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THE COMPUTING FORMULA FOR TWO CLASSES OF GENERALIZED EULER FUNCTIONS

LIAO Qun-ying[1], LUO Wen-li[2]

(*1.College of Mathematics, Sichuan Normal University, Chengdu 610066, China*)

(*2.The Middle School Attached to Sichuan University (No.12 Middle School), Chengdu 610061, China*)

Abstract: In this paper, we study the computing formula of the generalized Euler function. By using elementary methods and techniques, we obtain the computing formula of the generalized Euler function $\varphi_{pq}(n)$ for some cases and the computing formula of $\varphi_e(n)(e=p,p^2)$ for any prime factor $m|n$ with $m\equiv 1$ or $-1(\mathrm{mod}\ e)$ and $\gcd(m,e)=1$, where p and q are distinct primes, which are the generalizations for the corresponding main results given in [5].

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1 Introduction

In 18th century, as one of the most outstanding mathematician, Euler first defined the Euler function $\varphi(n)$ of a positive integer n to be the number of positive integers not greater than n but prime to n [1]. It's well known that as one of the important number theory functions, Euler function was applied widely. Euler function played a key role in RSA public-key cryptosystem since 1970's, and it is also one of the important tools to seek the theoretical basis for the generators of circle groups. There were many interesting open problems on Euler function [2]. For example, Carmichael conjectured that for any positive integer n, there exists a positive integer m such that $m\neq n$ and $\varphi(m)=\varphi(n)$. And then Schinzel conjectured that for any fixed positive integer k, the equation $\varphi(n+k)=\varphi(n)$ has infinitely many positive integer solutions for n.

On the other hand, in 1938, for any odd prime p, Lehmer [3] established the following important congruence identity

$$\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i}\equiv -2q_2(p)+pq_2^2(p)(\mathrm{mod}\ p^2), \tag{1.1}$$

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Biography: Liao Qunying (1974–), female, born in Yibin, Sichuan, professor, major in coding and cryptology theory.

where $q_r(n)$ denotes the Euler quotient, i.e., $q_r(n)=\frac{r^{\varphi(n)}-1}{n}, n$ and $r\geq 2$ are both natural numbers with $\gcd(n,r)=1$.

By using (1.1) and the others similar congruences identity, Lehmer obtained many ways to prove the first case of the well-known Fermat's last theorem [4]. Until 2002 and 2007, basing on (1.1) and the other congruence identities given by Lehmer, Cai, etc [5, 6] generalized the modulo from the square of a prime to the square of any positive integer, and defined the generalized Euler function for any positive integer n to be

$$\varphi_e(n)=\sum_{i=1,\gcd(i,n)=1}^{[\frac{n}{e}]}1,$$

i.e., $\varphi_e(n)$ is the number of positive integers not greater than $[\frac{n}{e}]$ but prime to n, where e is a positive integer and $[x]$ is the greatest integer which is not greater than x. It's easy to verify that $\varphi_1(n)=\varphi(n)$ is the known Euler function of n, and

$$\varphi_e(n)=\sum_{d|n}\mu(\frac{n}{d})[\frac{d}{e}], \tag{1.2}$$

where $\mu(n)$ is the Möbius function, i.e.,

$$\mu(n)=\begin{cases}1, & n=1;\\ (-1)^k, & n\geq 2 \quad \text{and} \quad \alpha_1=\cdots=\alpha_k=1;\\ 0, & n\geq 2 \text{ and there is some } \alpha_i>1 (1\leq i\leq k),\end{cases} \tag{1.3}$$

when $n=\prod\limits_{i=1}^{k}p_i^{\alpha_i}(\alpha_i\geq 1)$ is a positive integer and $p_1,\cdots,p_k$ are distinct primes. Easily to see that for any positive integer $n\geq 2$,

$$\sum_{d|n}\mu(\frac{n}{d})=0. \tag{1.4}$$

On the other hand, for $e=1$, the following computing formula for the generalized Euler function is well-known

$$\varphi(n)=\prod_{i=1}^{k}(p_i^{\alpha_i}-p_i^{\alpha_i-1}).$$

Therefore one can naturally to ask the following

Question For any fixed positive integer e, determine the explicit algorithm formula for the generalized Euler function $\varphi_e(n)$.

Fixed a positive integer $n=\prod\limits_{i=1}^{k}p_i^{\alpha_i}\geq 2$, where $p_1,\cdots,p_k$ are distinct primes and $\alpha_1,\cdots,\alpha_k$ are positive integers, denote $\Omega(n)=\sum\limits_{i=1}^{k}\alpha_i$ and $\omega(n)=k$. Especially, $\Omega(1)=\omega(1)=0$.

In recent years, Cai etc [7, 8] obtained the accurate calculation formula for $\varphi_e(n)$ $(e = 2, 3, 4, 6)$, and then, by using properties for Legendre or Jacobi symbols, they also got some necessary and sufficient conditions for that $\varphi_e(n)$ and $\varphi_e(n+1)(e = 2, 3, 4)$ are both odd or even numbers.

Proposition 1.1 [7, 8] Let $p_1, \cdots, p_k$ be distinct primes, $\alpha_1, \cdots, \alpha_k$ be positive integers, and $n_1 = \prod\limits_{i=1}^{k} p_i^{\alpha_i}$.

(1) If $\gcd(p_i, 3) = 1$ $(i = 1, \cdots, k)$ and $n = 3^{\alpha} n_1 > 3$, then

$$\varphi_3(n) = \begin{cases} \frac{\varphi(n)}{3} + \frac{(-1)^{\Omega(n)} 2^{\omega(n)-\alpha-1}}{3}, & \alpha \in \{0, 1\} \text{ and } p_i \equiv 2 \ (\mathrm{mod}\ 3)\ (i = 1, \cdots, k); \\ \frac{\varphi(n)}{3}, & \text{otherwise.} \end{cases}$$

(2) If α is a nonnegative integer and $n = 2^{\alpha} n_1 > 4$, then

$$\varphi_4(n) = \begin{cases} \frac{\varphi(n)}{4} + \frac{(-1)^{\Omega(n)} 2^{\omega(n)-\alpha}}{4}, & \alpha \in \{0, 1\} \text{ and } p_i \equiv 3 \ (\mathrm{mod}\ 4)\ (i = 1, \cdots, k); \\ \frac{\varphi(n)}{4}, & \text{otherwise.} \end{cases}$$

(3) If $\gcd(p_i, 6) = 1 (i = 1, \cdots, k)$ and $n = 2^{\alpha} 3^{\beta} n_1 > 6$, then

$$\varphi_6(n) = \begin{cases} \frac{1}{6}\varphi(n) + \frac{(-1)^{\Omega(n)} 2^{\omega(n)+1-\beta}}{6}, & \alpha = 0, \beta \in \{0, 1\} \text{ and } p_i \equiv 5 \ (\mathrm{mod}\ 6)\ (i = 1, \cdots, k); \\ \frac{1}{6}\varphi(n) + \frac{(-1)^{\Omega(n)} 2^{\omega(n)-1-\beta}}{6}, & \alpha = 1, \beta \in \{0, 1\} \text{ and } p_i \equiv 5 \ (\mathrm{mod}\ 6)\ (i = 1, \cdots, k); \\ \frac{1}{6}\varphi(n) - \frac{(-1)^{\Omega(n)} 2^{\omega(n)-\beta}}{6}, & \alpha \geq 2, \beta \in \{0, 1\} \text{ and } p_i \equiv 5 \ (\mathrm{mod}\ 6)\ (i = 1, \cdots, k); \\ \frac{1}{6}\varphi(n), & \text{otherwise.} \end{cases}$$

Recently, we [9] obtained the formula for $\varphi_5(n)$ and some sufficient conditions for $2|\varphi_5(n)$. The present paper continues the study, based on the elementary methods and techniques, the computing formula for $\varphi_e(n)$ $(e = p, p^2, pq)$ is obtained, where p and q are distinct primes (Theorems 1.1–1.5).

