

OPTIMAL DIVIDEND AND FINANCING PROBLEMS FOR A DIFFUSION MODEL WITH EXCESS-OF-LOSS REINSURANCE STRATEGY AND TERMINAL VALUE

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Abstract: In this paper, we investigate the optimal dividend and financing policies problems with excess-of-loss reinsurance and a terminal value. In our model, by using approximate diffusion and dynamic programming and constructing of suboptimal problems, we obtain the HJB equation satisfied the general optimal problem and verification theorem. Assuming the proportional and fixed transaction costs and the terminal value at bankruptcy, we get the optimal value function, the optimal dividend policy, the optimal reinsurance strategy and the optimal finance strategy.

Keywords: diffusion process; dividend and financing; excess-of-loss reinsurance; expected value principle; terminal value

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1 Introduction

In the actuarial and mathematical insurance literature, the optimal dividend problem for an insurance company consists in finding a dividend strategy that maximizes the expected discounted dividends until the ruin. The diffusion model about this problem of optimal risk control and dividend distribution was widely studied since fifteen years ago. As we know, the diffusion models for companies can control risk exposure and dividend payment. For example, the papers such as Taksar and Zhou [1], Asmussen et al. [2] and Cadenillas et al. [3] considered that the company control the risk exposure by proportional reinsurance. Meanwhile, some increasing attention was paid to the dividend problem in term of excess-of-loss reinsurance. The literature includes Zhang and Zhou [4], Xu and Zhou [5] and reference therein.

When companies are on the verge of ruin, we can choose to bail out through capital injections or get out of business. When the surplus is null, we inject to prevent bankruptcy,

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see He and Liang [6] and Peng and Chen [7]. Leaving the business may lead to bankruptcy and the corresponding terminal value (say P). For $P \geq 0$, we view the terminal value as salvage value; for $P < 0$, it is viewed as the penalty value. The terminal value considered by the optimal control problem goes back to Karatzas et al. [8]. Taskar [9] firstly considers the risk control and dividend distribution problem with terminal value. Since then, more and more attention are paid on the control problem with terminal value. For instance, the optimal proportion reinsurance policy with terminal value can be found on Taskar [9]. Albrecher [10] and Liang and Young [11] found the optimal dividend strategy in diffusion risk model under a penalty (or salvage) for ruin. Some papers consider the optimal dividend and financing strategies with terminal value and reinsurance policies under different premium principle. For example, Yao and Yang [12] and Cheng and Zhao [13] investigated the optimal problem under the variance premium principle. Cheng and Wang [14] solved this problem under the exponential premium principle. Considering the excess-of-loss reinsurance, Xu and Zhou [5] and Liu and Hu [15] started to research on this optimal problem. As far as we know, very little work has considered the combined optimal dividend, excess-of-loss reinsurance and financing policies with terminal value.

Motivated by the above references, in this paper we study the optimal dividend, financing and excess-of-loss reinsurance with terminal value. The paper is organized as follows. In Section 2, we introduce the model formulation of the problem. In Section 3, we give the HJB equations about the optimisation problem and some properties of the value function. In Section 4, we consider the optimal control problem without capital injection. In Section 5, we solve the optimal problem that the company does not allow to ruin through financing.

2 Model Formulation and the Optimal Control Problem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a probability space, on which all stochastic quantities in this paper are well defined. Here $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration, which satisfies the usual conditions. To specify the diffusion model, firstly, we introduce the classical Cramer-Lundberg risk model. The surplus process of an insurance company satisfies

$$Z_t = x + ct - \sum_{i=1}^{N_t} Y_i, \quad (2.1)$$

where $Z_0 = x \geq 0$ is the initial reserve and $c \geq 0$ is the premium rate. $\{N_t\}_{t \geq 0}$ is a Poisson process with constant intensity λ . Y_i is the size of the i th claim, where Y_i are independent and identically distribution positive random variables with common distribution function F . Assume that mean $\mu_1 = E[Y_i] > 0$ and the second moment $\mu_2 = E[Y_i^2] > 0$ are finite.

In this paper, excess-of-loss reinsurance is available, so let a be the excess-of-loss retention level. Assume the insurer purchases reinsurance contract R to cede the risk of claim, that is to say, the insurer covers $R(Y_i)$ for claim Y_i , and $R(Y_i) = Y_i \wedge a$. Then define the corresponding $\mu_R^{(1)}$ and $\mu_R^{(2)}$. The premium rate for the reinsurance covers c^R , under the

expected value principle, it has $c^R = (1 + \theta)E[\sum_{i=1}^{N_t}(Y_i - R(Y_i))] = (1 + \theta)\lambda(\mu^{(1)} - \mu_R^{(1)})$, where θ is the safety loading, and

$$\mu_R^{(1)}(a) = E[R(Y_i)] = \int_0^a \bar{F}(x)dx, \quad \mu_R^{(2)}(a) = E[R^2(Y_i)] = \int_0^a 2x\bar{F}(x)dx,$$

where $\bar{F}(x) = P(Y_i > x) = 1 - F(x)$. Define

$$N := \inf\{x \geq 0 : \bar{F}(x) = 0\}. \tag{2.2}$$

Thus the function $\mu_R^{(1)}$ and $\mu_R^{(2)}$ are increasing on $[0, N]$, meanwhile they are constants on $[N, \infty)$, which equal to $\mu_R^{(1)}(N)$ and $\mu_R^{(2)}(N)$. So the surplus process in the presence of reinsurance R can be written as

$$Z_t^R = x + (c - c^R)t - \sum_{i=1}^{N_t} R(Y_i). \tag{2.3}$$

By the same argument as in the Asmussen et al. [2], we can approximate the process (2.3) by a pure diffusion process $\{X_t^R\}$ with the same coefficient and volatility. Specifically, $\{X_t^R\}$ satisfies

$$X_t^R = x + [\theta\lambda\mu_R^{(1)}(a) + c - (1 + \theta)\lambda\mu^{(1)}]t + \sqrt{\lambda\mu_R^{(2)}(a)}B_t, \quad X_0^R = x, \tag{2.4}$$

where $\{B_t\}$ is a standard Brownian motion, adapted to the filtration $\{\mathcal{F}_t\}$. Assume the retention level can be dynamically adjusted to control the risk expose. At each time, the insurer chooses the retention level $a = a(t)$.

Next, we consider dividend payment and capital injection in the above processes. Denote L_t as the cumulative amount of dividends paid up to time t . The capital injection process $G_t = \sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}}\eta_n$ is described by a sequence of random variables $\{\eta_n, n = 1, 2, \dots\}$ and a sequence of increasing stopping times $\{\tau_n, n = 1, 2, \dots\}$, which are corresponded to the amounts of capital injections and the times, respectively. Then given an admissible control strategy $\pi = (a^\pi, L^\pi, G^\pi)$, the dynamics of the control surplus process $\{X_t^\pi\}$ with initial reserve $x > 0$ can be written by

$$X_t^\pi = X_t^R - L_t^\pi + G_t^\pi \quad \text{with} \quad X_0^\pi = x - L_0^\pi + G_0^\pi. \tag{2.5}$$

In this paper, using the excess-of-loss reinsurance strategy means that among the admissible reinsurance policies the retention level $\{a_t\}$ can be adjusted dynamically to reach the whole optimization. In addition, we assume that the insure company premium under expected premium principle is $c = (1 + \theta)\lambda\mu^{(1)}$, which is the case of cheap reinsurance. Thus, the process $\{X_t^R\}$ can be written as the following stochastic process

$$X_t^R = x + \int_0^t \theta\lambda\mu_R^{(1)}(a_t)dt + \int_0^t \sqrt{\lambda\mu_R^{(2)}(a_t)}dB_t, \quad X_0^R = x. \tag{2.6}$$

Then we give the following definition of an admissible strategy which the insurer can select.

