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FURTHER DISCUSSION ON UNCERTAINTY PRINCIPLES FOR THE α-FOCK SPACE F_α^2

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Abstract: In this article, for $\alpha > 0$, we characterize several versions uncertainty principles of self-adjoint operators and linear operators for the α-fock space F_α^2 in the complex plane. By using the general result from functional analysis, we find two linear operators $Tf = \frac{f'}{\alpha}$ and $T^* = zf$ to construct two self-adjoint operators A and B such that $[A, B]$ is a scalar multiple of the identity operator on F_α^2, and obtain some more accurate results about the uncertainty principles for the α-fock space F_α^2, where T^* is the adjoint of T, $[A, B] = AB - BA$ is the commutator of A and B, which extends and completes the results of Qu [1] and Zhu [2].

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1 Introduction

Let $\mathbb{C}$ be the complex plane, for any positive parameter α, we consider

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z)$$

be the Gaussian measure on $\mathbb{C}$, where $dA(z) = dxdy$ is the Euclidean area measure on the complex plane. We define the α-fock space F_α^2 as follow:

$$F_\alpha^2 = L^2(\mathbb{C}, d\lambda_\alpha) \cap H(\mathbb{C}),$$

where $H(\mathbb{C})$ is the space of all entire functions. It is easy to show that F_α^2 is a Hilbert space with the following inner product inherited from $L^2(\mathbb{C}, d\lambda_\alpha)$:

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda_\alpha(z),$$

accordingly define the norm $\|f\|_{2,\alpha}$ by

$$\|f\|_{2,\alpha} = \Big(\frac{\alpha}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} d\lambda_\alpha(z)\Big)^{\frac{1}{2}}.$$

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It is well know that the Fock space has become one of the vitally important mathematical tools of quantum physics. Thus, it is significative to study the uncertainty principle for the Fock space. In fact, it was an extensive interest in study of uncertainty principles for fock space F^2. In particular, some versions uncertainty principles of self-adjoint operators for the fock space F^2 were obtained, for some details, see [1, 2]. In addition, an inequality of uncertainty principle about the average value and the covariance of self-adjoint operators for the fock space F^2 also was proved in [1], and this result is due to the uncertainty principle of signal analysis, see for example [3]. On the other hand, The uncertainty principles of linear operators for the fock space F^2 also can be found in the article [4]. Moreover, we invite the interested reader to see [5–7] for other perspectives in the study of uncertainty principles see [8–13]. Based on these work, our goal here is to introduce a positive parameter α and extend uncertainty inequalities on two fronts. For one thing, we introduce a positive parameter α and obtain two different forms uncertainty principles of self-adjoint operators for the α-fock space F_α^2, see Section 2. For another thing, we study the uncertainty principles of linear operators for α-fock space F_α^2, see Section 3.

Note that all results discussed in this article are on the complex plane $\mathbb{C}$, there is no explanation below.

2 Uncertainty Principles of Self-Adjoint Operators

In [14], an uncertainty principle about self-adjoint operators from functional analysis is stated as follows.

Theorem 1 [14] Suppose A and B are self-adjoint operators, possibly unbounded, on Hilbert space H. Then

$$\|(A-aI)f\|\|(B-bI)f\| \geq \frac{1}{2} \mid \langle [A,B]f,f\rangle \mid \tag{2.1}$$

for all $f \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$ and all $a, b \in \mathbb{C}$, where $\mathrm{Dom}(AB)$ and $\mathrm{Dom}(BA)$ are the domains of the operators AB and BA, respectively. Here $[A,B] = AB - BA$ is the commutator of A and B, and I is the identity operator. Furthermore, equality in (2.1) holds if and only if $(A-aI)f$ and $(B-bI)f$ are purely imaginary scalar multiples of one another.

Proof This result is very useful and widely known, see page 27 of [14] for a proof.

Later, a sharper inequality about (2.1) was provided in [15].

Theorem 2 [15] Suppose A and B are self-adjoint operators, possibly unbounded, on Hilbert space H. Then we have

$$\|(A-aI)f\|\|(B-bI)f\| \geq \frac{1}{2}\sqrt{|\langle [A,B]f,f\rangle|^2 + |\langle [A-aI,B-bI]_+f,f\rangle|^2} \tag{2.2}$$

for all $f \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$ and all $a, b \in \mathbb{C}$. Here $[A-aI, B-bI]_+ = (A-aI)(B-bI) + (B-bI)(A-aI)$ and I is the identity operator.

Proof This is proved. See [15] for some details.

Combining with the above theorems, we construct two natural self-adjoint operators A and B such that $[A,B]$ is a scalar multiple of the identity operator which based on the operator of multiplication by z and constant multiple of the differentiation operator on F^2_α, then an uncertainty principle arises. Next, we collect two lemmas which provided a crucial evidence in proving Theorem 3.

Lemma 1 Let $T : F^2_\alpha \to F^2_\alpha$ be the constant multiple of the differentiation operator, that is $(Tf)(z) = \frac{f'(z)}{\alpha}$, then its adjoint T^* is given by $(T^*f)(z) = zf(z)$.

Proof This is proved. See [16] for some details.

It is very easy to check that

$$[T, T^*]f = \frac{(zf)'}{\alpha} - \frac{zf'}{\alpha} = \frac{f}{\alpha}. \tag{2.3}$$

Thus we consider the following two self-adjoint operators on F^2_α:

$$A = T + T^*, B = i(T - T^*),$$

that is

$$Af = \frac{f'}{\alpha} + zf, Bf = i\Big(\frac{f'}{\alpha} - zf\Big) \tag{2.4}$$

It follows from [1, 2] that, for a function $f \in F^2_\alpha$, if $\dfrac{f'}{\alpha} \in F^2_\alpha$, then both Af and Bf are well defined. If both $\dfrac{f'}{\alpha} + zf$ and $\dfrac{f'}{\alpha} - zf$ are in F^2_α, it's obvious that $\dfrac{f'}{\alpha}$ and zf are in F^2_α. Therefore, the intersection of the domains of A and B consists of those function f such that $\dfrac{f'}{\alpha}$ (or zf) is still in F^2_α. It's possible to identify the domains of AB, BA, and their intersection as well.

Lemma 2 For the operators A and B defined above, we have $[A, B] = -\frac{2}{\alpha}iI$, where I is the identity operator on F^2_α and i is the imaginary unit.

Proof From (2.3) and (2.4), we have

$$AB - BA = 2i(T^*T - TT^*) = -\frac{2}{\alpha}iI.$$

This proves the desired result.

We now derive the first version of the uncertainty principle about self-adjoint operators on F^2_α.