For convenience, throughout the paper, we assume that p, q, $p_1, \cdots, p_k$ are distinct primes, $\alpha_1, \cdots, \alpha_k$ are positive integers, α and β are both nonnegative integers, and

$$n_1 = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

Theorem 1.1 If $n = p^{\alpha} n_1 > p$, then

$$\varphi_p(n) = \begin{cases} \frac{\varphi(n)}{p} + \frac{(p-2)(-1)^{\Omega(n)} 2^{\omega(n)-\alpha-1}}{p}, & \alpha \in \{0, 1\} \text{ and } p_i \equiv -1 \ (\mathrm{mod}\ p)\ (i = 1, \cdots, k); \\ \frac{\varphi(n)}{p}, & \alpha \geq 2 \text{or } \alpha \in \{0, 1\} \text{ and } p_i \equiv 1 \ (\mathrm{mod}\ p)\ (i = 1, \cdots, k). \end{cases}$$

Theorem 1.2 If $n = p^{\alpha} n_1 > p^2$, then

$$\varphi_{p^2}(n) = \begin{cases} \varphi_{p^2}(n_1), & \alpha = 0; \\ \varphi_p(n_1) - \varphi_{p^2}(n_1), & \alpha = 1; \\ \varphi(n_1) - \varphi_p(n_1), & \alpha = 2; \\ \frac{1}{p^2}\varphi(n), & \alpha \geq 3. \end{cases}$$

Theorem 1.3 For $n=p^{\alpha}n_1>p^2$ and $\alpha\le 2$.

(1) If $\alpha=0$, then

$$\varphi_{p^2}(n)=\begin{cases}\frac{1}{p^2}\varphi(n), & p_i\equiv 1(\operatorname{mod} p^2)(i=1,\cdots,k).\\ \frac{1}{p^2}\varphi(n)+\frac{(p^2-2)(-1)^{\Omega(n)}2^{\omega(n)-1}}{p^2}, & p_i\equiv -1\ (\operatorname{mod} p^2)\ (i=1,\cdots,k).\end{cases}$$

(2) If $\alpha=1$, then

$$\varphi_{p^2}(n)=\begin{cases}\frac{\varphi(n)}{p^2}, & p_i\equiv 1(\operatorname{mod} p^2)(i=1,\cdots,k);\\ \frac{\varphi(n)}{p^2}+\frac{(p-1)(-1)^{\Omega(n)}2^{\omega(n)-1}}{p^2}, & p_i\equiv -1\ (\operatorname{mod} p^2)\ (i=1,\cdots,k).\end{cases}$$

(3) If $\alpha=2$, then

$$\varphi_{p^2}(n)=\begin{cases}\frac{\varphi(n)}{p^2}, & p_i\equiv 1(\operatorname{mod} p)(i=1,\cdots,k).\\ \frac{\varphi(n)}{p^2}+\frac{(p-2)(-1)^{\Omega(n)-1}2^{\omega(n)-2}}{p}, & p_i\equiv -1\ (\operatorname{mod} p)\ (i=1,\cdots,k).\end{cases}$$

Theorem 1.4 If $n=p^{\alpha}q^{\beta}n_1>pq$, then

$$\varphi_{pq}(n)=\begin{cases}\varphi_{pq}(n_1), & \alpha=\beta=0;\\ \varphi_q(n_1)-\varphi_{pq}(n_1), & \alpha=1,\beta=0;\\ \varphi_p(n_1)-\varphi_{pq}(n_1), & \alpha=0,\beta=1;\\ \varphi(n_1)+\varphi_{pq}(n_1)-\varphi_p(n_1)-\varphi_q(n_1), & \alpha=\beta=1;\\ \varphi_q(p^{\alpha-1}n_1), & \alpha\ge 2,\beta=0;\\ \frac{1}{p(q-1)}\varphi(n)-\varphi_q(p^{\alpha-1}n_1), & \alpha\ge 2,\beta=1;\\ \frac{1}{pq}\varphi(n), & \alpha\ge 2,\beta\ge 2.\end{cases}$$

Theorem 1.5 For $n=p^{\alpha}q^{\beta}n_1>pq$.

(1) If $\alpha=\beta=0$, then

$$\varphi_{pq}(n)=\begin{cases}\frac{\varphi(n)}{pq}, & p_i\equiv 1\ (\operatorname{mod} pq)\ (i=1,\cdots,k);\\ \frac{1}{pq}\varphi(n)+\frac{(pq-2)(-1)^{\Omega(n)}2^{\omega(n)-1}}{pq}, & p_i\equiv -1\ (\operatorname{mod} pq)\ (i=1,\cdots,k).\end{cases}$$

(2) If $\alpha=1$ and $\beta=0$, then

$$\varphi_{pq}(n)=\begin{cases}\frac{\varphi(n)}{pq}, & p_i\equiv 1\ (\operatorname{mod} pq)\ (i=1,\cdots,k);\\ \frac{\varphi(n)}{pq}+\frac{(p-1)(-1)^{\Omega(n)}2^{\omega(n)-1}}{pq}, & p_i\equiv -1\ (\operatorname{mod} pq)\ (i=1,\cdots,k).\end{cases}$$

(3) If $\alpha=0$ and $\beta=1$, then

$$\varphi_{pq}(n)=\begin{cases}\frac{\varphi(n)}{pq}, & p_i\equiv 1\ (\operatorname{mod} pq)\ (i=1,\cdots,k);\\ \frac{\varphi(n)}{pq}+\frac{(q-1)(-1)^{\Omega(n)}2^{\omega(n)-1}}{pq}, & p_i\equiv -1\ (\operatorname{mod} pq)\ (i=1,\cdots,k).\end{cases}$$

(4) If $\alpha=\beta=1$, then

$$\varphi_{pq}(n)=\begin{cases}\frac{\varphi(n)}{pq}, & p_i\equiv 1\ (\operatorname{mod} pq)\ (i=1,\cdots,k);\\ \frac{\varphi(n)}{pq}+\frac{(2p+2q-pq-2)}{pq}(-1)^{\Omega(n)}2^{\omega(n)-3}, & p_i\equiv -1\ (\operatorname{mod} pq)\ (i=1,\cdots,k).\end{cases}$$