Definition 2.1 A strategy π is said to be admissible if

- (i) $0 \leq a_t^\pi \leq a, t \geq 0$.
- (ii) $\{L_t^\pi\}$ is an increasing and \mathcal{F}_t -adapted càdlàg process with $L_0^\pi = 0$ and satisfies $\Delta L_t^\pi = L_t^\pi - L_{t-}^\pi \leq X_{t-}^\pi$ for $t \geq 0$.
- (iii) $\{\tau_n^\pi\}$ is a sequence of stopping times with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ and $0 \leq \tau_1^\pi < \tau_2^\pi < \dots < \tau_n^\pi < \dots$ a.s..
- (iv) $\eta_n^\pi \geq 0, n = 1, 2, \dots$ is measurable with respect to $\mathcal{F}_{\tau_n^\pi}$.
- (v) $P(\lim_{n \rightarrow \infty} \tau_n^\pi < t) = 0, \forall t > 0$.

Condition (ii) implies the reserve available is more than the total amount of dividends at the moment. Condition (v) demands that the capital injections may not occur infinitely in a finite interval. Denote by Π the set of these admissible strategies. For each admissible policies $\pi \in \Pi$, the ruin time of the control surplus process is defined as $T^\pi = \inf\{t \geq 0 : X_t^\pi < 0\}$, which is first time that surplus becomes negative. Then we can study the following problem.

We measure the company's value associated with strategy $\pi \in \Pi$ using the following performance function $V(x; \pi)$, which is the expected sum of discounted terminal value and the discounted dividends less the expected discounted costs of refinancing until the time of ruin

$$V(x; \pi) = E_x(\beta_1 \int_0^{T^\pi} e^{-\delta s} dL_s^\pi + P e^{-\delta T^\pi} - \sum_{n=1}^{\infty} e^{-\delta \tau_n^\pi} (\beta_2 \eta_n^\pi + K) I_{\{\tau_n^\pi \leq T^\pi\}}). \quad (2.7)$$

E_x denotes the expectation conditional on $X_0^\pi = x$, and $\delta > 0$ is the discount rate. In the financing process, we assume that the shareholders need to pay $\beta_2 \eta + K$ to meet the capital injection of η , where $\beta_2 > 1$ measures the proportion costs, $K > 0$ is the fixed costs. In the dividend distribution process, proportion costs transaction are taken into account through the value β_1 , $\beta_1 \in (0, 1)$ means that the shares can get $\beta_1 l$ if the company pays l as dividends.

The objection is to find the optimal value function

$$V(x) = \sup_{\pi \in \Pi} V(x; \pi) \quad (2.8)$$

and the associated optimal strategy π^* such that $V(x) = V(x; \pi^*)$.

3 The Hamilton-Jacobi-Bellman (HJB) Equation and Verification Theorem

The solution of the optimal value function and the optimal strategy are based on the HJB (Hamilton-Jacobi-Bellman) equation. In order to deriving this equation, we first define some operators.

For the function $v(x) : [0, \infty) \mapsto \mathbb{R}$, we define the financing operator \mathcal{M} by

$$\mathcal{M}v(x) = \sup_{y \geq 0} \{v(x + y) - \beta_2 y - K\},$$

which means the value of the police that consists of choosing the best immediate capital injection. The differential operator \mathcal{A}^a is denoted by

$$\mathcal{A}^{(a)}v(x) = \frac{1}{2}\lambda\mu_R^{(2)}(a)v''(x) + \theta\lambda\mu_R^{(1)}(a)v'(x) - \delta v(x).$$

Assuming that the value function $V(x)$ is the twice continuously differentiable, and using the standard stochastic control theory (see, e.g., Fleming and Soner [16]), we can characterize the HJB equation of the control problem as

$$\max\{\max_{0 \leq a \leq A} \{\mathcal{A}^a v(x)\}, \beta_1 - v'(x), \mathcal{M}v(x) - v(x)\} = 0; \tag{3.1}$$

$$\max\{\mathcal{M}v(0) - v(0), P - v(0)\} = 0. \tag{3.2}$$

Then we will give some property of the value function.

Lemma 3.1 The value function defined by (2.8) is increasing on $[0, \infty)$ with

$$\beta_1(x - y) \leq V(x) - V(y) \leq \beta_2(x - y) + K \tag{3.3}$$

and satisfies the following boundary condition

$$\beta_1x + P \leq V(x) \leq \beta_1x + \beta_1\theta\lambda\mu_R^{(1)}/\delta + PI_{\{P>0\}}. \tag{3.4}$$

Proof Given $\varepsilon \geq 0$, take a admissible strategy $\pi_1 \in \Pi$ such that $V(x; \pi_1) \geq V(x) - \varepsilon$. For each $x \geq y \geq 0$, we define a new strategy $\tilde{\pi}_1 \in \Pi$ as follows: issue equities with the amount of $x - y$ and then follow the strategy π_1 . The strategy π_1 is admissible and we have

$$V(y) \geq V(y; \tilde{\pi}_1) = V(x; \pi_1) - \beta_2(x - y) - K \geq V(x) - \varepsilon - \beta_2(x - y) - K,$$

ε is arbitrary. So we obtain the second inequality. The first inequality can be similarly proved.

Consider a new strategy π_2 . When we pay all of the surplus as dividends and claim the terminal value, we can get the corresponding performance function $V(x; \pi_2) = \beta_1x + P$. Because of the optimality of the value function, we obtain $V(x) \geq V(x; \pi_2) = \beta_1x + P$.

The surplus process $\{X_t^R\}$ with only reinsurance is described by (2.6), we have

$$E_x\left(\int_0^{T^\pi} e^{-\delta s} dX_s^R\right) = E_x\left(\int_0^{T^\pi} e^{-\delta s} (\theta\lambda\mu_R^{(1)}) ds\right) \leq \theta\lambda\mu_R^{(1)}/\delta.$$

By Itô's formula,

$$e^{-\delta T^\pi} X_{T^\pi}^\pi = x - \delta \int_0^{T^\pi} e^{-\delta s} X_s^\pi ds + \int_0^{T^\pi} e^{-\delta s} dX_s^\pi.$$

We know that $X_{T^\pi}^\pi = 0$ and $X_t^\pi \geq 0$, for $t \leq T^\pi$. Taking the expectation on both sides, we have

$$-E_x\left(\int_0^{T^\pi} e^{-\delta s} dX_s^\pi\right) = x - \delta E_x\left(\int_0^{T^\pi} e^{-\delta s} X_s^\pi ds\right) \leq x,$$

so

$$\begin{aligned}
 V(x; \pi) &= E_x(\beta_1 \int_0^{T^\pi} e^{-\delta s} dL_s^\pi + P e^{-\delta T^\pi} - \sum_{n=1}^\infty e^{-\delta \tau_n^\pi} (\beta_2 \eta_n^\pi + K) I_{\{\tau_n^\pi \leq T^\pi\}}) \\
 &\leq \beta_1 E_x(\int_0^{T^\pi} e^{-\delta s} dL_s^\pi - \sum_{n=1}^\infty e^{-\delta \tau_n^\pi} \eta_n^\pi I_{\{\tau_n^\pi \leq T^\pi\}}) + P I_{\{P > 0\}} \\
 &= \beta_1 [E_x(\int_0^{T^\pi} e^{-\delta s} dX_s^R) - E_x(\int_0^{T^\pi} e^{-\delta s} dX_s^\pi)] + P I_{\{P > 0\}} \\
 &\leq \beta_1 x + \beta_1 \theta \lambda \mu_R^{(1)} / \delta + P I_{\{P > 0\}}. \tag{3.5}
 \end{aligned}$$

Hence, we can get the boundary condition.