Theorem 3 Let $f \in F^2_\alpha$ and $f', f'' \in F^2_\alpha$, then we have

$$\begin{aligned}\Big\|\frac{f'}{\alpha} + zf - af\Big\|_{2,\alpha}\Big\|\frac{f'}{\alpha} - zf + ibf\Big\|_{2,\alpha} \geq \Big[\frac{1}{\alpha^2}\|f\|^4_{2,\alpha} + \Big|i\Big\langle\frac{f''}{\alpha^2}, f\Big\rangle \\ -i\langle z^2f, f\rangle - (b + ai)\Big\langle\frac{f'}{\alpha}, f\Big\rangle - (b - ai)\langle zf, f\rangle + ab\|f\|^2_{2,\alpha}\Big|^2\Big]^{\frac{1}{2}}\end{aligned} \tag{2.5}$$

for all $a, b \in \mathbb{C}$. Here f'' is the second derivative of f.

Proof From (2.4), we get

$$\|(A - aI)f\|_{2,\alpha} = \Big\|\frac{f'}{\alpha} + zf - af\Big\|_{2,\alpha},$$

$$\|(B - bI)f\|_{2,\alpha} = \Big\|i\Big(\frac{f'}{\alpha} - zf\Big) - bf\Big\|_{2,\alpha} = \Big\|\frac{f'}{\alpha} - zf + ibf\Big\|_{2,\alpha}.$$

Also from Lemma 2, we have $|\langle[A,B]f,f\rangle|^2=\dfrac{4}{\alpha^2}\|f\|^4_{2,\alpha}$,

$$\begin{aligned}[A-aI,B-bI]_+&=(A-aI)(B-bI)+(B-bI)(A-aI)\\&=AB+BA-2bA-2aB+2abI\\&=2i(TT-T^*T^*)-2b(T+T^*)-2ai(T-T^*)+2abI.\end{aligned}$$

This implies that

$$\begin{aligned}\langle[A-aI,B-bI]_+f,f\rangle=&\Big\langle\Big[2i\Big(\frac{f''}{\alpha^2}-z^2f\Big)-2b\Big(\frac{f'}{\alpha}+zf\Big)-2ai\Big(\frac{f'}{\alpha}-zf\Big)+2abf\Big],f\Big\rangle\\=&2i\Big\langle\frac{f''}{\alpha^2},f\Big\rangle-2i\langle z^2f,f\rangle-(2b+2ai)\Big\langle\frac{f'}{\alpha},f\Big\rangle\\&-(2b-2ai)\langle zf,f\rangle+2ab\|f\|^2_{2,\alpha}.\end{aligned}$$

From the above, the inequality in (2.5) follows from (2.2).

This completes the proof of the theorem.

In order to prove Corollary 1, a discussion about the minimization is also needed.

Lemma 3 If fix some function $f\in F^2_\alpha$, for any $a,b\in\mathbb{C}$ and the operators T and T^* defined above, we have

$$\min_{a\in\mathbb{C}}\|(T+T^*)f-af\|^2_{2,\alpha}=\|(T+T^*)f\|^2_{2,\alpha}-\frac{|\langle(T+T^*)f,f\rangle|^2}{\|f\|^2_{2,\alpha}}$$

and the minimum is attained when

$$a=\frac{\langle(T+T^*)f,f\rangle}{\|f\|^2_{2,\alpha}}.$$

Similarly, we have

$$\min_{b\in\mathbb{C}}\|(T-T^*)f+ibf\|^2_{2,\alpha}=\|(T-T^*)f\|^2_{2,\alpha}-\frac{|\langle(T-T^*)f,f\rangle|^2}{\|f\|^2_{2,\alpha}}.$$

Also the minimum is attained when

$$b=\frac{\langle(T-T^*)f,f\rangle}{\|f\|^2_{2,\alpha}}i.$$

Proof The conclusion is obviously established. We omit the details.

If f is a unit vector in F^2_α, we obtain the following corollary of the uncertainty principle.

Corollary 1 If f is a unit vector in F^2_α, $f',f''\in F^2_\alpha$, then we have

$$\begin{aligned}&\Big(\Big\|\frac{f'}{\alpha}+zf\Big\|^2_{2,\alpha}-\Big|\Big\langle\frac{f'}{\alpha}+zf,f\Big\rangle\Big|^2\Big)\Big(\Big\|\frac{f'}{\alpha}-zf\Big\|^2_{2,\alpha}-\Big|\Big\langle\frac{f'}{\alpha}-zf,f\Big\rangle\Big|^2\Big)\\\geq\ &\frac{1}{\alpha^2}+\Big|\Big\langle\frac{f''}{\alpha^2},f\Big\rangle-\langle z^2f,f\rangle-\Big(\Big\langle\frac{f'}{\alpha},f\Big\rangle\Big)^2+(\langle zf,f\rangle)^2\Big|^2.\end{aligned}$$

Proof Since f is a unit vector, from Theorem 3 and its proof, we get

$$|\langle[A,B]f,f\rangle|^2=\frac{4}{\alpha^2}.$$

Also

$$\begin{aligned}\langle [A-aI, B-bI]_+ f, f\rangle =& 2i\Big\langle \frac{f''}{\alpha^2}, f\Big\rangle - 2i\langle z^2 f, f\rangle - (2b+2ai)\Big\langle \frac{f'}{\alpha}, f\Big\rangle \\ & - (2b-2ai)\langle zf, f\rangle + 2ab,\end{aligned}$$

which easily implies that

$$\begin{aligned}& \Big\|\frac{f'}{\alpha} + zf - af\Big\|_{2,\alpha}\Big\|\frac{f'}{\alpha} - zf + ibf\Big\|_{2,\alpha} \\ \geq \ & \Big[\frac{1}{\alpha^2} + \Big|i\Big\langle \frac{f''}{\alpha^2}, f\Big\rangle - i\langle z^2 f, f\rangle - (b+ai)\Big\langle \frac{f'}{\alpha}, f\Big\rangle - (b-ai)\langle zf, f\rangle + ab\Big|^2\Big]^{\frac{1}{2}}.\end{aligned}$$

Then it follows from combining this with the minimization argument of Lemma 3 and

$$a = \Big\langle \frac{f'}{\alpha} + zf, f\Big\rangle, b = \Big\langle \frac{f'}{\alpha} - zf, f\Big\rangle i.$$

This completes the proof of corollary.

Corollary 2 Let $f \in F^2_\alpha$, and $f', f'' \in F^2_\alpha$, then we have

$$\Big\|\frac{f'}{\alpha} + zf\Big\|_{2,\alpha}\Big\|\frac{f'}{\alpha} - zf\Big\|_{2,\alpha} \geq \Big(\frac{\|f\|^4_{2,\alpha}}{\alpha^2} + \Big|\Big\langle \frac{f''}{\alpha^2}, f\Big\rangle - \langle z^2 f, f\rangle\Big|^2\Big)^{\frac{1}{2}}.$$

Proof This follows directly from Theorem 3 by setting $a = b = 0$.

In fact, we can improve the argument above to obtain a more interesting result of uncertainty principle.