(5) If $\alpha \geq 2$ and $\beta = 0$, then

$$\varphi_{pq}(n) = \begin{cases} \frac{1}{pq}\varphi(n), & p \equiv p_i \equiv 1 \ (\mathrm{mod}\ q)\ (i = 1, \cdots, k); \\ \frac{1}{pq}\varphi(n) - \frac{(q-2)(-1)^{\Omega(n)} \cdot 2^{\omega(n)-1}}{q}, & p \equiv p_i \equiv -1 \ (\mathrm{mod}\ q)\ (i = 1, \cdots, k). \end{cases}$$

(6) If $\alpha \geq 2$ and $\beta = 1$, then

$$\varphi_{pq}(n) = \begin{cases} \frac{1}{pq}\varphi(n), & p \equiv p_i \equiv 1 \ (\mathrm{mod}\ q)\ (i = 1, \cdots, k); \\ \frac{1}{pq}\varphi(n) - \frac{(q-2)(-1)^{\Omega(n)} 2^{\omega(n)-2}}{q}, & p \equiv p_i \equiv -1 \ (\mathrm{mod}\ q)\ (i = 1, \cdots, k). \end{cases}$$

(7) If $\beta \geq 2$ and $\alpha \in \{0, 1\}$, then

$$\varphi_{pq}(n) = \begin{cases} \frac{1}{pq}\varphi(n), & q \equiv p_i \equiv 1 \ (\mathrm{mod}\ p)\ (i = 1, \cdots, k); \\ \frac{1}{pq}\varphi(n) - \frac{(p-2)(-1)^{\Omega(n)} \cdot 2^{\omega(n)-\alpha-1}}{p}, & q \equiv p_i \equiv -1 \ (\mathrm{mod}\ p)\ (i = 1, \cdots, k). \end{cases}$$

Remark By taking $p = 3$ in Theorem 1.1, or $p = 2$ in Theorems 1.2–1.3, one can get (1) or (2) of Proposition 1.1, respectively. And by taking $p = 2$ and $q = 3$ in Theorems 1.4–1.5, one can get (3) of Proposition 1.1. The details is left to interested readers.

2 Proofs for Main Results

Proof for Theorem 1.1 (1) If $\alpha = 0$ and $p_i \equiv -1$ (mod p) ($i = 1, \cdots, k$), i.e., $n = n_1$ and for any $d \mid n_1, d \equiv \pm 1(\mathrm{mod}\ p)$. Then by (1.2)–(1.4), we have

$$\begin{aligned}
\varphi_p(n) &= \sum_{d \mid n_1} \mu\left(\frac{n_1}{d}\right)\left[\frac{d}{p}\right] \\
&= \sum_{d \mid n_1, d \equiv 1 (\mathrm{mod}\ p)} \mu\left(\frac{n_1}{d}\right) \cdot \left(\frac{d-1}{p}\right) + \sum_{d \mid n_1, d \equiv -1 (\mathrm{mod}\ p)} \mu\left(\frac{n_1}{d}\right) \cdot \left(\frac{d-(p-1)}{p}\right) \\
&= \frac{1}{p}\left(\sum_{\substack{d \mid n_1 \\ d \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) d + \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) d\right) \\
&\quad - \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) + \frac{1}{p} \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) - \frac{1}{p} \sum_{\substack{d \mid n_1 \\ d \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) \\
&= \frac{1}{p}\varphi(n_1) - \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) + \frac{1}{p} \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) - \frac{1}{p} \sum_{\substack{d \mid n_1 \\ d \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) \\
&= \frac{1}{p}\varphi(n_1) - \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) + \frac{2}{p} \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) - \frac{1}{p} \sum_{d \mid n_1} \mu\left(\frac{n_1}{d}\right) \\
&= \frac{1}{p}\varphi(n_1) - \left(\frac{p-2}{p}\right) \cdot \sum_{\substack{d \mid n_1 \\ d \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right) \\
&= \frac{1}{p}\varphi(n_1) + \left(\frac{p-2}{p}\right) \cdot \sum_{\substack{d \mid n_1 \\ d \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{d}\right)
\end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{\substack{(\prod\limits_{i=1}^{k} p_i^{\alpha_i-1})t \mid n_1 \\ (\prod\limits_{i=1}^{k} p_i^{\alpha_i-1})t \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{n_1}{(\prod\limits_{i=1}^{k} p_i^{\alpha_i-1})t}\right) \\
&= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{\substack{t \mid \prod\limits_{i=1}^{k} p_i \\ (-1)^{\Omega(n)-k} t \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{\prod\limits_{i=1}^{k} p_i}{t}\right).
\end{aligned}
$$

(a) If $2 \mid \Omega(n)$ and $2 \mid k$, then by (2.1) and $n = n_1$, we have $\Omega(n) = \Omega(n_1), \omega(n) = \omega(n_1)$ and

$$
\begin{aligned}
\varphi_p(n) &= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{\substack{t \mid \prod\limits_{i=1}^{k} p_i \\ (-1)^{\Omega(n)-k} t \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{\prod\limits_{i=1}^{k} p_i}{t}\right) \\
&= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{\substack{t \mid \prod\limits_{i=1}^{k} p_i \\ t \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{\prod\limits_{i=1}^{k} p_i}{t}\right) \\
&= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{j=0}^{\frac{k}{2}} \tbinom{k}{2j}(-1)^{k-2j} = \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot 2^{k-1} \\
&= \tfrac{1}{p}\varphi(n) + \tfrac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p} \\
&= \tfrac{1}{p}\varphi(n) + \tfrac{(p-2)(-1)^{\Omega(n)}2^{\omega(n)-\alpha-1}}{p}.
\end{aligned}
$$

(b) If $2 \mid \Omega(n)$ and $2 \nmid k$, then by (2.1) we have

$$
\begin{aligned}
\varphi_p(n) &= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{\substack{t \mid \prod\limits_{i=1}^{k} p_i \\ (-1)^{\Omega(n)-k} t \equiv 1 (\mathrm{mod}\ p)}} \mu\left(\frac{\prod\limits_{i=1}^{k} p_i}{t}\right) \\
&= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{\substack{t \mid \prod\limits_{i=1}^{k} p_i \\ t \equiv -1 (\mathrm{mod}\ p)}} \mu\left(\frac{\prod\limits_{i=1}^{k} p_i}{t}\right) \\
&= \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot \sum_{j=1}^{\frac{k+1}{2}} \tbinom{k}{2j-1}(-1)^{k-2j+1} = \tfrac{1}{p}\varphi(n) + \left(\tfrac{p-2}{p}\right)\cdot 2^{k-1} \\
&= \tfrac{1}{p}\varphi(n) + \tfrac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p} \\
&= \tfrac{1}{p}\varphi(n) + \tfrac{(p-2)(-1)^{\Omega(n)}2^{\omega(n)-\alpha-1}}{p}.
\end{aligned}
$$

For the case $2 \nmid \Omega(n)$ and $2 \nmid k$ or $2 \nmid \Omega(n)$ and $2 \mid k$, in the same proof, we can get

$$
\varphi_p(n) = \frac{1}{p}\varphi(n) + \frac{(p-2)(-1)^{\Omega(n)}2^{\omega(n)-\alpha-1}}{p}. \tag{2.2}
$$