Theorem 3.1 (Verification theorem) Let $v(x)$ be a twice continuously differentiable, increasing and concave solution of HJB equations (3.1) and (3.2), and $v'(x)$ is bounded, then for each $\pi \in \Pi$, we have $v(x) \geq V(x; \pi)$. So $v(x) \geq V(x)$ for all $x \geq 0$. Moreover, if there exists some strategy $\pi^* = (a^{\pi^*}, L^{\pi^*}, G^{\pi^*})$ such that $v(x) = V(x; \pi^*)$, then $v(x) = V(x)$ and π^* is optimal.

Proof For each given strategy $\pi = (a^\pi, L^\pi, G^\pi) \in \Pi$, let's define $\Lambda_L^\pi = \{s : L_{s-}^\pi \neq L_s^\pi\}$, $\Lambda_G^\pi = \{s : G_{s-}^\pi \neq G_s^\pi\} = \{\tau_1^\pi, \tau_2^\pi, \dots, \tau_n^\pi, \dots\}$. Let $\hat{L}_t^\pi = \sum_{s \in \Lambda_L^\pi, s \leq t} (L_s^\pi - L_{s-}^\pi)$ be the discontinuous part of L_t^π and $\tilde{L}_t^\pi = L_t^\pi - \hat{L}_t^\pi$ be the continuous part of L_t^π . Similarly, \hat{G}_t^π and \tilde{G}_t^π stand for the discontinuous and continuous parts of G_t^π , respectively. Then, using Itô's formula, we have

$$\begin{aligned}
 e^{-\delta(t \wedge T^\pi)} v(X_{t \wedge T^\pi}^\pi) - v(x) &= \int_0^{t \wedge T^\pi} e^{-\delta s} \mathcal{A}^{a^\pi} v(X_s^\pi) ds + \int_0^{t \wedge T^\pi} \sqrt{\lambda} \mu_R^{(2)} e^{-\delta s} v'(X_s^\pi) dB_s \\
 &\quad - \int_0^{t \wedge T^\pi} e^{-\delta s} v'(X_s^\pi) d\tilde{L}_s^\pi + \int_0^{t \wedge T^\pi} e^{-\delta s} v'(X_s^\pi) d\tilde{G}_s^\pi \\
 &\quad + \sum_{s \in \Lambda_L^\pi \cup \Lambda_G^\pi, s \leq t \wedge T^\pi} e^{-\delta s} (v(X_s^\pi) - v(X_{s-}^\pi)). \tag{3.6}
 \end{aligned}$$

The sum of discontinuous parts of last term can be written as

$$\begin{aligned}
 &\sum_{s \in \Lambda_L^\pi \cup \Lambda_G^\pi, s \leq t \wedge T^\pi} e^{-\delta s} (v(X_s^\pi) - v(X_{s-}^\pi)) \\
 &= \sum_{s \in \Lambda_L^\pi, s \leq t \wedge T^\pi} e^{-\delta s} (v(X_s^\pi) - v(X_{s-}^\pi)) + \sum_{s \in \Lambda_G^\pi, s \leq t \wedge T^\pi} e^{-\delta s} (v(X_s^\pi) - v(X_{s-}^\pi)) \\
 &\leq - \sum_{s \in \Lambda_L^\pi, s \leq t \wedge T^\pi} e^{-\delta s} \beta_1 (L_s^\pi - L_{s-}^\pi) + \sum_{n=1}^\infty e^{-\delta \tau_n^\pi} (\beta_2 \eta_n^\pi + K) I_{\{\tau_n^\pi \leq t \wedge T^\pi\}}, \tag{3.7}
 \end{aligned}$$

where the inequality holds, because $v(x)$ satisfies the HJB (3.1) with $v'(x) \geq \beta_1$ and $\mathcal{M}v(x) \leq v(x)$. Moreover, in consideration of (3.1), the first term of the right hand side of (3.6) is

non-positive. Therefore, by substituting (3.7) into (3.6), we have

$$\begin{aligned}
 e^{-\delta(t \wedge T^\pi)} v(X_{t \wedge T^\pi}^\pi) \leq & v(x) + \int_0^{t \wedge T^\pi} \sqrt{\lambda} \mu_R^{(2)} e^{-\delta s} v'(X_s^\pi) dB_s \\
 & - \int_0^{t \wedge T^\pi} e^{-\delta s} \beta_1 dL_s^\pi + \sum_{n=1}^\infty e^{-\delta \tau_n^\pi} (\beta_2 \eta_n^\pi + K) I_{\{\tau_n^\pi \leq t \wedge T^\pi\}}.
 \end{aligned} \tag{3.8}$$

Because $v(x)$ is an increasing function and $v(0) \geq P$, we obtain

$$\begin{aligned}
 e^{-\delta(t \wedge T^\pi)} P + \int_0^{t \wedge T^\pi} e^{-\delta s} \beta_1 dL_s^\pi - \sum_{n=1}^\infty e^{-\delta \tau_n^\pi} (\beta_2 \eta_n^\pi + K) I_{\{\tau_n^\pi \leq t \wedge T^\pi\}} \\
 \leq v(x) + \int_0^{t \wedge T^\pi} \sqrt{\lambda} \mu_R^{(2)} e^{-\delta s} v'(X_s^\pi) dB_s.
 \end{aligned} \tag{3.9}$$

Taking expectation and limitation on both sides yields

$$v(x) \geq E_x(\beta_1 \int_0^{T^\pi} e^{-\delta s} dL_s^\pi + P e^{-\delta T^\pi} - \sum_{n=1}^\infty e^{-\delta \tau_n^\pi} (\beta_2 \eta_n^\pi + K) I_{\{\tau_n^\pi \leq T^\pi\}}). \tag{3.10}$$

Consequently, $v(x) \geq V(x)$ follows.