Corollary 3 Let $f \in F^2_\alpha$ and $f', f'' \in F^2_\alpha$. For any $\delta > 0$, then we have

$$\frac{\delta}{2}\Big\|\frac{f'}{\alpha} + zf\Big\|^2_{2,\alpha} + \frac{1}{2\delta}\Big\|\frac{f'}{\alpha} - zf\Big\|^2_{2,\alpha} \geq \Big(\frac{\|f\|^4_{2,\alpha}}{\alpha^2} + \Big|\Big\langle \frac{f''}{\alpha^2}, f\Big\rangle - \langle z^2 f, f\rangle\Big|^2\Big)^{\frac{1}{2}}.$$

Proof From Corollary 2, we have the following estimates

$$\begin{aligned}\Big(\frac{\|f\|^4_{2,\alpha}}{\alpha^2} + \Big|\Big\langle \frac{f''}{\alpha^2}, f\Big\rangle - \langle z^2 f, f\rangle\Big|^2\Big)^{\frac{1}{2}} &\leq \Big\|\frac{f'}{\alpha} + zf\Big\|_{2,\alpha}\Big\|\frac{f'}{\alpha} - zf\Big\|_{2,\alpha} \\ &= \Big\|\sqrt{\delta}\Big(\frac{f'}{\alpha} + zf\Big)\Big\|_{2,\alpha}\Big\|\Big(\frac{f'}{\alpha} - zf\Big)/\sqrt{\delta}\Big\|_{2,\alpha} \\ &\leq \frac{1}{2}\Big[\Big\|\sqrt{\delta}\Big(\frac{f'}{\alpha} + zf\Big)\Big\|^2_{2,\alpha} + \Big\|\Big(\frac{f'}{\alpha} - zf\Big)/\sqrt{\delta}\Big\|^2_{2,\alpha}\Big] \\ &= \frac{\delta}{2}\Big\|\frac{f'}{\alpha} + zf\Big\|^2_{2,\alpha} + \frac{1}{2\delta}\Big\|\frac{f'}{\alpha} - zf\Big\|^2_{2,\alpha},\end{aligned}$$

which proves the desired result.

Extraordinarily, we now consider several versions of the uncertainty principle which are based on the geometric notions of angle and distance.

Corollary 4 Let $f \in F^2_\alpha$, not identically zero, and $\theta_\pm$ are the angles between f and $\frac{f'}{\alpha} \pm zf$ in F^2_α. Then we have

$$\begin{aligned}&\Big\|\frac{f'}{\alpha}+zf\Big\|_{2,\alpha}\Big\|\frac{f'}{\alpha}-zf\Big\|_{2,\alpha}|\sin(\theta_+)\sin(\theta_-)|\\ \geq\ &\left[\frac{\|f\|^4_{2,\alpha}}{\alpha^2}+\left|\Big\langle\frac{f''}{\alpha^2},f\Big\rangle-\langle z^2f,f\rangle-\frac{(\langle f'/\alpha,f\rangle)^2}{\|f\|^2_{2,\alpha}}+\frac{(\langle zf,f\rangle)^2}{\|f\|^2_{2,\alpha}}\right|^2\right]^{\frac{1}{2}},\end{aligned}$$

here $f', f'' \in F^2_\alpha$.

Proof In fact, we have

$$\begin{aligned}\Big\|\frac{f'}{\alpha}+zf\Big\|^2_{2,\alpha}-\frac{\langle f'/\alpha+zf,f\rangle^2}{\|f\|^2_{2,\alpha}}&=\Big\|\frac{f'}{\alpha}+zf\Big\|^2_{2,\alpha}\left[1-\left|\frac{\langle f'/\alpha+zf,f\rangle}{\|f'/\alpha+zf\|_{2,\alpha}\|f\|_{2,\alpha}}\right|^2\right]\\&=\Big\|\frac{f'}{\alpha}+zf\Big\|^2_{2,\alpha}(1-\cos^2(\theta_+))\\&=\Big\|\frac{f'}{\alpha}+zf\Big\|^2_{2,\alpha}\sin^2(\theta_+).\end{aligned}$$

The same arguement shows that

$$\Big\|\frac{f'}{\alpha}-zf\Big\|^2_{2,\alpha}-\frac{\langle f'/\alpha-zf,f\rangle^2}{\|f\|^2_{2,\alpha}}=\Big\|\frac{f'}{\alpha}-zf\Big\|^2_{2,\alpha}\sin^2(\theta_-).$$

Applying Corollary 1 with

$$a=\frac{\langle f'/\alpha+zf,f\rangle}{\|f\|^2_{2,\alpha}}$$

and

$$b=\frac{\langle f'/\alpha-zf,f\rangle}{\|f\|^2_{2,\alpha}}i.$$

Then we can obtain the desired result.

Corollary 5 Suppose f is a unit vector in F^2_α, $\theta_\pm$ are the angles between f and $\frac{f'}{\alpha} \pm zf$ in F^2_α, and $f', f'' \in F^2_\alpha$. Then for any $\delta > 0$, we have

$$\begin{aligned}&\Big(\frac{\delta}{2}\Big\|\frac{f'}{\alpha}+zf\Big\|^2_{2,\alpha}+\frac{1}{2\delta}\Big\|\frac{f'}{\alpha}-zf\Big\|^2_{2,\alpha}\Big)|\sin(\theta_+)\sin(\theta_-)|\\ \geq\ &\left[\frac{1}{\alpha^2}+\left|\Big\langle\frac{f''}{\alpha^2},f\Big\rangle-\langle z^2f,f\rangle-\Big(\Big\langle\frac{f'}{\alpha},f\Big\rangle\Big)^2+(\langle zf,f\rangle)^2\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Proof This desired result is clear by using the proofs of Corollaries 3 and 4.

Corollary 6 Suppose f a unit vector in F^2_α, $\theta_\pm$ are the angles between f and $\frac{f'}{\alpha} \pm zf$ in F^2_α, and $f', f'' \in F^2_\alpha$, then we have

$$\begin{aligned}&\Big(\Big\|\frac{f'}{\alpha}\Big\|^2_{2,\alpha}+\Big\|zf\Big\|^2_{2,\alpha}\Big)|\sin(\theta_+)\sin(\theta_-)|\\ \geq\ &\left[\frac{1}{\alpha^2}+\left|\Big\langle\frac{f''}{\alpha^2},f\Big\rangle-\langle z^2f,f\rangle-\Big(\Big\langle\frac{f'}{\alpha},f\Big\rangle\Big)^2+(\langle zf,f\rangle)^2\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Proof This follows directly from Corollary 5 by $\delta = 1$.

Motivated by Corollary 4, here we get the following results.