(2) If $\alpha = 1$ and for any $i = 1, \cdots, k, p_i \equiv -1 (\text{mod } p)$, i.e., $n = pn_1$ and for any $d \mid n_1, d \equiv \pm 1 (\text{mod } p)$. Then by (1.2)–(1.3), (2.2) and $\gcd(p, n_1) = 1$, we have

$$\varphi(n) = (p-1)\varphi(n_1),\ \Omega(n) = \Omega(n_1) + 1, \omega(n) = \omega(n_1) + 1,$$

and then

$$\begin{aligned} \varphi_p(n) \quad &= \varphi_p(pn_1) = \sum_{d \mid pn_1} \mu\left(\tfrac{pn_1}{d}\right)\left[\tfrac{d}{p}\right] \\ &= \sum_{d_1 \mid n_1} \mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{d_1}{p}\right] + \sum_{pd_1 \mid pn_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{pd_1}{p}\right] \\ &= -\sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{d_1}{p}\right] + \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1 \\ &= -\varphi_p(n_1) + \varphi(n_1) \\ &= -\tfrac{1}{p}\varphi(n_1) - \tfrac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p} + \varphi(n_1) \\ &= \left(\tfrac{p-1}{p}\right) \cdot \varphi(n_1) + \tfrac{(p-2)(-1)^{\Omega(n_1)+1}2^{\omega(n)-2}}{p} \\ &= \tfrac{1}{p}\varphi(n) + \tfrac{(p-2)(-1)^{\Omega(n)}2^{\omega(n)-\alpha-1}}{p}. \end{aligned} \tag{2.3}$$

(3) If $\alpha \geq 2$, i.e., $n = p^\alpha n_1$, then by $\gcd(p, n_1) = 1$ and (1.2)–(1.3), we have

$$\varphi(n) = \varphi(p^\alpha n_1) = (p^\alpha - p^{\alpha-1})\varphi(n_1),$$

and then

$$\begin{aligned} \varphi_p(n) \quad &= \varphi_p(p^\alpha n_1) = \sum_{d \mid p^\alpha n_1} \mu\left(\tfrac{p^\alpha n_1}{d}\right)\left[\tfrac{d}{p}\right] \\ &= \sum_{d_1 \mid n_1} \mu\left(\tfrac{p^\alpha n_1}{d_1}\right)\left[\tfrac{d_1}{p}\right] + \sum_{d_1 \mid n_1, 1 \leq \beta \leq \alpha} \mu\left(\tfrac{p^\alpha n_1}{p^\beta d_1}\right)\left[\tfrac{p^\beta d_1}{p}\right] \\ &= \sum_{\substack{d_1 \mid n_1 \\ 1 \leq \beta \leq \alpha}} \mu\left(\tfrac{p^{\alpha-\beta} n_1}{d_1}\right) p^{\beta-1} d_1 \\ &= p^{\alpha-1} \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1 + p^{\alpha-2} \sum_{d_1 \mid n_1} \mu\left(\tfrac{pn_1}{d_1}\right) d_1 \\ &= p^{\alpha-1} \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1 - p^{\alpha-2} \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1 \\ &= (p^{\alpha-1} - p^{\alpha-2}) \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1 \\ &= (p^{\alpha-1} - p^{\alpha-2})\varphi(n_1) = \tfrac{\varphi(n)}{p}. \end{aligned} \tag{2.4}$$

(4) If $\alpha = 0$ and $p_i \equiv 1 (\text{mod } p)(i = 1, \cdots, k)$, i.e., $n = n_1$ and for any $d \mid n_1, d \equiv 1 (\text{mod } p)$. Then by (1.2) and (1.4), we have

$$\varphi_p(n) = \sum_{d \mid n_1} \mu\left(\frac{n}{d}\right)\left[\frac{d}{p}\right] = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \cdot \left(\frac{d-1}{p}\right) = \frac{1}{p}\sum_{d \mid n} \mu\left(\frac{n}{d}\right) d - \frac{1}{p}\sum_{d \mid n} \mu\left(\frac{n}{d}\right) = \frac{1}{p}\varphi(n). \tag{2.5}$$

(5) If $\alpha = 1$ and $p_i \equiv 1 (\text{mod } p)(i = 1, \cdots, k)$, i.e., $n = pn_1$, then $\varphi(n) = (p-1)\varphi(n_1)$

and for any $d \mid n_1, d \equiv 1 (\mathrm{mod}\ p)$. Thus by (1.2)–(1.3), $\gcd(p, n_1) = 1$ and (4) we have

$$\begin{aligned}\varphi_p(n) &= \sum_{d \mid pn_1} \mu\left(\tfrac{pn_1}{d}\right)\left[\tfrac{d}{p}\right]\\ &= \sum_{d_1 \mid n_1} \mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{d_1}{p}\right] + \sum_{pd_1 \mid pn_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{pd_1}{p}\right]\\ &= -\sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{d_1}{p}\right] + \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1\\ &= -\varphi_p(n_1) + \varphi(n_1) = -\tfrac{1}{p}\varphi(n_1) + \varphi(n_1)\\ &= \left(\tfrac{p-1}{p}\right) \cdot \varphi(n_1) = \tfrac{1}{p}\varphi(n).\end{aligned} \tag{2.6}$$

Now from (2.2)–(2.6), Theorem 1.1 is proved.

Proof for Theorem 1.2 (1) For the case $\alpha = 0$, the result is obvious.

(2) If $\alpha = 1$, i.e., $n = pn_1$, then by $\gcd(p, n_1) = 1$ and (1.2)–(1.3), we have

$$\begin{aligned}\varphi_{p^2}(n) &= \sum_{d \mid n} \mu\left(\tfrac{n}{d}\right)\left[\tfrac{d}{p^2}\right] = \sum_{d \mid pn_1} \mu\left(\tfrac{pn_1}{d}\right)\left[\tfrac{d}{p^2}\right]\\ &= \sum_{d_1 \mid n_1} \mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{d_1}{p^2}\right] + \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{d_1}{p}\right]\\ &= -\varphi_{p^2}(n_1) + \varphi_p(n_1).\end{aligned} \tag{2.7}$$

(3) If $\alpha = 2$, i.e., $n = p^2 n_1$, then by $\gcd(p, n_1) = 1$ and (1.2)–(1.3), we have

$$\begin{aligned}\varphi_{p^2}(n) &= \sum_{d \mid p^2 n_1} \mu\left(\frac{p^2 n_1}{d}\right)\left[\tfrac{d}{p^2}\right]\\ &= \sum_{d_1 \mid n_1} \mu\left(\frac{p^2 n_1}{d_1}\right)\left[\tfrac{d_1}{p^2}\right] + \sum_{d_1 \mid n_1} \mu\left(\frac{p^2 n_1}{pd_1}\right)\left[\tfrac{pd_1}{p^2}\right] + \sum_{d_1 \mid n_1} \mu\left(\frac{p^2 n_1}{p^2 d_1}\right)\left[\tfrac{p^2 d_1}{p^2}\right]\\ &= -\sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{d_1}{p}\right] + \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1\\ &= -\varphi_p(n_1) + \varphi(n_1).\end{aligned} \tag{2.8}$$