4 The Case without Capital Injection

In this section, we consider the first case when the policy without capital injection is optimal. Hence, the corresponding boundary condition is $v(0) = P$ and $\mathcal{M}v(0) \leq v(0)$. Let $x_1 = \inf\{x; v'(x) = \beta_1\}$. Then, the solution $v(x)$ satisfies the following equations

$$\max_{0 \leq a_t \leq N} \{\mathcal{A}^a v(x)\} = 0, \quad 0 \leq x < x_1, \tag{4.1}$$

$$v'(x) - \beta_1 = 0, \quad x \geq x_1, \tag{4.2}$$

$$v(0) = P, \tag{4.3}$$

$$\mathcal{M}v(0) - v(0) \leq 0. \tag{4.4}$$

In fact, these equations indicate that the continuous region is $(0, x_1)$, and the dividend region is $[x_1, \infty)$. For $0 \leq x < x_1$, differentiating (4.1) with respect to a , we can obtain

$$\lambda \bar{F}(a) [a v''(x) + \theta v'(x)] = 0, \tag{4.5}$$

then let the derivative equal to zero yields

$$a(x) = -\theta \frac{v'(x)}{v''(x)}, \quad \text{if } v''(x) \neq 0. \tag{4.6}$$

In view of (2.2), the maximizer of the left-side of (4.1) satisfies the above equation.

Substituting (4.6) into (4.1) leads to

$$\theta \lambda h(a) v'(x) - \delta v(x) = 0, \tag{4.7}$$

where $h(a) = \mu_R^{(1)}(a) - \frac{\mu_R^{(2)}(a)}{2a}$. It is easy to get $h'(a) = \frac{\mu_R^{(2)}(a)}{2a^2}$, $h(a) \geq 0$ and $h(\infty) = \mu^{(1)}$. Differentiating (4.7) yields

$$\theta\lambda[h'(a)a'(x) - \frac{\delta}{\theta\lambda} - \theta\frac{h(a)}{a}]v'(x) = 0, \quad (4.8)$$

from (4.8), we have

$$a'(x) = \frac{\frac{2\delta}{\theta\lambda}a^2 + 2\theta ah(a)}{\mu_R^{(2)}(a)}. \quad (4.9)$$

Let

$$Q(x) = \int_0^x \frac{\mu_R^{(2)}(y)}{\frac{2\delta}{\theta\lambda}y^2 + 2\theta yh(y)} dy. \quad (4.10)$$

It is easy to obtain that $Q'(x) \geq 0$, so the inverse function of $Q(x)$ exists, we have

$$a(x) = Q^{-1}(x + k). \quad (4.11)$$

There exists some constant k such that

$$Q(a(0)) = k. \quad (4.12)$$

Since $a'(x) \geq 0$, the $a(x)$ is a strictly increasing function. Next, we suppose that there exists a reinsurance level x_0 is smaller than that for dividend x_1 , that is, the insurer will keep all the claims when the surplus exceeds x_0 , and we have $Q(N) = x_0 + k$. Thus, from (4.12), we get

$$x_0 = Q(N) - Q(a(0)). \quad (4.13)$$

For $0 \leq x < x_0$, applying (4.3) and (4.6), we obtain

$$v(x) = k_3 \int_0^x e^{\int_y^{x_0} \frac{\theta}{a(s)} ds} dy + P, \quad 0 \leq x < x_0, \quad (4.14)$$

where k_3 is unknown coefficient.

Meanwhile, for $x_0 \leq x < x_1$, we know that the HJB equation becomes a second-order ordinary differential equation

$$\frac{1}{2}\lambda\mu^{(2)}v''(x) + \theta\lambda\mu^{(1)}v'(x) - \delta v(x) = 0. \quad (4.15)$$

Then the solution of (4.15) is of the form

$$v(x) = k_1 e^{r+(x-x_1)} + k_2 e^{r-(x-x_1)}, \quad x_0 \leq x < x_1, \quad (4.16)$$

where k_1 and k_2 need to be determined, and

$$r_{\pm} = \frac{-\theta\mu^{(1)} \pm \sqrt{(\theta\mu^{(1)})^2 + 2\delta\mu^{(2)}}}{\mu^{(2)}}. \tag{4.17}$$

Finally, for $x \geq x_1$, we have $v'(x) = \beta_1$, and $v(x)$ is continuous, we see

$$v(x) = \beta_1(x - x_1) + v(x_1), \quad x \geq x_1. \tag{4.18}$$

Thus, the solution of HJB equation is

$$v(x) = \begin{cases} k_3 \int_0^x e^{\int_y^{x_0} \frac{\theta}{a(s)} ds} dy + P, & 0 \leq x \leq x_0, \\ k_1 e^{r_+(x-x_1)} + k_2 e^{r_-(x-x_1)}, & x_0 \leq x \leq x_1, \\ \beta_1(x - x_1) + v(x_1), & x \geq x_1. \end{cases} \tag{4.19}$$

According to the principle of smooth fit and calculating the first and the second derivatives of $v(x)$ at x_0 and x_1 , respectively, we have

$$k_1 r_+ + k_2 r_- = \beta_1, \tag{4.20}$$

$$k_1 (r_+)^2 + k_2 (r_-)^2 = 0, \tag{4.21}$$

$$k_1 r_+ e^{r_+(x_0-x_1)} + k_2 r_- e^{r_-(x_0-x_1)} = k_3, \tag{4.22}$$

$$k_1 (r_+)^2 e^{r_+(x_0-x_1)} + k_2 (r_-)^2 e^{r_-(x_0-x_1)} = -\frac{\theta}{N} k_3. \tag{4.23}$$

Solving (4.20) and (4.21) leads to

$$k_1 = \frac{-\beta_1 r_-}{r_+(r_+ - r_-)} > 0, \quad k_2 = \frac{\beta_1 r_+}{r_-(r_+ - r_-)} < 0. \tag{4.24}$$

Meanwhile, by multiplying (4.22) by $-\frac{\theta}{N}$ and subtracting it from (4.23) yields

$$e^{(r_+ - r_-)(x_0 - x_1)} = -\frac{k_2 r_- (Nr_- + \theta)}{k_1 r_+ (Nr_+ + \theta)} = \frac{N + \frac{\theta}{r_-}}{N + \frac{\theta}{r_+}}. \tag{4.25}$$

Thus, we can obtain

$$x_1 = x_0 + \frac{1}{r_+ - r_-} \ln\left(\frac{N + \frac{\theta}{r_+}}{N + \frac{\theta}{r_-}}\right), \tag{4.26}$$

where $x_1 > x_0$ holds since $N + \frac{\theta}{r_-} > N - \frac{\mu^{(2)}}{2\mu^{(1)}}$ and

$$\mu^{(2)} = \int_0^N x^2 dF(x) < N \int_0^N x dF(x) = N\mu^{(1)}.$$

So

$$N + \frac{\theta}{r_-} > N - \frac{N}{2} = \frac{N}{2} > 0. \tag{4.27}$$

Using (4.22) and (4.24), we get

$$k_3 = \frac{\beta_1 N}{N + \frac{\theta}{r_+}} \left(\frac{N + \frac{\theta}{r_+}}{N + \frac{\theta}{r_-}} \right)^{\frac{-r_-}{r_+ - r_-}} > 0. \quad (4.28)$$

Lemma 4.1 $\beta_1 < k_3 < 2\beta_1$.