Corollary 7 Suppose f is any function in F_α^2, not identically zero, and $f', f'' \in F_\alpha^2$, then we have

$$\begin{aligned}&\operatorname{dist}\Big(\frac{f'}{\alpha}+zf,[f]\Big)\operatorname{dist}\Big(\frac{f'}{\alpha}-zf,[f]\Big)\\ \geq\ & \left[\frac{\|f\|_{2,\alpha}^4}{\alpha^2}+\Big|\Big\langle\frac{f''}{\alpha^2},f\Big\rangle-\langle z^2f,f\rangle-\frac{\big(\langle f'/\alpha,f\rangle\big)^2}{\|f\|_{2,\alpha}^2}+\frac{(\langle zf,f\rangle)^2}{\|f\|_{2,\alpha}^2}\Big|^2\right]^{\frac12},\end{aligned}$$

where $[f] = \mathbb{C}f$ is the one-dimensional subspace of F_α^2 spanned by f and $d(g, X)$ denotes the distance in F_α^2 from g to X.

Proof This is an equivalent state of Corollary 4, because

$$\operatorname{dist}\Big(\frac{f'}{\alpha}+zf,[f]\Big)=\Big\|\frac{f'}{\alpha}+zf\Big\|_{2,\alpha}|\sin(\theta_+)|$$

and

$$\operatorname{dist}\Big(\frac{f'}{\alpha}-zf,[f]\Big)=\Big\|\frac{f'}{\alpha}-zf\Big\|_{2,\alpha}|\sin(\theta_-)|.$$

Hence the result is clear from Corollary 4.

Now we can do a significant extension. Actually, all conclusions above that we have done for the α-fock space F_α^2 remains valid for any operator T and its adjoint operator T^* which satisfies $[T, T^*] = mI$, here m is a positive constant.

Corollary 8 If f is any function in F_α^2, Suppose T is any operator on F_α^2 such that $[T, T^*] = mI$, then we have

$$\begin{aligned}&\|Tf+T^*f-af\|_{2,\alpha}\|Tf-T^*f+ibf\|_{2,\alpha}\\ \geq\ & [m^2\|f\|_{2,\alpha}^4+|i(TT-T^*T^*)f-b(T+T^*)f-ai(T-T^*)f+abf|^2]^{\frac12},\end{aligned}$$

here m is a positive constant.

Proof This follows from the proofs of Lemma 2 and Theorem 3.

It is worth paying attention to the case that when the function f' (or equivalently, the function zf) also belongs to the α-fock space F_α^2, which is not always the case, the interested reader could see [1] for some details. When the function f' is not in F_α^2, each of the left-hand sides of the inequalities above is infinite, Hence the inequality always becomes valid.

Next we will obtain a different version uncertainty principle of self-adjoint operators, for this purpose, we first give the following definition which also be found in [1].

Definition 1 [1] If $f \in F_\alpha^2$, suppose P is any self-adjoint operator on F_α^2. Then the average value and the covariance of operator P are defined respectively by

$$\langle P\rangle_f=\langle Pf,f\rangle=\int_{\mathbb{C}}\overline{f}Pf\,d\lambda_\alpha(z) \tag{2.6}$$

and

$$\sigma_P^2(f)=\int_{\mathbb{C}}\overline{f}(P-\langle P\rangle I)^2f\,d\lambda_\alpha(z). \tag{2.7}$$

The following lemma plays an important role in proving Theorem 4.

Lemma 4 Let $f \in F^2_\alpha$, not identically zero, and $f' \in F^2_\alpha$, then we have

$$\begin{aligned}\sigma_A^2(f) &= \int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{f'/\alpha + zf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\\ &\quad + \int_{\mathbb{C}} \Big|\mathrm{Re}\Big(\frac{f'/\alpha + zf}{f}\Big) - \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big) d\lambda_\alpha(z)\Big|^2 |f|^2 d\lambda_\alpha(z),\\ \sigma_B^2(f) &= \int_{\mathbb{C}} \Big[\mathrm{Im}\Big(i\frac{f'/\alpha - zf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\\ &\quad + \int_{\mathbb{C}} \Big|\mathrm{Re}\Big(i\frac{f'/\alpha - zf}{f}\Big) - \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big) d\lambda_\alpha(z)\Big|^2 |f|^2 d\lambda_\alpha(z).\end{aligned}$$

Proof By (2.6) and (2.7), we conclude that

$$\begin{aligned}\sigma_A^2(f) &= \int_{\mathbb{C}} \overline{f}(A - \langle A\rangle I)^2 f d\lambda_\alpha(z) = \int_{\mathbb{C}} |(A - \langle A\rangle I) f|^2 d\lambda_\alpha(z)\\ &= \int_{\mathbb{C}} \Big|\Big(\frac{Af}{f} - \langle A\rangle\Big) f\Big|^2 d\lambda_\alpha(z) = \int_{\mathbb{C}} \Big|\frac{Af}{f} - \langle A\rangle\Big|^2 |f|^2 d\lambda_\alpha(z)\\ &= \int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Af}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z) + \int_{\mathbb{C}} \Big[\mathrm{Re}\Big(\frac{Af}{f}\Big) - \langle A\rangle\Big]^2 |f|^2 d\lambda_\alpha(z)\\ &= \int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{f'/\alpha + zf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\\ &\quad + \int_{\mathbb{C}} \Big|\mathrm{Re}\Big(\frac{f'/\alpha + zf}{f}\Big) - \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big) d\lambda_\alpha(z)\Big|^2 |f|^2 d\lambda_\alpha(z).\end{aligned}$$

The same procedure may be easily adapted to obtain that

$$\begin{aligned}\sigma_B^2(f) &= \int_{\mathbb{C}} \Big[\mathrm{Im}\Big(i\frac{f'/\alpha - zf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\\ &\quad + \int_{\mathbb{C}} \Big|\mathrm{Re}\Big(i\frac{f'/\alpha - zf}{f}\Big) - \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big) d\lambda_\alpha(z)\Big|^2 |f|^2 d\lambda_\alpha(z).\end{aligned}$$

This proves the desired estimate.

Carefully examining the proof of Lemma 4, we obtain the following characterization.

Theorem 4 Let $f \in F^2_\alpha$, not identically zero, and $f' \in F^2_\alpha$, then we have

$$\begin{aligned}\sigma_A^2(f)\sigma_B^2(f) \geq &\Big|\int_{\mathbb{C}} \Big[\mathrm{Re}\Big(\frac{\overline{f}f'}{\alpha}\Big) - x\overline{f}f\Big]\Big[\frac{f'/\alpha + zf}{f} - \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big) d\lambda_\alpha(z)\Big] d\lambda_\alpha(z)\Big|^2\\ &+ \Big[\int_{\mathbb{C}} \Big|\frac{f'}{\alpha} + zf - \Big(\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big) d\lambda_\alpha(z)\Big) f\Big|\\ &\times \Big|\mathrm{Re}\Big(i\frac{f'/\alpha - zf}{f}\Big) - i\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big) d\lambda_\alpha(z)\Big| |f| d\lambda_\alpha(z)\Big]^2\end{aligned}$$

for all $f \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$. Here $z = x + iy$.