(4) If $\alpha \geq 3$, i.e., $n = p^\alpha n_1$, then by $\gcd(p, n_1) = 1$ and (1.2)–(1.3), we have $\varphi(n) = p^2(p^{\alpha-2} - p^{\alpha-3})\varphi(n_1)$, and then

$$\begin{aligned}\varphi_{p^2}(n) &= \sum_{d \mid p^\alpha n_1} \mu\left(\tfrac{p^\alpha n_1}{d}\right)\left[\tfrac{d}{p^2}\right]\\ &= \sum_{d_1 \mid n_1} \mu\left(\tfrac{p^\alpha n_1}{d_1}\right)\left[\tfrac{d_1}{p^2}\right] + \sum_{d_1 \mid n_1, 1 \leq \alpha_1 \leq \alpha} \mu\left(\frac{p^\alpha n_1}{p^{\alpha_1} d_1}\right)\left[\frac{p^{\alpha_1} d_1}{p^2}\right]\\ &= \sum_{d_1 \mid n_1} \mu\left(\tfrac{pn_1}{d}\right)\left[\frac{p^{\alpha-1} d_1}{p^2}\right] + \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\frac{p^\alpha d_1}{p^2}\right]\\ &= -p^{\alpha-3} \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1 + p^{\alpha-2} \sum_{d_1 \mid n_1} \mu\left(\tfrac{n_1}{d_1}\right) d_1\\ &= -p^{\alpha-3}\varphi(n_1) + p^{\alpha-2}\varphi(n_1)\\ &= p^{\alpha-3}(p-1)\varphi(n_1) = \tfrac{1}{p^2}\varphi(n).\end{aligned} \tag{2.9}$$

Now from (2.7)–(2.9), we complete the proof of Theorem 1.2.

Proof for Theorem 1.3 (1) If $\alpha = 0$, i.e., $n = n_1$, and then $\gcd(n, p) = 1$. Suppose that $p_i \equiv 1 \pmod{p^2}$ $(i = 1, \cdots, k)$, then for any $d \mid n, d \equiv 1 \pmod{p^2}$, thus by (1.2)–(1.4), we have

$$\varphi_{p^2}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left[\frac{d}{p^2}\right] = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(\frac{d-1}{p^2}\right) = \frac{1}{p^2}\varphi(n) - \frac{1}{p^2}\sum_{d|n}\mu\left(\frac{n}{d}\right) = \frac{1}{p^2}\varphi(n). \tag{2.10}$$

Suppose that $p_i \equiv -1 \pmod{p^2}$ $(i = 1, \cdots, k)$, i.e, for any $d \mid n, d \equiv \pm 1 \pmod{p^2}$, and then by (1.2), (1.4) and the proof of Theorem 1.1 (1), we can get

$$\begin{aligned} \varphi_{p^2}(n) &= \sum_{d|n} \mu\left(\tfrac{n}{d}\right)\left[\tfrac{d}{p^2}\right] \\ &= \sum_{d|n, d\equiv 1 (\mathrm{mod}\ p^2)} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d-1}{p^2}\right) + \sum_{d|n, d\equiv -1 (\mathrm{mod}\ p^2)} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d-(p^2-1)}{p^2}\right) \\ &= \tfrac{1}{p^2}\varphi(n) - \tfrac{1}{p^2}\sum_{d|n, d\equiv 1 (\mathrm{mod}\ p^2)} \mu\left(\tfrac{n}{d}\right) - \left(\tfrac{p^2-1}{p^2}\right)\sum_{d|n, d\equiv -1 (\mathrm{mod}\ p^2)} \mu\left(\tfrac{n}{d}\right) \\ &= \tfrac{1}{p^2}\varphi(n) + \left(\tfrac{p^2-2}{p^2}\right)\sum_{d|n, d\equiv 1 (\mathrm{mod}\ p^2)} \mu\left(\tfrac{n}{d}\right) \\ &= \tfrac{1}{p^2}\varphi(n) + \left(\tfrac{p^2-2}{p^2}\right)(-1)^{\Omega(n)}2^{\omega(n)-1}. \end{aligned} \tag{2.11}$$

Now from (2.10)–(2.11), we complete the proof of (1).

(2) If $\alpha = 1$, i.e, $n = pn_1$, then by $\gcd(p, n_1) = 1$, we have

$$\varphi(n) = (p-1)\varphi(n_1), \Omega(n) = \Omega(n_1) + 1, \omega(n) = \omega(n_1) + 1. \tag{2.12}$$

And so by Theorems 1.1–1.2, (2.12) and (1), we can obtain

$$\begin{aligned} &\varphi_{p^2}(n) = \varphi_p(n_1) - \varphi_{p^2}(n_1) \\ = &\begin{cases} \frac{\varphi(n_1)}{p} - \frac{\varphi(n_1)}{p^2}, & p_i \equiv 1 (\mathrm{mod}\ p^2) (i = 1, \cdots, k); \\ \frac{\varphi(n_1)}{p} - \frac{\varphi(n_1)}{p^2} + \frac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p} - \frac{(p^2-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p^2}, & p_i \equiv -1 (\mathrm{mod}\ p^2)\ (i = 1, \cdots, k); \end{cases} \\ = &\begin{cases} \frac{\varphi(n)}{p^2}, & p_i \equiv 1 (\mathrm{mod}\ p^2)\ (i = 1, \cdots, k); \\ \frac{\varphi(n)}{p^2} + \frac{(p-1)(-1)^{\Omega(n_1)+1}2^{\omega(n_1)}}{p^2}, & p_i \equiv -1 (\mathrm{mod}\ p^2)\ (i = 1, \cdots, k); \end{cases} \\ = &\begin{cases} \frac{\varphi(n)}{p^2}, & p_i \equiv 1 (\mathrm{mod}\ p^2)\ (i = 1, \cdots, k); \\ \frac{\varphi(n)}{p^2} + \frac{(p-1)(-1)^{\Omega(n)}2^{\omega(n)-1}}{p^2}, & p_i \equiv -1 (\mathrm{mod}\ p^2)\ (i = 1, \cdots, k). \end{cases} \end{aligned}$$

This completes the proof of (2).

(3) If $\alpha = 2$, i.e, $n = p^2 n_1$, then by $\gcd(p, n_1) = 1$, we have

$$\varphi(n) = p(p-1)\varphi(n_1), \Omega(n) = \Omega(n_1) + 2, \omega(n) = \omega(n_1) + 1. \tag{2.13}$$

Thus by Theorems 1.1–1.2 and (2.13), in the same proof as that of Theorem 1.3 (2), (3) is immediate.

This completes the proof of Theorem 1.3.

Proof for Theorem 1.4 (1) If $\alpha = 0, \beta = 0$, the result is obvious.