Proof According to the concavity of the log function, we get

$$\begin{aligned} \log k_3 &= \log \beta_1 + \log N + \left[\log \left(N + \frac{\theta}{r_+} \right) - \log \left(N + \frac{\theta}{r_-} \right) \right] \left(-\frac{r_-}{r_+ - r_-} \right) - \log \left(N + \frac{\theta}{r_+} \right) \\ &= \log \beta_1 + \log N - \left[\frac{r_+}{r_+ - r_-} \log \left(N + \frac{\theta}{r_+} \right) - \frac{r_-}{r_+ - r_-} \log \left(N + \frac{\theta}{r_-} \right) \right] \\ &> \log \beta_1 + \log N - \log \left[\frac{r_+}{r_+ - r_-} \left(N + \frac{\theta}{r_+} \right) - \frac{r_-}{r_+ - r_-} \left(N + \frac{\theta}{r_-} \right) \right] \\ &= \log \beta_1 + \log N - \log N = \log \beta_1, \end{aligned}$$

which implies $\beta_1 < k_3$. In addition, we have $k_3 = \beta_1 \left(\frac{N + \frac{\theta}{r_-}}{N + \frac{\theta}{r_+}} \right)^{\frac{r_+}{r_+ - r_-}} - \frac{\theta k_3}{r_- N} < \beta_1 - \frac{\theta k_3}{r_- N}$.

Together with (4.27), we derive that $k_3 < \beta_1 \left(1 + \frac{\theta}{r_- N} \right)^{(-1)} = \frac{N}{N + \frac{\theta}{r_-}} < \frac{\beta_1 N}{N/2} = 2\beta_1$.

Next, we come to determine the value $a(0)$, x_0 and x_1 . For $0 \leq x < x_0$, combining with the boundary value $v(0) = P$ and let $x = 0$ in (4.7), we obtain

$$\theta \lambda h(a(0)) v'(0) = \delta P. \quad (4.29)$$

From (4.13) and (4.26), we can know once the $a(0)$ is determined, the value of x_0 and x_1 can be calculated. In view of (4.14), we have

$$v'(x) = k_3 e^{\int_x^{x_0} \frac{\theta}{a(s)} ds}. \quad (4.30)$$

Then, together with the (4.29), we obtain

$$\theta \lambda k_3 h(a(0)) e^{\int_0^{x_0} \frac{\theta}{a(s)} ds} = \delta P. \quad (4.31)$$

Let's apply a variable change of $y = a(s)$ and use (4.9) and $a(x_0) = N$, we get

$$\theta \lambda k_3 g(a(0)) = \delta P, \quad (4.32)$$

where the function of $g(x)$ is written as $g(x) = h(x) e^{\int_x^N \frac{h'(y)}{\frac{\delta}{\theta^2 \lambda} y + h(y)} dy}$, $x > 0$. After some simple calculations, it is easy to know that $g(0+) = 0$, $g(N) = h(N) = \mu^{(1)} - \frac{\mu^{(2)}}{2N}$, and

$g'(x) = h'(x) e^{\int_x^N \frac{h'(y)}{\frac{\delta}{\theta^2 \lambda} y + h(y)} dy} \left[1 - \frac{h(x)}{\frac{\delta}{\theta^2 \lambda} x + h(x)} \right] > 0$. Then we obtain that the $g(x)$ is a strictly increasing function on $[0, N]$. Thus we can get that (4.32) determines a unique root

$a(0)$ on $[0, N]$ if the condition $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ holds. Thus, we will confirm (4.4) in the following cases.

(1) If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ and $\beta_2 \geq v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > \beta_1$, we are aware of $v'(x)$ is decreasing on $[0, \infty)$, so $v'(x) \leq \beta_2$ holds for $x \geq 0$. Thus $\mathcal{M}v(0) - v(0) = \max_{y \geq 0} \{v(y) - \beta_2 y - K\} - v(0) = -K < 0$, (4.4) holds. In this case Figure 1(a) is a graph of $v'(x)$.

(2) If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ and $v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > \beta_2 \geq k_3$, we know that $v'(x)$ is strictly decreasing from $v'(0)$ to $v'(x_1) = \beta_1$. Thus, there exists a unique number $\gamma_1 \in (0, x_0)$ such that $v'(\gamma_1) = \beta_2$. Then, we define an integral function

$$I(\gamma) = \int_0^\gamma (v'(x) - \beta_2) dx = v(\gamma) - v(0) - \beta_2 \gamma. \tag{4.33}$$

Obviously, (4.4) is valid if and only if when $K \geq I(\gamma_1)$. In this case Figure 1 (b) gives a graph of $v'(x)$.

(3) If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ and $v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > k_3 \geq \beta_2$, similar to (2), we know that $v'(x)$ is strictly decreasing from $v'(0)$ to $v'(x_1) = \beta_1$. Thus, there exists a unique number $\gamma_2 \in (x_0, x_1)$ such that $v'(\gamma_2) = \beta_2$. Then, we define an integral function

$$J(\gamma_2) = \int_{x_0}^{\gamma_2} (v'(x) - \beta_2) dx = v(\gamma_2) - v(x_0) - \beta_2(\gamma_2 - x_0), \tag{4.34}$$

we know $J(\gamma_2)$ is decreasing with regard to β_2 . (4.4) holds if and only if when $K \geq J(\gamma_2)$.

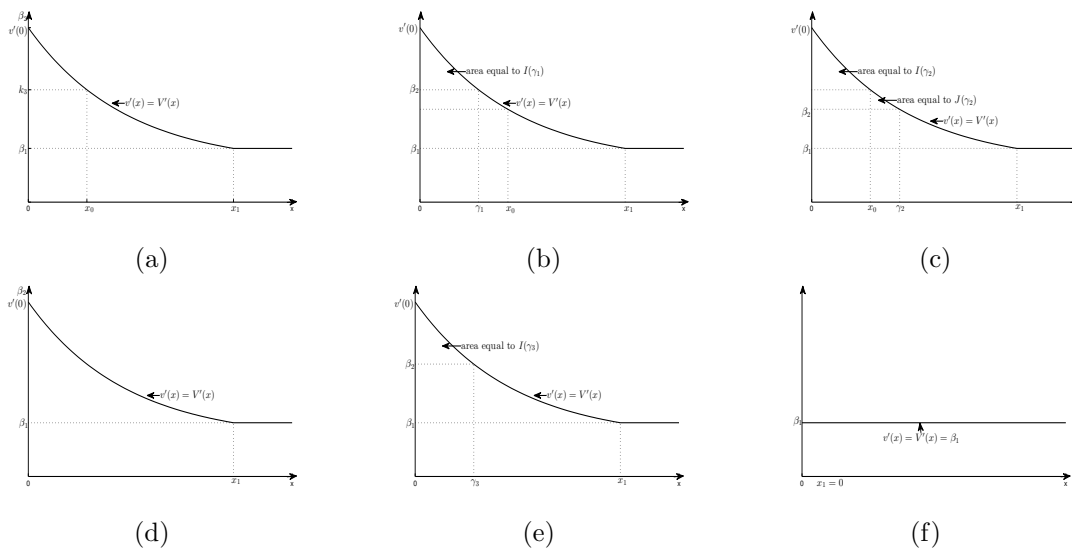


Figure 1: The graph of the derivative $v'(x) = V'(x)$.

In this case, Figure 1 (c) provides a graph of $v'(x)$.