Proof Recall that

$$\begin{aligned}\sigma_A^2(f)\sigma_B^2(f) = &\int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Bf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\sigma_A^2(f)\\ &+ \int_{\mathbb{C}} \Big|\mathrm{Re}\Big(\frac{Bf}{f}\Big) - \langle B\rangle\Big|^2 |f|^2 d\lambda_\alpha(z)\sigma_A^2(f).\end{aligned}$$

Hence we divide the proof into two steps.

For one thing, we consider the inequality that

$$\begin{aligned}\int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Bf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\sigma_A^2(f) \geq &\Big|\int_{\mathbb{C}} \Big[\mathrm{Re}\Big(\frac{\overline{f}f'}{\alpha}\Big) - x\overline{f}f\Big]\Big[\frac{f'/\alpha + zf}{f}\\ &- \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big)d\lambda_\alpha(z)\Big]d\lambda_\alpha(z)\Big|^2.\end{aligned}$$

Actually, we have

$$\begin{aligned}&\int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Bf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z)\sigma_A^2(f) = \int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Bf}{f}\Big)\Big]^2 |f|^2 d\lambda_\alpha(z) \int_{\mathbb{C}} |(A - \langle A\rangle I)f|^2 f d\lambda_\alpha(z)\\
\geq&\Big|\int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Bf}{f}\Big)\Big]\overline{f}(A - \langle A\rangle I)f d\lambda_\alpha(z)\Big|^2 = \Big|\int_{\mathbb{C}} \Big[\mathrm{Im}\Big(\frac{Bf}{f}\Big)\Big]|f|^2\frac{1}{f}(Af - \langle A\rangle f)d\lambda_\alpha(z)\Big|^2\\
=&\Big|\int_{\mathbb{C}} \mathrm{Im}\Big(\frac{Bf}{f}\Big)|f|^2\Big(\frac{Af}{f} - \langle A\rangle\Big)d\lambda_\alpha(z)\Big|^2 = \Big|\int_{\mathbb{C}} \mathrm{Im}(\overline{f}Bf)\Big(\frac{Af}{f} - \langle A\rangle\Big)d\lambda_\alpha(z)\Big|^2\\
=&\Big|\int_{\mathbb{C}} \frac{1}{2i}(\overline{f}Bf - f\overline{Bf})\Big(\frac{Af}{f} - \langle A\rangle\Big)d\lambda_\alpha(z)\Big|^2\\
=&\frac{1}{4}\Big|\int_{\mathbb{C}} \Big[\overline{f}i\Big(\frac{f'}{\alpha} - zf\Big) + if\overline{\Big(f'/\alpha - zf\Big)}\Big]\Big(\frac{Af}{f} - \langle A\rangle\Big)d\lambda_\alpha(z)\Big|^2\\
=&\frac{1}{4}\Big|\int_{\mathbb{C}} \Big(\frac{\overline{f}f'}{\alpha} - z\overline{f}f + \frac{\overline{f'}f}{\alpha} - \overline{z}\overline{f}f\Big)\Big(\frac{Af}{f} - \langle A\rangle\Big)d\lambda_\alpha(z)\Big|^2\\
=&\Big|\int_{\mathbb{C}} \Big[\mathrm{Re}\Big(\frac{\overline{f}f'}{\alpha}\Big) - x\overline{f}f\Big]\Big[\frac{f'/\alpha + zf}{f} - \int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big)d\lambda_\alpha(z)\Big]\Big|^2.\end{aligned}$$

For another thing, we consider the another inequality that

$$\begin{aligned}&\Big[\int_{\mathbb{C}}\Big|\frac{f'}{\alpha} + zf - \Big(\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} + zf\Big)d\lambda_\alpha(z)\Big)f\Big|\\
&\times\Big|\mathrm{Re}\Big[i\frac{f'/\alpha - zf}{f}\Big] - i\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big)d\lambda_\alpha(z)\Big||f|d\lambda_\alpha(z)\Big]^2\\
=&\Big[\int_{\mathbb{C}} |Af - \langle A\rangle f|\Big|\mathrm{Re}\Big(i\frac{f'/\alpha - zf}{f}\Big) - i\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big)d\lambda_\alpha(z)\Big||f|d\lambda_\alpha(z)\Big]^2\\
\leq&\int_{\mathbb{C}} |Af - \langle A\rangle f|^2 d\lambda_\alpha(z)\int_{\mathbb{C}}\Big|\mathrm{Re}\Big(i\frac{f'/\alpha - zf}{f}\Big) - i\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big)d\lambda_\alpha(z)\Big|^2|f|^2 d\lambda_\alpha(z)\\
=&\sigma_A^2(f)\int_{\mathbb{C}}\Big|\mathrm{Re}\Big(i\frac{f'/\alpha - zf}{f}\Big) - i\int_{\mathbb{C}} \overline{f}\Big(\frac{f'}{\alpha} - zf\Big)d\lambda_\alpha(z)\Big|^2|f|^2 d\lambda_\alpha(z).\end{aligned}$$

This completes the proof of the theorem.

3 Uncertainty Principles of Linear Operators

In this section, we turn out our attention to the uncertainty principles of linear operators on F^2_α. To achieve that end, we let A^* and B^* be the adjoint of the operators A and B respectively. Throughout the article, we shall use the notation

$$\mathrm{Dom}(A|B) = \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA) \cap \mathrm{Dom}(A^*) \cap \mathrm{Dom}(B^*).$$

Definition 2 [17] Suppose A is linear operator with domain and range in the same complex Hilbert space H, for any nonzero $f \in \mathrm{Dom}(A)$, we defined

$$\Delta_f(A) = \Big(\|Af\|^2 - \frac{|\langle Af, f\rangle|^2}{\|f\|^2}\Big)^{\frac{1}{2}}, \tag{3.1}$$

which is equal to

$$\Delta_f(A) = \Big\|Af - \frac{\langle Af, f\rangle}{\|f\|^2} f\Big\|. \tag{3.2}$$

More interestingly, as a generalization of Lemma 3, we have the following lemma.