(2) If $\alpha=1,\beta=0$, i.e., $n=pn_1$, then by $\gcd(pq,n_1)=\gcd(p,q)=1$ and (1.2)–(1.3), we have

$$\begin{aligned}\varphi_{pq}(n) &= \sum_{d\mid pn_1}\mu\left(\tfrac{pn_1}{d}\right)\left[\tfrac{d}{pq}\right]\\ &= \sum_{d_1\mid n_1}\mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{d_1}{pq}\right]+\sum_{pd_1\mid pn_1}\mu\left(\tfrac{pn_1}{pd_1}\right)\left[\tfrac{pd_1}{pq}\right]\\ &= -\varphi_{pq}(n_1)+\varphi_q(n_1).\end{aligned}\tag{2.14}$$

(3) For the case $\alpha=1,\beta=0$, in the same proof of (2), the result is obvious.

(4) If $\alpha=\beta=1$, i.e., $n=pqn_1$, then by $\gcd(pq,n_1)=\gcd(p,q)=1$ and (1.2)–(1.3), in the same proof as that of (2), (4) is immediate.

(5) If $\alpha\geq 2$ and $\beta=0$, i.e., $n=p^{\alpha}n_1$, then by $\gcd(p,n_1)=1$ and (1.2)–(1.3), we can obtain

$$\begin{aligned}\varphi_{pq}(n) &= \sum_{d\mid p^{\alpha}n_1}\mu\left(\tfrac{p^{\alpha}n_1}{d}\right)\left[\tfrac{d}{pq}\right]\\ &= \sum_{d_1\mid n_1}\mu\left(\tfrac{p^{\alpha}n_1}{d_1}\right)\left[\tfrac{d_1}{pq}\right]+\sum_{d_1\mid n_1,1\leq\alpha_1\leq\alpha}\mu\left(\tfrac{p^{\alpha}n_1}{p^{\alpha_1}d_1}\right)\left[\tfrac{p^{\alpha_1}d_1}{pq}\right]\\ &= \sum_{d_1\mid n_1}\mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{pq}\right]+\sum_{d_1\mid n_1}\mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha}d_1}{pq}\right]\\ &= -\sum_{d_1\mid n_1}\mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha-2}d_1}{q}\right]+\sum_{d_1\mid n_1}\mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{q}\right]\\ &= -B_{\alpha-2}+B_{\alpha-1}.\end{aligned}\tag{2.15}$$

While by $\alpha\geq 2$ and (1.2)–(1.4), we know that

$$\begin{aligned}B_{\alpha-1} &= \sum_{d_1\mid n_1}\mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{q}\right]=\sum_{p^{\alpha-1}d_1\mid p^{\alpha-1}n_1}\mu\left(\tfrac{p^{\alpha-1}n_1}{p^{\alpha-1}d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{q}\right]\\ &= \varphi_q(p^{\alpha-1}n_1)-\sum_{d_1\mid n_1,0\leq\beta\leq\alpha-2}\mu\left(\tfrac{p^{\alpha-\beta-1}n_1}{d_1}\right)\left[\tfrac{p^{\beta}d_1}{q}\right]\\ &= \varphi_q(p^{\alpha-1}n_1)-\sum_{d_1\mid n_1}\mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{p^{\alpha-2}d_1}{q}\right]\\ &= \varphi_q(p^{\alpha-1}n_1)+\sum_{d_1\mid n_1}\mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha-2}d_1}{q}\right]=\varphi_q(p^{\alpha-1}n_1)+B_{\alpha-2},\end{aligned}\tag{2.16}$$

thus by (2.15)–(2.16), we have $\varphi_{pq}(n)=\varphi_q(p^{\alpha-1}n_1)$. Thus (5) is proved.

(6) If $\alpha\geq 2$ and $\beta=1$, i.e., $n=p^{\alpha}qn_1$, then by $\gcd(p,n_1)=1$, we have

$$\varphi(n)=p^{\alpha-1}(p-1)(q-1)\varphi(n_1).\tag{2.17}$$

Thus by (1.2)–(1.4), (2.16) and (2.17), we can get

$$\begin{aligned}\varphi_{pq}(n) &= \sum_{d\mid p^{\alpha}qn_1}\mu\left(\tfrac{p^{\alpha}qn_1}{d}\right)\left[\tfrac{d}{pq}\right]\\ &= \sum_{d_1\mid n_1}\mu\left(\tfrac{p^{\alpha}qn_1}{d_1}\right)\left[\tfrac{d_1}{pq}\right]+\sum_{d_1\mid n_1,1\leq\alpha_1\leq\alpha}\mu\left(\tfrac{p^{\alpha}qn_1}{p^{\alpha_1}d_1}\right)\left[\tfrac{p^{\alpha_1}d_1}{pq}\right]\\ &\quad+\sum_{d_1\mid n_1}\mu\left(\tfrac{p^{\alpha}n_1}{d_1}\right)\left[\tfrac{d_1}{p}\right]+\sum_{d_1\mid n_1,1\leq\alpha_1\leq\alpha}\mu\left(\tfrac{p^{\alpha}qn_1}{p^{\alpha_1}qd_1}\right)\left[\tfrac{p^{\alpha_1}qd_1}{pq}\right]\\ &= \sum_{d_1\mid n_1}\mu\left(\tfrac{pqn_1}{d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{pq}\right]+\sum_{d_1\mid n_1}\mu\left(\tfrac{qn_1}{d_1}\right)\left[\tfrac{p^{\alpha}d_1}{pq}\right]\\ &\quad+\sum_{d_1\mid n_1}\mu\left(\tfrac{pn_1}{d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{p}\right]+\sum_{d_1\mid n_1}\mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha}d_1}{p}\right]\end{aligned}$$

$$\begin{aligned}
&= \sum_{d_1|n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha-2}d_1}{q}\right] - \sum_{d_1|n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\tfrac{p^{\alpha-1}d_1}{q}\right]\\
&\quad - \sum_{d_1|n_1} \mu\left(\tfrac{n_1}{d_1}\right)(p^{\alpha-2}d_1) + \sum_{d_1|n_1} \mu\left(\tfrac{n_1}{d_1}\right)(p^{\alpha-1}d_1)\\
&= B_{\alpha-2} - B_{\alpha-1} - p^{\alpha-2}\varphi(n_1) + p^{\alpha-1}\varphi(n_1)\\
&= -\varphi_q(p^{\alpha-1}n_1) + \tfrac{1}{p(q-1)}\varphi(n).
\end{aligned}$$