If $P > \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, there is no solution to solve (4.32). In this case, it shows that there does not exist $x_0 > 0$ such that $a(x) < N$ for $x < x_0$. We set $x_0 = 0$, which means that the insurer will keep all claims, i.e., $a(x) \equiv N$ for $x \geq 0$. The suggested solution to (4.1)–(4.3) is of the form

$$v(x) = \begin{cases} k_1e^{r+(x-x_1)} + k_2e^{r-(x-x_1)}, & 0 \leq x \leq x_1, \\ \beta_1(x - x_1) + v(x_1), & x \geq x_1. \end{cases} \quad (4.35)$$

According to the principle of smooth fit at $x = x_1$ and the boundary condition $v(0) = P$, we have

$$k_1e^{-r+x_1} + k_2e^{-r-x_1} = P. \quad (4.36)$$

In order to prove the existence of x_1 , we define $\varphi(x) = k_1e^{-r+x} + k_2e^{-r-x}$. We find that $\varphi(0) = \frac{\theta\lambda\mu^{(1)}}{\delta}$, $\varphi'(x) < 0$ and $\varphi(\infty) = -\infty$. Therefore, when $\frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$ holds, (4.37) will have a unique root of $x_1 > 0$.

(4) If $\frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$ and $\beta_2 \geq v'(0) = k_1r_+e^{-x_1} + k_2r_-e^{-x_1} > \beta_1$, we have $v'(x) \leq \beta_2$ for $x \geq 0$, so $v'(x)$ is decreasing on $[0, \infty)$. Thus

$$\mathcal{M}v(0) - v(0) = \max_{y \geq 0} \{v(y) - \beta_2y - K\} - v(0) = -K < 0,$$

(4.4) holds. In this case, Figure 1 (d) is a graph of $v'(x)$.

(5) If $\frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$ and $v'(0) = k_1r_+e^{-x_1} + k_2r_-e^{-x_1} > \beta_2$, we know that $v'(x)$ is strictly decreasing from $v'(0)$ to $v'(x_1) = \beta_1$. Then there is a unique solution $\gamma_3 \in (0, x_1)$ such that $v'(\gamma_3) = \beta_2$. Thus, (4.4) holds if and only if when $K \geq I(\gamma_3)$. In this case, Figure 1 (e) is a graph of $v'(x)$.

(6) If $P > \frac{\theta\lambda\mu^{(1)}}{\delta}$, there is no solution of (4.36). Therefore, we set $x_1 = 0$, which means that the insurance company will pay all current surplus as dividend at once. The solution is of form $v(x) = \beta_1x + P$. In this case, Figure 1 (f) is a graph of $v'(x)$.

5 The Case with Capital Injection

In this section, we consider the case with capital injection, namely, it is optimal to inject capital to prevent bankruptcy only when the surplus is null. Thus, the solution of $g(x)$ should satisfy

$$\max_{0 \leq a_t \leq N} \{\mathcal{A}^a g(x)\} = 0, \quad 0 \leq x < b_1, \quad (5.1)$$

$$g'(x) - \beta_1 = 0, \quad x \geq b_1, \quad (5.2)$$

$$g(0) \geq P, \quad (5.3)$$

$$\mathcal{M}g(0) - g(0) = 0, \quad (5.4)$$

where $b_1 \geq 0$ is the level of the dividend policy. Similar to Section 4, we solve (5.1)–(5.2) as the following cases.

(1) If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, $v'(0) = k_3e^{\int_0^{x_0} \frac{\theta}{a(s)}ds} > \beta_2 \geq k_3$ and $K < I(\gamma_1)$, we determine a candidate solution of $g(x)$ about some parameter $p_1^* > 0$,

$$g(x) = v(x + p_1^*) = \begin{cases} k_3 \int_0^{x+p_1^*} e^{\int_z^{b_0} \frac{\theta}{a(s)}ds} dz + P, & 0 \leq x < b_0, \\ k_1e^{r+(x-b_1)} + k_2e^{r-(x-b_1)}, & b_0 \leq x < b_1, \\ \beta_1(x - b_1) + \frac{\theta\lambda\mu^{(1)}}{\delta}, & x \geq b_1, \end{cases} \tag{5.5}$$

where $b_0 = x_0 - p_1^*$, $b_1 = x_1 - p_1^*$; x_0 , x_1 and $v(x)$ are written by (4.13), (4.26) and (4.19), respectively. Correspondingly, we define the optimal reinsurance strategy by

$$a_1^{\pi^*}(x) = a^{\pi^*}(x + p_1^*) = \begin{cases} G^{-1}(x + G(a(0)) + p_1^*), & 0 \leq x < b_0, \\ N, & x \geq b_0. \end{cases} \tag{5.6}$$

We get that $g(x)$ and $a_1^{\pi^*}$ satisfy (5.1)–(5.4). Then we will determine $p_1^* > 0$ by using (5.4). Denote

$$\phi(p) = \phi(p; \gamma) := v(\gamma) - v(p) - \beta_2(\gamma - p) - K, \quad 0 \leq p \leq \gamma. \tag{5.7}$$

Because of $K < I(\gamma_1)$, we can get

$$\phi(0; \gamma_1) = v(\gamma_1) - v(0) - \beta_2\gamma_1 - K = I(\gamma_1) - K > 0. \tag{5.8}$$

In addition, we obtain

$$\phi(\gamma_1; \gamma_1) = -K < 0, \quad \phi'(p; \gamma_1) = \beta_2 - v'(p) < 0. \tag{5.9}$$

Therefore, there is a unique $p_1^* \in (0, \gamma_1)$ satisfying $\phi(p_1^*; \gamma_1) = 0$, that is

$$g(l) - g(0) - \beta_2l - K = 0, \tag{5.10}$$

where $l = \gamma_1 - p_1^* > 0$. Noting that $g'(l) = v'(\gamma_1) = \beta_2$, then we have

$$\mathcal{M}g(0) = \max_{y \geq 0} \{g(y) - \beta_2y - K\} = g(l) - \beta_2l - K = g(0). \tag{5.11}$$

Thus, (5.1)–(5.4) hold. In this case, Figure 2 (a) is a graph of $v'(x) = g'(x)$.

(2) If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, $v'(0) = k_3e^{\int_0^{x_0} \frac{\theta}{a(s)}ds} > k_3 \geq \beta_2$ and $J(\gamma_2) < K < I(\gamma_2)$, similar to (1), we define the following candidate solution

$$g_2(x) = v(x + p_2^*) = \begin{cases} k_3 \int_0^{x+p_2^*} e^{\int_z^{x_0} \frac{\theta}{a(s)}ds} dz + P, & 0 \leq x < u_0, \\ k_1e^{r+(x-u_1)} + k_2e^{r-(x-u_1)}, & u_0 \leq x < u_1, \\ \beta_1(x - u_1) + \frac{\theta\lambda\mu^{(1)}}{\delta}, & x \geq u_1, \end{cases} \tag{5.12}$$