Lemma 5 If fix some function $f \in F^2_\alpha$, suppose A and B are linear operators on F^2_α. Then for any $a, b \in \mathbb{C}$, we have

$$\min_{a\in\mathbb{C}} \|(A - aI)f\|_{2,\alpha} = \Delta_f(A), \tag{3.3}$$

$$\min_{a\in\mathbb{C}} \|(A^* - \overline{a}I)f\|_{2,\alpha} = \Delta_f(A^*). \tag{3.4}$$

Furthermore, the minimum of (3.3) and (3.4) are attained when $a = \dfrac{\langle Af, f\rangle}{\|f\|^2_{2,\alpha}}$. Similarly, we have

$$\min_{b\in\mathbb{C}} \|(B - bI)f\|_{2,\alpha} = \Delta_f(B), \tag{3.5}$$

$$\min_{b\in\mathbb{C}} \|(B^* - \overline{b}I)f\|_{2,\alpha} = \Delta_f(B^*). \tag{3.6}$$

Also the minimum of (3.5) and (3.6) are attained when

$$b = \frac{\langle Bf, f\rangle}{\|f\|^2_{2,\alpha}}.$$

Proof A direct calculation shows that

$$\begin{aligned}\|(A - aI)f\|^2_{2,\alpha} &= \langle (A - aI)f, (A - aI)f\rangle \\ &= \|Af\|^2_{2,\alpha} + |a|^2\|f\|^2_{2,\alpha} - \overline{a}\langle Af, f\rangle - a\langle f, Af\rangle \\ &= \|Af\|^2_{2,\alpha} + \|f\|^2_{2,\alpha}\Big[|a|^2 - 2\frac{\mathrm{Re}(\overline{a}\langle Af, f\rangle)}{\|f\|^2_{2,\alpha}}\Big],\end{aligned}$$

which easily implies that

$$\min_{a\in\mathbb{C}} \|(A-aI)f\|_{2,\alpha}^2 = \|Af\|_{2,\alpha}^2 - \frac{|\langle Af, f\rangle|^2}{\|f\|_{2,\alpha}^2}$$

or equivalently by (3.1) and (3.2), we have $\min_{a\in\mathbb{C}} \|(A-aI)f\|_{2,\alpha} = \Delta_f(A)$ and the minimum is attained when

$$|a|^2 = 2\frac{\mathrm{Re}(\overline{a}\langle Af, f\rangle)}{\|f\|_{2,\alpha}^2},$$

that is $a = \dfrac{\langle Af, f\rangle}{\|f\|_{2,\alpha}^2}$. The same argument may be valid to obtain (3.4)–(3.6).

This finishes the proof of the lemma.

The following theorem on the commutator is another generalization of the Heisenberg uncertainty principle.

Theorem 5 [17] Let A and B be linear operators with domain and range in the same complex Hilbert space H, for any nonzero $f \in \mathrm{Dom}(A|B)$, there holds

$$|\langle [A,B]f, f\rangle| \le \Delta_f(A)\Delta_f(B^*) + \Delta_f(A^*)\Delta_f(B). \tag{3.7}$$

Proof It is very easy to verify that for any nonzero $f \in \mathrm{Dom}(A|B)$,

$$\langle [A,B]f, f\rangle = \langle ABf, f\rangle - \langle BAf, f\rangle = \langle Bf, A^*f\rangle - \langle Af, B^*f\rangle.$$

This implies that

$$|\langle [A,B]f, f\rangle| \le |\langle Bf, A^*f\rangle| + |\langle Af, B^*f\rangle| \le \|Af\|\|B^*f\| + \|A^*f\|\|Bf\|. \tag{3.8}$$

For any $a, b \in \mathbb{C}$, we replace A and B above by $A - aI$ and $B - bI$, respectively to obtain that

$$|\langle [A,B]f, f\rangle| \le \|(A-aI)f\|\|(B^* - \overline{b}I)f\| + \|(A^* - \overline{a}I)f\|\|(B-bI)f\|.$$

According to Lemma 5, we conclude that

$$\begin{aligned} &\min_{a\in\mathbb{C}}\{\|Af - af\|\|B^*f - \overline{b}f\| + \|A^*f - \overline{a}f\|\|Bf - bf\|\} \\ = \ &\Delta_f(A)\Delta_f(B^*) + \Delta_f(A^*)\Delta_f(B) \end{aligned}$$

from which (3.8) follows. and the minimum value is attained uniquely at $a = \dfrac{\langle Af, f\rangle}{\|f\|_{2,\alpha}^2}$ and $b = \dfrac{\langle Bf, f\rangle}{\|f\|_{2,\alpha}^2}$. This completes the desired result.

From Theorem 5, we know that if we find out two linear operators and their adjoint on F_α^2, then an uncertainty principle arises. To this end, we still consider the operator T and its adjoint T^* which defined from Lemma 1. To simplify notation, let $A = T$, and $B = T^*$, namely $A = \dfrac{f'}{\alpha}$ and $B = zf$. It is obvious that A and B are linear operators on F_α^2, and $A^*f = Bf, B^*f = Af$. Then we characterize the first uncertainty principle of linear operators on F_α^2 as follows.

Theorem 6 Let $f, f' \in F_\alpha^2$, for any $a \in \mathbb{C}$, then we have

$$\left\|\frac{f'}{\alpha} - af\right\|_{2,\alpha}^2 + \|zf - \overline{a}f\|_{2,\alpha}^2 \geq \frac{1}{\alpha}\|f\|_{2,\alpha}^2. \tag{3.9}$$

Proof From Lemma 1, we have $[A,B]f = \dfrac{1}{\alpha}f$. Following the method used in the proof of Theorem 5, we have

$$|\langle [A,B]f, f\rangle| \leq \|Af - af\|_{2,\alpha}^2 + \|A^*f - \overline{a}f\|_{2,\alpha}^2.$$

More specifically

$$\frac{1}{\alpha}\|f\|_{2,\alpha}^2 \leq \left\|\frac{f'}{\alpha} - af\right\|_{2,\alpha}^2 + \|zf - \overline{a}f\|_{2,\alpha}^2.$$

This proves the desired result.

Note that when f' is not in F_α^2, the left-hand sides of the inequality of (3.9) is infinite, so the inequality becomes trivial.

When f is a unit vector in F_α^2 , then we obtain the following corollary.

Corollary 9 Suppose f is a unit vector in F_α^2 and $f' \in F_\alpha^2$, then we have

$$\|zf\|_{2,\alpha}^2 + \left\|\frac{f'}{\alpha}\right\|_{2,\alpha}^2 - 2\left|\left\langle \frac{f'}{\alpha}, f\right\rangle\right|^2 \geq \frac{1}{\alpha}.$$

Proof Since f is a unit vector, by Lemma 5, we get

$$\min_{a\in\mathbb{C}} \|(A - aI)f\|_{2,\alpha}^2 = \|Af\|_{2,\alpha}^2 - |\langle Af, f\rangle|^2 = \left\|\frac{f'}{\alpha}\right\|_{2,\alpha}^2 - \left|\left\langle \frac{f'}{\alpha}, f\right\rangle\right|^2$$

and

$$\begin{aligned}\min_{a\in\mathbb{C}} \|(A^* - \overline{a}I)f\|_{2,\alpha}^2 &= \|A^*f\|_{2,\alpha}^2 - |\langle A^*f, f\rangle|^2 \\ &= \|zf\|_{2,\alpha}^2 - |\langle zf, f\rangle|^2 = \|zf\|_{2,\alpha}^2 - \left|\left\langle \frac{f'}{\alpha}, f\right\rangle\right|^2.\end{aligned}$$

Then the desired corollary can be proved by Theorem 5.