(7) If $\alpha \geq 2$ and $\beta \geq 2$, then by $\gcd(pq, n_1) = \gcd(p, q) = 1$, and (1.2)–(1.4), we have

$$\varphi(n) = p^{\alpha-1}q^{\beta-1}(p-1)(q-1)\varphi(n_1),$$

and then

$$\begin{aligned}
\varphi_{pq}(n) &= \sum_{d|p^\alpha q^\beta n_1} \mu\left(\frac{p^\alpha q^\beta n_1}{d}\right)\left[\frac{d}{pq}\right]\\
&= \sum_{d_1|n_1} \mu\left(\frac{p^\alpha q^\beta n_1}{d_1}\right)\left[\frac{d_1}{pq}\right] + \sum_{d_1|n_1, 1\leq\alpha_1\leq\alpha} \mu\left(\frac{p^\alpha q^\beta n_1}{p^{\alpha_1}d_1}\right)\left[\frac{p^{\alpha_1}d_1}{pq}\right]\\
&\quad + \sum_{d_1|n_1, 1\leq\beta_1\leq\beta} \mu\left(\frac{p^\alpha q^\beta n_1}{q^{\beta_1}d_1}\right)\left[\frac{q^{\beta_1}d_1}{pq}\right] + \sum_{d_1|n_1, 1\leq\alpha_1\leq\alpha, 1\leq\beta_1\leq\beta} \mu\left(\frac{p^\alpha q^\beta n_1}{p^{\alpha_1}q^{\beta_1}d_1}\right)\left[\frac{p^{\alpha_1}q^{\beta_1}d_1}{pq}\right]\\
&= \sum_{d_1|n_1} \mu\left(\tfrac{qn_1}{d_1}\right)\left[\frac{p^\alpha q^{\beta-1}d_1}{pq}\right] + \sum_{d_1|n_1} \mu\left(\tfrac{n_1}{d_1}\right)\left[\frac{p^\alpha q^\beta d_1}{pq}\right]\\
&\quad + \sum_{d_1|n_1} \mu\left(\tfrac{pn_1}{d_1}\right)\left[\frac{p^{\alpha-1}q^\beta d_1}{pq}\right] + \sum_{d_1|n_1} \mu\left(\tfrac{pqn_1}{d_1}\right)\left[\frac{p^{\alpha-1}q^{\beta-1}d_1}{pq}\right]\\
&= -p^{\alpha-1}q^{\beta-2}\varphi(n_1) + p^{\alpha-1}q^{\beta-1}\varphi(n_1) - p^{\alpha-2}q^{\beta-1}\varphi(n_1) + p^{\alpha-2}q^{\beta-2}\varphi(n_1)\\
&= p^{\alpha-2}q^{\beta-2}(-p + pq - q + 1)\varphi(n_1)\\
&= p^{\alpha-2}(p-1)q^{\beta-2}(q-1)\varphi(n_1) = \tfrac{1}{pq}\varphi(n).
\end{aligned}$$

This completes the proof of (7).

Proof for Theorem 1.5 (1) For the case $\alpha = \beta = 0$, i.e., $n = n_1$.

(i) If $p_i \equiv 1 (\bmod\ pq) (i = 1, \cdots, k)$, then for any $d \mid n, d \equiv 1 (\bmod\ pq)$. Thus by (1.2) and (1.4), we have

$$\begin{aligned}
\varphi_{pq}(n) &= \sum_{d|n} \mu\left(\tfrac{n}{d}\right)\left[\tfrac{d}{pq}\right] = \sum_{d|n} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d-1}{pq}\right)\\
&= \tfrac{1}{pq}\cdot\sum_{d|n} \mu\left(\tfrac{n}{d}\right) d - \tfrac{1}{pq}\cdot\sum_{d|n} \mu\left(\tfrac{n}{d}\right) = \tfrac{1}{pq}\varphi(n).
\end{aligned} \tag{2.18}$$

(ii) If $p_i \equiv -1 (\bmod\ pq) (i = 1, \cdots, k)$, i.e., for any $d \mid n, d \equiv \pm 1 (\bmod\ pq)$. Then by (1.2)–(1.4) and the proof of Theorem 1.1 (1), we have

$$\begin{aligned}
\varphi_{pq}(n) &= \sum_{d|n, d\equiv 1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)\left[\tfrac{d}{pq}\right] + \sum_{d|n, d\equiv -1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)\left[\tfrac{d}{pq}\right]\\
&= \sum_{d|n, d\equiv 1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d-1}{pq}\right) + \sum_{d|n, d\equiv -1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d+1-pq}{pq}\right)\\
&= \sum_{d|n} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d-1}{pq}\right) + \sum_{d|n_1, d\equiv -1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)\left(\tfrac{d-1+2-pq}{pq}\right)\\
&= \tfrac{1}{pq}\varphi(n) + \left(\tfrac{2-pq}{pq}\right)\sum_{d|n, d\equiv -1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)\\
&= \tfrac{1}{pq}\varphi(n) + \left(\tfrac{pq-2}{pq}\right)\sum_{d|n, d\equiv 1 (\bmod\ pq)} \mu\left(\tfrac{n}{d}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{pq}\varphi(n) + \left(\frac{pq-2}{pq}\right) \sum_{\substack{(\prod\limits_{i=1}^{k} p_i^{\alpha_i-1})t \mid n \\ (\prod\limits_{i=1}^{k} p_i^{\alpha_i-1})t \equiv 1 (\bmod\ pq)}} \mu\left(\frac{n}{(\prod\limits_{i=1}^{k} p_i^{\alpha_i-1})t}\right) \\
&= \frac{1}{pq}\varphi(n) + \left(\frac{pq-2}{pq}\right) \sum_{\substack{t \mid \prod\limits_{i=1}^{k} p_i \\ (-1)^{\Omega(n)-k} t \equiv 1 (\bmod\ pq)}} \mu\left(\frac{\prod\limits_{i=1}^{k} p_i}{t}\right) \\
&= \frac{1}{pq}\varphi(n) + \left(\frac{pq-2}{pq}\right)(-1)^{\Omega(n)} 2^{\omega(n)-1}.
\end{aligned} \tag{2.19}$$

From (2.18)–(2.19), we complete the proof of (1).

(2) For $\alpha = 1$ and $\beta = 0$, i.e., $n = pn_1$, then by $\gcd(pq, n_1) = \gcd(p, q) = 1$, we have

$$\varphi(n) = (p-1)\varphi(n_1), \Omega(n) = \Omega(n_1) + 1, \omega(n) = \omega(n_1) + 1.$$

(i) If $p_i \equiv 1 (\bmod\ pq)$, then $p_i \equiv 1 (\bmod\ q) (i = 1, \cdots, k)$. And so by Theorem 1.1, Theorem 1.4 and (2.18), we have

$$\varphi_{pq}(n) = \varphi_q(n_1) - \varphi_{pq}(n_1) = \frac{\varphi(n_1)}{q} - \frac{\varphi(n_1)}{pq} = \frac{\varphi(n)}{q(p-1)} - \frac{\varphi(n)}{pq(p-1)} = \frac{\varphi(n)}{pq}. \tag{2.20}$$

(ii) If $p_i \equiv -1 (\bmod\ pq)$, then $p_i \equiv -1 (\bmod\ q) (i = 1, \cdots, k)$, thus by Theorem 1.1, Theorem 1.4, and (2.19), we have

$$\begin{aligned}
\varphi_{pq}(n) &= \varphi_q(n_1) - \varphi_{pq}(n_1) \\
&= \left(\frac{\varphi(n_1)}{q} + \frac{(q-2)(-1)^{\Omega(n_1)} 2^{\omega(n_1)-1}}{q}\right) - \left(\frac{1}{pq}\varphi(n_1) + \left(\frac{pq-2}{pq}\right)(-1)^{\Omega(n_1)} 2^{\omega(n_1)-1}\right) \\
&= \frac{\varphi(n)}{q(p-1)} - \frac{1}{pq(p-1)}\varphi(n) + \left(\frac{p-1}{pq}\right)(-1)^{\Omega(n_1)+1} 2^{\omega(n_1)} \\
&= \frac{1}{pq}\varphi(n) + \left(\frac{p-1}{pq}\right)(-1)^{\Omega(n)} 2^{\omega(n)-1}.
\end{aligned} \tag{2.21}$$

Now from (2.20)–(2.21), we complete the proof of (2).