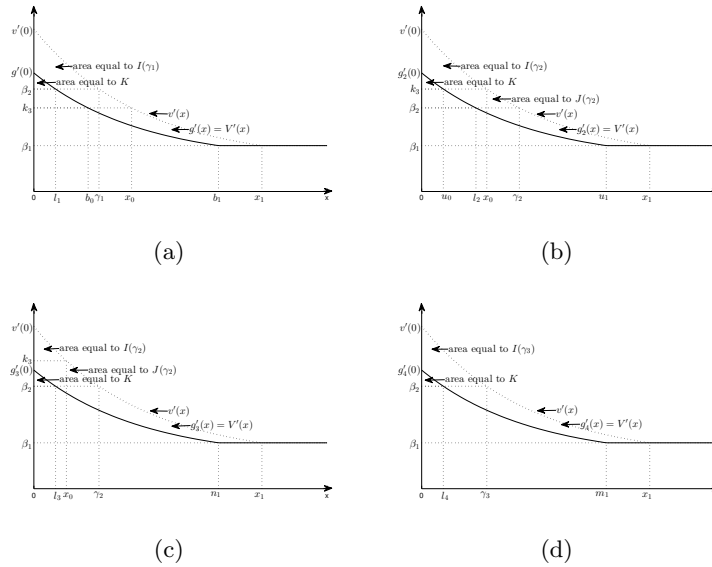


Figure 2: The graph of the derivative $g'(x) = V'(x)$.

where $u_0 = x_0 - p_2^* > 0$, $u_1 = x_1 - p_2^* > 0$ and $p_2^* \in (0, x_0)$ is the unique solution to $\phi(p; \gamma_2) = 0$, i.e.,

$$g_2(l_2) - g_2(0) - \beta_2 l_2 - K = 0, \tag{5.13}$$

where $l_2 = \gamma_2 - p_2^* > 0$. Meanwhile, define a optimal reinsurance policy by

$$a_{\pi_2^*}(x) = a^{\pi^*}(x + p_2^*) = \begin{cases} G^{-1}(x + G(a(0)) + p_2^*), & 0 \leq x < u_0, \\ N, & x \geq u_0. \end{cases} \tag{5.14}$$

Thus, (5.1)–(5.4) hold. In this case, Figure 2 (b) gives a graph of $v'(x) = g_2'(x)$.

(3) If $P \leq \frac{\theta \lambda}{\delta} k_3 (\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, $v'(0) = k_3 e^{\int_0^{x_0} \frac{\theta}{m(s)} ds} > k_3 \geq \beta_2$ and $0 < K \leq J(\gamma_2) < I(\gamma_2)$, similar to (1) and (2), we define a candidate solution

$$g_3(x) = v(x + p_3^*) = \begin{cases} k_1 e^{r+(x-n_1)} + k_2 e^{r-(x-n_1)}, & 0 \leq x < n_1, \\ \beta_1(x - n_1) + \frac{\theta \lambda \mu^{(1)}}{\delta}, & x \geq n_1, \end{cases} \tag{5.15}$$

where $n_1 = x_1 - p_3^* > 0$, and $p_3^* \in [x_0, \gamma_2)$ is the unique solution to $\phi(p; \gamma_2) = 0$, i.e.,

$$g_3(l_3) - g_3(0) - \beta_2 l_3 - K = 0, \tag{5.16}$$

where $l_3 = \gamma_2 - p_3^* > 0$. The optimal reinsurance strategy is $a_{\pi_3^*}(x) \equiv N$. Thus, (5.1)–(5.4) hold. In this case, Figure 2 (c) is a graph of $v'(x) = g_3'(x)$.

(4) If $\frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$, $v'(0) = k_1r_+ + k_2r_- > \beta_2$ and $K < I(\gamma_2)$, similar to (1), (2) and (3), we define a candidate solution

$$g_4(x) = v(x + p_4^*) = \begin{cases} k_1e^{r_+(x-m_1)} + k_2e^{r_-(x-m_1)}, & 0 \leq x < m_1, \\ \beta_1(x - m_1) + \frac{\theta\lambda\mu^{(1)}}{\delta}, & x \geq m_1, \end{cases} \tag{5.17}$$

where $m_1 = x_1 - p_4^* > 0$, and $p_4^* \in (0, \gamma_3)$ is the unique solution to $\phi(p; \gamma_3) = 0$, i.e.,

$$g_4(l_4) - g_4(0) - \beta_2l_4 - K = 0, \tag{5.18}$$

where $l_4 = \gamma_3 - p_4^* > 0$. The optimal reinsurance strategy is $a_4^{\pi^*}(x) \equiv N$. Thus, (5.1)–(5.4) hold. In this case, Figure 2 (d) provides a graph of $v'(x) = g_4'(x)$.

Based on the above analysis, we identify the explicit solution to the value function and construct the associated optimal strategy in the following. In order to the following theorem, let

$$L_t^{\pi^*}(u) = (x - u)_+ + \int_0^t I_{\{X_s^{\pi^*} \geq u\}} dL_s^{\pi^*}. \tag{5.19}$$

Theorem 5.1 The value function $V(x)$ and the corresponding optimal reinsurance policy $\pi^* = (a^{\pi^*}, L^{\pi^*}, G^{\pi^*})$ are given in the following different cases.

Case 1 If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ and $\beta_2 \geq v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > \beta_1$, then the value function $V(x)$ coincides with $v(x)$ in (4.19), and the optimal retention level is

$$a^{\pi^*} = \begin{cases} G^{-1}(x + G(a(0))), & 0 \leq x < x_0, \\ N, & x \geq x_0. \end{cases} \tag{5.20}$$

Case 2 If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ and $v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > \beta_2 \geq k_3$, then the value function and the optimal retention level take the same forms as those in Case 1.

Case 3 If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$ and $v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > k_3 \geq \beta_2$, then the value function and the optimal retention level take the same forms as Case 1.

Case 4 If $\frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$ and $\beta_2 \geq v'(0) = k_1r_+e^{-x_1} + k_2r_-e^{-x_1} > \beta_1$, then the value function $V(x)$ coincides with $v(x)$ in (4.37), and the optimal retention level is $a^{\pi^*} \equiv N$ for all $x \geq 0$, which means that the insurance don't take any reinsurance.

Case 5 If $\frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$ and $v'(0) = k_1r_+e^{-x_1} + k_2r_-e^{-x_1} > \beta_2$, then the value and the optimal retention level take the same forms as Case 4.

Case 6 If $P > \frac{\theta\lambda\mu^{(1)}}{\delta}$, then $V(x) = \beta_1x + P$. Paying all of the surplus as dividend is the optimal policy, and then claim the terminal value P at once.

Case 7 If $P \leq \frac{\theta\lambda}{\delta}k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, $v'(0) = k_3e \int_0^{x_0} \frac{\theta}{a(s)} ds > \beta_2 \geq k_3$ and $K < I(\gamma_1)$, then the value function $V(x)$ coincides with $g(x)$ and the optimal retention level in (5.5)

and (5.6), respectively. It is optimal to refinance if and only if the surplus is null, and the surplus immediately jumps to $l_1 = \gamma_1 - p_1^*$ when it reaches 0 by issuing equities. So, G^{π^*} is written as

$$\begin{cases} \int_0^\infty I_{\{t: X_t^{\pi^*} > 0\}} dG_t^{\pi^*} = 0, \\ \tau_1^{\pi^*} = \inf\{t \geq 0 : X_{t-}^{\pi^*} = 0\}, \\ \tau_n^{\pi^*} = \inf\{t > \tau_{n-1}^{\pi^*} : X_{t-}^{\pi^*} = 0\}, \quad n = 2, 3, \dots, \\ l_n^{\pi^*} \equiv l_1 = \gamma_1 - p_1^*, \quad n = 1, 2, \dots. \end{cases} \tag{5.21}$$

Case 8 If $P \leq \frac{\theta\lambda}{\delta} k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, $v'(0) = k_3 e^{\int_0^{x_0} \frac{\theta}{a(s)} ds} > k_3 \geq \beta_2$ and $J(\gamma_2) < K < I(\gamma_2)$, then the value function $V(x)$ is identical to $g_2(x)$ in (5.12), G^{π^*} is written as $l_n^{\pi^*} \equiv l_2^{\pi^*} = \gamma_2^{\pi^*} - p_2^{\pi^*}$, $n = 1, 2, \dots$ and the optimal reinsurance strategy $a_2^{\pi^*}$ is characterised by (5.14).