The results in Theorem 6 and Corollary 9 rely upon Theorem 5, on the other hand, the proof of Theorem 5 rely upon (3.8). While in [13], the author try to find two operators U and V to reduce the upper bound in (3.8) and require that $f \in \mathrm{Dom}(A|B) \cap \mathrm{Dom}(A|U) \cap \mathrm{Dom}(B|V) \cap \mathrm{Dom}(V|U)$ and such that

$$[A,U]f = [B,V]f = [V,U]f = 0, \tag{3.10}$$

then we need find two linear operators on F_α^2 which satisfies (2.17), this is a formidable task. For any $a, b_0, b_1 \in \mathbb{C}$, we might as well suppose V to be a multiple of the identity, namely $V = aI$, and let another operator $U = b_0I + b_1A$. Then, it is very easy to check that

$$[B,V]f = (aB - aB)f = 0.$$

Also

$$[A,U]f = A(Uf) - U(Af) = A\Big(b_0 f + \frac{b_1 f'}{\alpha}\Big) - U\Big(\frac{f'}{\alpha}\Big) = 0,$$

$[V,U]f = (aU - aU)f = 0$, which shows that U and V satisfies (3.10). Hence we can obtain the following theorem.

Theorem 7 Let $f, f' \in F^2_\alpha$, for any $a, b_0, b_1 \in \mathbb{C}$, then we have

$$\begin{aligned}&\Big\{\Big\|\frac{f'}{\alpha} - \overline{b_0}f - \overline{b_1}zf\Big\|^2_{2,\alpha} + \Big\|zf - b_0 f - \frac{b_1 f'}{\alpha}\Big\|^2_{2,\alpha}\Big\}^{\frac{1}{2}}\\ &\times\Big\{\Big\|zf - \frac{\langle zf, f\rangle}{\|f\|^2_{2,\alpha}}f\Big\|^2_{2,\alpha} + \Big\|\frac{f'}{\alpha} - \frac{\langle f'/\alpha, f\rangle}{\|f\|^2_{2,\alpha}}f\Big\|^2_{2,\alpha}\Big\}^{\frac{1}{2}} \geq \frac{1}{\alpha}\|f\|^2_{2,\alpha}\end{aligned}$$

for all $f \in \mathrm{Dom}(A|B) \cap \mathrm{Dom}(A|U) \cap \mathrm{Dom}(B|V) \cap \mathrm{Dom}(V|U)$, and $a, b_0, b_1 \in \mathbb{C}$.

Proof By (3.10), we get

$$\begin{aligned}\langle [A,B]f, f\rangle =& \langle [A,B]f, f\rangle + \langle [B,V]f, f\rangle + \langle [U,A]f, f\rangle + \langle [V,U]f, f\rangle\\ =& \langle Bf, A^*f\rangle - \langle Af, B^*f\rangle + \langle Vf, B^*f\rangle - \langle Bf, V^*f\rangle\\ &+ \langle Af, U^*f\rangle - \langle Uf, A^*f\rangle + \langle Uf, V^*f\rangle - \langle Vf, U^*f\rangle\\ =& \langle Bf - Uf, A^*f - V^*f\rangle - \langle Af - Vf, B^*f - U^*f\rangle.\end{aligned}$$

This implies that

$$|\langle [A,B]f, f\rangle| \leq |\|Bf - Uf\|_{2,\alpha}|\|A^*f - V^*f\|_{2,\alpha} + |\|Af - Vf\|_{2,\alpha}|\|B^*f - U^*f\|_{2,\alpha}.$$

It is very easy to verify that

$$\begin{aligned}&|\langle [A,B]f, f\rangle| = \frac{1}{\alpha}\|f\|^2_{2,\alpha},\\ &\|Bf - Uf\|_{2,\alpha} = \Big\|zf - b_0 f - \frac{b_1 f'}{\alpha}\Big\|_{2,\alpha},\\ &\|A^*f - V^*f\|_{2,\alpha} = \|zf - \overline{a}f\|_{2,\alpha},\\ &\|Af - Vf\|_{2,\alpha} = \Big\|\frac{f'}{\alpha} - af\Big\|_{2,\alpha},\\ &\|B^*f - U^*f\|_{2,\alpha} = \Big\|\frac{f'}{\alpha} - \overline{b_0}f - \overline{b_1}zf\Big\|_{2,\alpha}.\end{aligned}$$

Then we conclude that

$$\begin{aligned}\frac{1}{\alpha}\|f\|^2_{2,\alpha} \quad\leq\quad & \|zf - \overline{a}f\|_{2,\alpha}\Big\|zf - b_0 f - \frac{b_1 f'}{\alpha}\Big\|_{2,\alpha}\\ &+\Big\|\frac{f'}{\alpha} - af\Big\|_{2,\alpha}\Big\|\frac{f'}{\alpha} - \overline{b_0}f - \overline{b_1}zf\Big\|_{2,\alpha}.\end{aligned} \tag{3.11}$$

From Lemma 5, we have

$$\begin{aligned}&\min_{a\in\mathbb{C}}\|zf - \overline{a}f\|_{2,\alpha} = \Big\|zf - \frac{\langle zf, f\rangle}{\|f\|^2_{2,\alpha}}f\Big\|_{2,\alpha},\\ &\min_{a\in\mathbb{C}}\Big\|\frac{f'}{\alpha} - af\Big\|_{2,\alpha} = \Big\|\frac{f'}{\alpha} - \frac{\langle f'/\alpha, f\rangle}{\|f\|^2_{2,\alpha}}f\Big\|_{2,\alpha}\end{aligned}$$

and the minimum is attained when $a = \dfrac{\langle f'/\alpha, f\rangle}{\|f\|_{2,\alpha}^2}$. Then combining this with the Cauchy-Schwarz inequality to the left hand side of (3.11) to obtain the inequality that

$$\left\{\left\|\frac{f'}{\alpha} - \overline{b_0}f - \overline{b_1}zf\right\|_{2,\alpha}^2 + \left\|zf - b_0f - \frac{b_1f'}{\alpha}\right\|_{2,\alpha}^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\left\|zf - \frac{\langle zf, f\rangle}{\|f\|_{2,\alpha}^2}f\right\|_{2,\alpha}^2 + \left\|\frac{f'}{\alpha} - \frac{\langle f'/\alpha, f\rangle}{\|f\|_{2,\alpha}^2}f\right\|_{2,\alpha}^2\right\}^{\frac{1}{2}} \geq \frac{1}{\alpha}\|f\|_{2,\alpha}^2.$$

This completes the proof of the desired theorem.

When f is a unit vector, we have the following corollary.

