla u_\varepsilon : \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \nabla\chi_{1\eta_1} * \chi_{2\eta_2} \otimes \omega_n^\varepsilon\ \varphi dxdt\\
&+\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)\nabla u_\varepsilon : \omega_n^\varepsilon \otimes \nabla\varphi\ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2} dxdt\\
=&L_1+L_2+L_3.
\end{aligned}$$

It is easy to check that

$$|L_2+L_3| \le C\varepsilon \to 0 \ \text{ as } \ \varepsilon \to 0.$$

On the other hand,

$$\begin{aligned}
L_1 =&-\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)u_\varepsilon \cdot \triangle\omega_n^\varepsilon\ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2}\ \varphi dxdt\\
&-\int_0^T\int_\Omega \mu'(\mathbb{T}_\varepsilon)\nabla\mathbb{T}_\varepsilon \otimes u_\varepsilon : \nabla\omega_n^\varepsilon\ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2}\ \varphi dxdt\\
&-\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)u_\varepsilon \otimes \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \nabla\chi_{1\eta_1} * \chi_{2\eta_2} : \nabla\omega_n^\varepsilon\ \varphi dxdt\\
&-\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)u_\varepsilon \otimes \nabla\varphi : \nabla\omega_n^\varepsilon\ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2} dxdt\\
=&L_{11}+L_{12}+L_{13}+L_{14}.
\end{aligned}$$

We also have

$$|L_{12}+L_{13}+L_{14}| \le C\varepsilon \to 0 \ \text{ as } \ \varepsilon \to 0.$$

To L_{11}, by virtue of the cell problem, we have

$$\begin{aligned}
L_{11} =&\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)\varepsilon^{-2}u_\varepsilon \cdot e_n\ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2}\ \varphi dxdt\\
&-\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)u_\varepsilon \cdot \varepsilon^{-1}\nabla\pi^\varepsilon\ [\mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2} - \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon})]\ \varphi dxdt\\
&-\int_0^T\int_\Omega \mu(\mathbb{T}_\varepsilon)u_\varepsilon\mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) \cdot \varepsilon^{-1}\nabla\pi^\varepsilon\ \varphi dxdt\\
=&L_{111}+L_{112}+L_{113}.
\end{aligned}$$

We continue discussing the convergence of those three terms.

$$|L_{112}+L_{113}| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Finally, passing the limit with $\varepsilon \to 0$, the limit of L_{111} is given by

$$\begin{aligned}&\int_0^T \int_\Omega \mu(\mathbb{T}_\varepsilon)\varepsilon^{-2}u_\varepsilon \cdot e_n \ \mathbb{T}_\varepsilon(\widehat{\rho_\varepsilon}) * \chi_{1\eta_1} * \chi_{2\eta_2} \ \varphi dxdt \\ \to &\int_0^T \int_\Omega \mu(\Xi)u \cdot e_n T_m(\rho) * \chi_{1\eta_1} * \chi_{2\eta_2}\varphi dxdt.\end{aligned} \tag{3.6}$$

Let $\eta_1, \eta_2 \to 0$ and $m \to +\infty$ and by using the arbitrary property of φ, (3.4), (3.5) and (3.6) lead to the homogenized system of the momentum. That is

$$\mu(\Xi)u = \mathcal{A}(-\nabla p(\rho, \Xi) + \rho f) \ \text{ on the set } \ \{\rho > 0\}. \tag{3.7}$$

Step 3 Passing the limit in the energy equation.

Integrating the energy equation in $\Omega \times (0, t)$ and passing the limit, we have

$$\frac{\partial(\rho e)}{\partial t} = \rho u \cdot f \qquad \text{a.e. in } \Omega \times (0, T). \tag{3.8}$$

We also have

$$p_\varepsilon \to p = a\rho^\gamma + b\rho^\beta \Xi, e_\varepsilon \to e = \frac{a}{\gamma - 1}\rho^{\gamma-1} + c\Xi^{\frac{3}{2}}, s_\varepsilon \to s = 3c\sqrt{\Xi} - \frac{b}{\beta - 1}\rho^{\beta-1}.$$

Then the following Gibbs' equation holds

$$\Xi Ds(\rho, \Xi) = De(\rho, \Xi) + p(\rho, \Xi)D(\frac{1}{\rho}) \quad \text{on the set } \{\rho > 0\}. \tag{3.9}$$

Now we finish Theorem 1.2.

References

[1] Sanchez-Palencia E. Nonhomogeneous media and vibration theory[M]. Lecture Notes Phys., Vol. 127, Berlin, Heidelberg: Springer-Verlag, 1980.