Case 9 If $P < \frac{\theta\lambda}{\delta} k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N})$, $v'(0) = k_3 e^{\int_0^{x_0} \frac{\theta}{m(s)} ds} > k_3 \geq \beta_2$ and $0 < K \leq J(\gamma_2) < I(\gamma_2)$, then the value function $V(x)$ coincides with $g_3(x)$ in (5.15), G^{π^*} is written as $l_n^{\pi^*} \equiv l_3^{\pi^*} = \gamma_2^{\pi^*} - p_3^{\pi^*}$, $n = 1, 2, \dots$ and the optimal reinsurance strategy is $a_\pi^*(x) \equiv N$.

Case 10 If $\frac{\theta\lambda}{\delta} k_3(\mu^{(1)} - \frac{\mu^{(2)}}{2N}) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$, $v'(0) = k_1 r_+ k_2 r_- > \beta_2$ and $K < I(\gamma_3)$, then the value function $V(x)$ is identical to $g_4(x)$ in (5.17), G^{π^*} is written as $l_n^{\pi^*} \equiv l_4^{\pi^*} = \gamma_3^{\pi^*} - p_4^{\pi^*}$, $n = 1, 2, \dots$ and the optimal reinsurance strategy is $a_\pi^*(x) \equiv N$.

Proof Here we only prove Case 7 in detail as an example and the other cases also can be proved by this method. By verifying the second derivative of $g(x)$, we can check the concavity of $g(x)$ function. For $0 \leq x < b_0$, we have

$$g''(x) = -k_3 \frac{\theta}{a(x)} e^{\int_{x+p_1^*}^{b_0} \frac{\theta}{a(s)} ds} < 0.$$

For $b_0 \leq x < b_1$, we obtain

$$\begin{aligned} g''(x) &= k_1(r_+)^2 e^{r_+(x-b_1)} + k_2(r_-)^2 e^{r_-(x-b_1)} \\ &= e^{r_-(x-b_1)} [k_1(r_+)^2 e^{(r_+-r_-)(x-b_1)} + k_2(r_-)^2] \\ &= \frac{r_+ r_-}{r_+ - r_-} e^{r_-(x-b_1)} [1 - e^{(r_+-r_-)(x-b_1)}] < 0. \end{aligned} \tag{5.22}$$

In addition, we can prove that the value function $g(x)$ is indeed an increasing, concave and twice continuously differentiable solution to equations (3.1) and (3.2). Using Theorem 3.1, we can show that the solution to HJB eq.(3.1) and (3.2) exactly equal to the value function. Thus, $g(x) \geq V(x)$ holds according to Theorem 3.1. Here we omit the details when $g(x)$ satisfies (3.1) and (3.2). Then, from (5.6), (5.19) and (5.21) we will check the

optimal policy $\pi^* = (a^{\pi^*}, L^{\pi^*}, G^{\pi^*}) \in \Pi$, where $u = b_1$ in (5.19). Because of $\mathcal{A}_1^{a^{\pi^*}} g(X_t^{\pi^*}) = 0$ for $0 \leq X_t^{\pi^*} \leq b_1$, it follows

$$\int_0^{t \wedge T^{\pi^*}} e^{-\delta s} \mathcal{A}_1^{a^{\pi^*}} g(X_s^{\pi^*}) ds = \int_0^{t \wedge T^{\pi^*}} e^{-\delta s} \mathcal{A}_1^{a^{\pi^*}} g(X_s^{\pi^*}) I_{\{0 \leq X_s^{\pi^*} \leq b_1\}} ds = 0. \tag{5.23}$$

Meanwhile, (5.6), (5.19) and (5.21) indicate that

$$\begin{aligned} & \sum_{s \in \Lambda_L^{\pi^*} \cup \Lambda_G^{\pi^*}, s \leq t \wedge T^{\pi^*}} e^{-\delta s} (g(X_s^{\pi^*}) - g(X_{s-}^{\pi^*})) \\ = & \sum_{s \in \Lambda_L^{\pi^*}, s \leq t \wedge T^{\pi^*}} e^{-\delta s} (g(X_s^{\pi^*}) - g(X_{s-}^{\pi^*})) I_{\{X_s^{\pi^*} = b_1\}} + \sum_{s \in \Lambda_G^{\pi^*}, s \leq t \wedge T^{\pi^*}} e^{-\delta s} (g(X_s^{\pi^*}) - g(X_{s-}^{\pi^*})) I_{\{X_{s-}^{\pi^*} = 0\}} \\ = & - \sum_{s \in \Lambda_L^{\pi^*}, s \leq t \wedge T^{\pi^*}} e^{-\delta s} \beta_1 (L_s^{\pi^*} - L_{s-}^{\pi^*}) + \sum_{n=1}^{\infty} e^{-\delta \tau_n^{\pi^*}} (\beta_2 \eta_n^{\pi^*} + K) I_{\{\tau_n^{\pi^*} \leq t \wedge T^{\pi^*}\}}. \end{aligned} \tag{5.24}$$

Thus, from (3.6), substituting π, T^π, u with $\pi^*, T^{\pi^*} = \infty, b_1$ and taking expectations, we get

$$g(x) = E_x[e^{-\delta t} g(X_t^{\pi^*})] + E_x(\beta_1 \int_0^t e^{-\delta s} dL_s^{\pi^*} - \sum_{n=1}^{\infty} e^{-\delta \tau_n^{\pi^*}} (\beta_2 \eta_n^{\pi^*} + K) I_{\{\tau_n^{\pi^*} \leq t\}}). \tag{5.25}$$

Let $t \rightarrow \infty$, the first term on the right hand side vanishes, therefore, we have

$$g(x) = E_x(\beta_1 \int_0^\infty e^{-\delta s} dL_s^{\pi^*} - \sum_{n=1}^{\infty} e^{-\delta \tau_n^{\pi^*}} (\beta_2 \eta_n^{\pi^*} + K) I_{\{\tau_n^{\pi^*} \leq \infty\}}) = V(x; \pi^*), \tag{5.26}$$

considering $g(x) \geq V(x)$, we check that $g(x) = V(x) = V(x; \pi^*)$.

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具有停止损失再保险策略和最终值的扩散模型的最优分红与注资问题

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摘要: 本文研究了具有停止损失再保险和最终值的最优分红和融资策略问题. 通过运用近似扩散和动态规划及构造次最优问题的方法, 得到了解决一般最优问题所应符合的HJB方程和验证定理. 假设有比例和固定交易费用以及在破产时刻产生最终值, 得到了相应的最优值函数, 最优分红策略, 再保险策略以及融资策略.

关键词: 扩散过程; 分红与注资; 停止损失再保险; 期望值原则; 最终值

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