Corollary 10 Let f is a unit vector in F_α^2, $f' \in F_\alpha^2$, then we have

$$\left\{\left\|\frac{f'}{\alpha} - \overline{b_0}f - \overline{b_1}zf\right\|_{2,\alpha}^2 + \left\|zf - b_0f - \frac{b_1f'}{\alpha}\right\|_{2,\alpha}^2\right\}^{\frac{1}{2}}$$
$$\times\left(\|zf\|_{2,\alpha}^2 + \left\|\frac{f'}{\alpha}\right\|_{2,\alpha}^2 - 2|\langle zf, f\rangle|^2\right)^{\frac{1}{2}} \geq \frac{1}{\alpha}.$$

Proof When f is a unit vector on F_α^2, and by minimization argument, we have

$$\min_{a\in\mathbb{C}}\left\|\frac{f'}{\alpha} - af\right\|_{2,\alpha} = \left(\left\|\frac{f'}{\alpha}\right\|_{2,\alpha}^2 - \left|\left\langle\frac{f'}{\alpha}, f\right\rangle\right|^2\right)^{\frac{1}{2}} = \left(\left\|\frac{f'}{\alpha}\right\|_{2,\alpha}^2 - |\langle zf, f\rangle|^2\right)^{\frac{1}{2}},$$
$$\min_{a\in\mathbb{C}}\|zf - \overline{a}f\|_{2,\alpha} = (\|zf\|_{2,\alpha}^2 - |\langle zf, f\rangle|^2)^{\frac{1}{2}}.$$

Then we may reach the conclusion just like the computation we performed in the proof of Theorem 7.

For the general case, we can modify the argument above to obtain something more interesting. Before this, we take the the operators U and V to be of the form

$$U = \sum_{k=0}^{n} b_kA^k, V = aI, \tag{3.12}$$

where n is a positive integer and $b_0, b_1, \cdots, b_n, a \in \mathbb{C}$. Let f be an element in

$$D_n(A|B) = D(AB) \cap D(BA) \cap D(A^*) \cap D(B^n) \cap D((B^*)^n).$$

Note that $D_1(A|B) = D(A|B)$.

Theorem 8 Let f is k-th derived function on F_α^2 and $f^{(k)} \in F_\alpha^2$, U and V are defined by (3.12), then we have

$$\left\{\left\|\frac{f'}{\alpha} - \sum_{k=0}^{n}\overline{b_k}z^kf\right\|_{2,\alpha}^2 + \left\|zf - \sum_{k=0}^{n}\frac{b_kf^{(k)}}{\alpha^k}\right\|_{2,\alpha}^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\left\|zf - \frac{\langle zf, f\rangle}{\|f\|_{2,\alpha}^2}f\right\|_{2,\alpha}^2 + \left\|\frac{f'}{\alpha} - \frac{\langle f'/\alpha, f\rangle}{\|f\|_{2,\alpha}^2}f\right\|_{2,\alpha}^2\right\}^{\frac{1}{2}} \geq \frac{1}{\alpha}\|f\|_{2,\alpha}^2$$

for all $f \in \mathrm{Dom}_n(A|B)$, and $b_0, b_1, \cdots, b_n, a \in F_\alpha^2$.

Proof By Theorem 7 and its proof, we have

$$|\langle [A,B]f,f\rangle| \le |\|Bf-Uf\|_{2,\alpha}|\|A^*f-V^*f\|_{2,\alpha}+|\|Af-Vf\|_{2,\alpha}|\|B^*f-U^*f\|_{2,\alpha}.$$

Calculate directly that

$$\begin{aligned}
&\langle [A,B]f,f\rangle=\frac{1}{\alpha}\|f\|_{2,\alpha}^2,\\
&\|Bf-Uf\|_{2,\alpha}=\Big\|zf-\sum_{k=0}^{n}\frac{b_kf^{(k)}}{\alpha^k}\Big\|,\\
&\|A^*f-V^*f\|_{2,\alpha}=\|zf-\overline{a}f\|_{2,\alpha},\\
&\|Af-Vf\|_{2,\alpha}=\Big\|\frac{f'}{\alpha}-af\Big\|_{2,\alpha},\\
&\|B^*f-U^*f\|_{2,\alpha}=\Big\|\frac{f'}{\alpha}-\sum_{k=0}^{n}\overline{b_k}z^kf\Big\|_{2,\alpha}.
\end{aligned}$$

Consequently, combining with the minimization argument of Lemma 5 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\Big\{\Big\|zf-\sum_{k=0}^{n}\frac{b_kf^{(k)}}{\alpha^k}\Big\|_{2,\alpha}^2+\Big\|\frac{f'}{\alpha}-\sum_{k=0}^{n}\overline{b_k}z^kf\Big\|_{2,\alpha}^2\Big\}^{\frac12}\\
&\times\Big\{\Big\|zf-\frac{\langle zf,f\rangle}{\|f\|_{2,\alpha}^2}f\Big\|_{2,\alpha}^2+\Big\|\frac{f'}{\alpha}-\frac{\langle f'/\alpha,f\rangle}{\|f\|_{2,\alpha}^2}f\Big\|_{2,\alpha}^2\Big\}^{\frac12}\ge\frac{1}{\alpha}\|f\|_{2,\alpha}^2.
\end{aligned}$$

The proof of the theorem is completed.

In addition, it is also possible to obtain an inequality of the case that f is a unit vector.

Corollary 11 Let f is n-th derived function on F_α^2 and $f^{(n)}\in F_\alpha^2$, $\|f\|_{2,\alpha}=1$, suppose U and V are defined by (2.19), then we have

$$\begin{aligned}
&\Big\{\Big\|\frac{f'}{\alpha}-\sum_{k=0}^{n}\overline{b_k}z^kf\Big\|_{2,\alpha}^2+\Big\|zf-\sum_{k=0}^{n}\frac{b_kf^{(k)}}{\alpha^k}\Big\|_{2,\alpha}^2\Big\}^{\frac12}\\
&\times\Big(\|zf\|_{2,\alpha}^2+\Big\|\frac{f'}{\alpha}\Big\|_{2,\alpha}^2-2|\langle zf,f\rangle|^2\Big)^{\frac12}\ge\frac{1}{\alpha}
\end{aligned}$$

for all $f\in \mathrm{Dom}_n(A|B)$, and $b_0,b_1,\cdots,b_n,a\in F_\alpha^2$.

Proof This follow directly from Corollary 10 and Theorem 8 and their proofs.

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加权**Fock** 空间F_α^2 上的测不准原理的推广

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摘 要: 对 $\alpha > 0$, 本文主要研究了复平面上的加权 Fock 空间 F_α^2 上的自伴算子和线性算子的测不准原理. 利用泛函分析中的一般性原理, 在 F_α^2 上构造了两个线性算子 $Tf = \frac{f'}{\alpha}$ 和 $T^* = zf$. 进一步, 构造了满足条件的两个自伴算子 A 和 B , 使得 $[A, B]$ 为恒等算子的常数倍, 得到了 F_α^2 上更精确的算子的测不准原理形式, 其中 T^* 是 T 的对偶算子, $[A, B] = AB - BA$ 为 A 和 B 的换位置. 本文的结果推广并完善了屈非非和朱克和在文献 [1] 和 [2] 中的结果.

关键词: 加权 Fock 空间; 测不准原理; 线性算子; 自伴算子; 高斯测度

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