(3) For $\alpha = 0$ and $\beta = 1$, in the same proof as that of (2), the result is immediate.

(4) For $\alpha = \beta = 1$, i.e., $n = pqn_1$, then by $\gcd(pq, n_1) = \gcd(p, q) = 1$, we have

$$\varphi(n) = (p-1)(q-1)\varphi(n_1), \Omega(n) = \Omega(n_1) + 2, \omega(n) = \omega(n_1) + 2.$$

(i) If $p_i \equiv 1 (\bmod\ pq)$, i.e., $p_i \equiv 1 (\bmod\ q)$ and $p_i \equiv 1 (\bmod\ q) (i = 1, \cdots, k)$. Then by Theorem 1.1, Theorem 1.4 and (2.18), we have

$$\begin{aligned}
\varphi_{pq}(n) &= \varphi(n_1) + \varphi_{pq}(n_1) - \varphi_q(n_1) - \varphi_p(n_1) \\
&= \varphi(n_1) + \frac{\varphi(n_1)}{pq} - \frac{\varphi(n_1)}{q} - \frac{\varphi(n_1)}{p} \\
&= \frac{(p-1)(q-1)\varphi(n_1)}{pq} \\
&= \frac{\varphi(n)}{pq}.
\end{aligned} \tag{2.22}$$

(ii) If $p_i \equiv -1 (\mathrm{mod}\ pq)$, then by Theorem 1.1, Theorem 1.4 and (2.19), we have

$$\begin{aligned}\varphi_{pq}(n) &= \varphi(n_1) + \varphi_{pq}(n_1) - \varphi_q(n_1) - \varphi_p(n_1)\\ &= \varphi(n_1) + \left(\tfrac{1}{pq}\varphi(n_1) + \left(\tfrac{pq-2}{pq}\right)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}\right)\\ &\quad - \left(\tfrac{\varphi(n_1)}{q} + \tfrac{(q-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{q}\right) - \left(\tfrac{\varphi(n_1)}{p} + \tfrac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p}\right)\\ &= \tfrac{(p-1)(q-1)\varphi(n_1)}{pq} + \tfrac{(2p+2q-pq-2)}{pq}(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}\\ &= \tfrac{\varphi(n)}{pq} + \tfrac{(2p+2q-pq-2)}{pq}(-1)^{\Omega(n)}2^{\omega(n)-3}.\end{aligned} \tag{2.23}$$

Now from (2.22)–(2.23), we complete the proof of (4).

(5) For $\alpha \geq 2$ and $\beta = 0$, i.e., $n = p^{\alpha}n_1$, then by $\gcd(n_1, pq) = \gcd(p, q) = 1$, we have

$$\varphi(n) = p\varphi(p^{\alpha-1}n_1), \Omega(n) = \Omega(p^{\alpha-1}n_1) + 1, \omega(n) = \omega(p^{\alpha-1}n_1). \tag{2.24}$$

(i) If $p \equiv p_i \equiv 1 (\mathrm{mod}\ q)(i = 1, \cdots, k)$, then by Theorem 1.1, Theorem 1.4 and (2.24), we have

$$\varphi_{pq}(n) = \varphi_q(p^{\alpha-1}n_1) = \frac{\varphi(p^{\alpha-1}n_1)}{q} = \frac{\varphi(n)}{pq}. \tag{2.25}$$

(ii) If $p \equiv p_i \equiv -1 (\mathrm{mod}\ q)(i = 1, \cdots, k)$, then by Theorem 1.1, Theorem 1.4 and (2.24), we have

$$\begin{aligned}\varphi_{pq}(n) &= \varphi_q(p^{\alpha-1}n_1)\\ &= \tfrac{\varphi(p^{\alpha-1}n_1)}{q} + \tfrac{(q-2)(-1)^{\Omega(p^{\alpha-1}n_1)}2^{\omega(p^{\alpha-1}n_1)-1}}{q}\\ &= \tfrac{\varphi(n)}{pq} - \tfrac{(q-2)(-1)^{\Omega(n)}2^{\omega(n)-1}}{q}.\end{aligned} \tag{2.26}$$

Now from (2.25)–(2.26), we complete the proof of (5).

(6) For $\alpha \geq 2$ and $\beta = 1$, i.e., $n = p^{\alpha}qn_1$, then by $\gcd(n_1, pq) = \gcd(p, q) = 1$, we have

$$\varphi(n) = \frac{\varphi(p^{\alpha-1}n_1)}{p(q-1)}, \Omega(n) = \Omega(p^{\alpha-1}n_1) + 2, \omega(n) = \omega(p^{\alpha-1}n_1) + 1. \tag{2.27}$$

(i) If $p \equiv p_i \equiv 1 (\mathrm{mod}\ q)(i = 1, \cdots, k)$, then by Theorem 1.1, Theorem 1.4 and (2.27), we have

$$\begin{aligned}\varphi_{pq}(n) &= \tfrac{\varphi(n)}{p(q-1)} - \varphi_q(p^{\alpha-1}n_1) = \tfrac{\varphi(n)}{p(q-1)} - \tfrac{\varphi(p^{\alpha-1}n_1)}{q}\\ &= \tfrac{\varphi(n)}{p(q-1)} - \tfrac{\varphi(n)}{pq(q-1)} = \tfrac{\varphi(n)}{pq}.\end{aligned} \tag{2.28}$$

(ii) If $p \equiv p_i \equiv -1 (\mathrm{mod}\ q)(i = 1, \cdots, k)$, then by Theorem 1.1, Theorem 1.4 and (2.27), in the same proof as that of case (5) (ii), one can get

$$\varphi_{pq}(n) = \frac{\varphi(n)}{pq} - \frac{(q-2)(-1)^{\Omega(n)}2^{\omega(n)-2}}{q}. \tag{2.29}$$

Now from (2.28)–(2.29), we complete the proof of (6).

(7) If $\beta \geq 2$ and $\alpha = 0$ or 1, in the same proofs as those of (4) and (5), the result is obvious.

From the above, Theorem 1.5 is proved.

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两类广义Euler函数的计算公式

廖群英[1], 罗文力[2]

(1.四川师范大学数学学院, 四川 成都 610066)

(2.四川大学附属中学 (成都市第十二中学), 四川 成都 610061)

摘要: 本文研究了广义Euler函数的计算公式. 利用初等的方法和技巧, 给出了两类特殊广义Euler函数的准确计算公式, 即 $\varphi_{pq}(n)$ 以及 $\varphi_e(n)(e=p,p^2)$, 其中 n 的任意素因数 $m\equiv 1$ 或者 $-1(\text{mod } e)$ 且 $\gcd(m,e)=1, p,q$ 是不同的素数. 这些结果是文献[5]相应结果的直接推广.

关键词: Euler函数; 广义Euler函数; Möbius函数

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