[2] Keller J B. Darcy's law for flow in porous media and the two-space method[C]. Nonl. Par. Diff. Equ. Eng. Appl. Sci. (Proceedings of the Conference, Kingston, R.I., June 4-8, 1979), New York: Marcel Dekker Inc., 1980.

[3] Bensoussan A, Lions J L, Papanicolaou G. Asymptotic analysis for periodic structures[M]. Studies in Mathematics and its Applications. Amsterdam, New York: North-Holland Publishing Co., 1978.

[4] Tartar L. Incompressible fluid flow in a porous medium convergence of the homogenization process[M]. Appendix Lecture Notes Phys., Vol. 127, Berlin: Springer Velag, 1980.

[5] Allaire G. Homogenization of the Stokes flow in a connected porous medium[J]. Asymptot. Anal., 1989, 2: 203–222.

[6] Allaire G. Homogenization and two-scale convergence[J]. SIAM J. Math. Anal., 1992, 23(6): 1482–1518.

[7] Allaire G. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes[J]. Arch. Rat. Mech. Anal., 1991, 113(3): 209–259.

[8] Allaire G. Homogenization of the Navier-Stokes equations with a slip boundary condition[J]. Comm. Pure. Appl. Math., 1991, 44: 605–641.

[9] Mikelic A. Homogenization of nonstationary Navier-Stokes equations in a domain with a grained boundary[J]. Ann. Mat. Pura. Ed. Appl., 1991: 158(4): 167–179.

[10] Mikelic A, Aganovic I. Homogenization of stationary of miscible fluids in a domain boundary[J]. SIAM J. Math. Anal., 1988, 19: 287–294.

[11] Feireisl E, Novotný A, Takahashi T. Homogenization and singular limits for the complete Navier-Stokes-Fourier system[J]. J. Math. Pures Appl., 2010, 94: 33–57.

[12] Zhao Hongxing, Yao Zhengan. Homogenization of the time discretized compressible Navier-Stokes system[J]. Nonl. Anal., 2012, 75: 2486–2498.

[13] Masmoudi N. Homogenization of the compressible Navier-Stokes equations in a porous medium[J]. ESAIM Control Optim. Calc. Var., 2002, 8: 885–906.

[14] Brinkman H C. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles[J]. Appl. Sci. Res., 1951, 2: 155.

[15] Rodrigues J F, Sequeira A. Mathematical topics in fluid mechanics[M]. Taylor & Francis Ltd., 1992.

[16] Bogovskii M E. Solution of some vector analysis problems connected with operators div and grad[J]. Tr. Sem. S. L. Sobolev, 1980, 80(1): 5–40 (in Russian).

[17] Galdi G P. An introduction to the mathematical theory of the Navier-Stokes equations[M]. Vol. I, New York: Springer-Verlag, 1994.

[18] Nečas J. Sur les normes équivalentes dans $W_p^k(\Omega)$ et sur la coercivité des formes formellement positives[J]. Séminaire Equations aux Dérivées Partielles, les Presses de l'Université de Montréal, 1966, 102–128.

[19] Lipton R, Avellanda M. Darcy's law for slow viscous flow past a stationary array of bubbles[J]. Proc. Roy. Soc. Edinbergh Sect. A, 1990, 114(1-2): 71–79.

[20] Allaire G. Homogenization and two-scale convergence[J]. SIAM J. Math. Anal., 1992, 23(6): 1482–1518.

[21] Nguetseng G. A general convergence result for a functional related to the theory of homogenization[J]. SIAM J. Math. Anal., 1989, 20(3): 608–623.

[22] Temam R. Navier-Stokes equations[M]. Amsterdam: North-Holland Publishing Co., 1979.

多孔介质中可压缩Navier-Stokes流的渐近行为

袁国智, 赵红星

(广州大学数学与信息科学学院, 广东 广州 510006)

摘 要: 本文研究了多孔介质中完全非线性Navier-Stokes流的渐近行为. 利用标准的能量以及双尺度收敛, 对于二维或三维的情形下, 证明了流体的密度与温度的强收敛性, 进一步论证了当特征尺度$\varepsilon \to 0$时, 流体的均匀化结果. 从另一个层面解释了文献中的结果.

关键词: 渐近分析; 均匀化; Navier-Stokes流; Gibbs 方程

MR(2010)**主题分类号**: 76D05; 76M50; 74Q10 **中图分类号**: O13; O29