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TWO ALGORITHM FRAMEWORKS BASED ON DISCRETIZATION METHOD AND LOCAL REDUCTION FOR SEMI-INFINITE PROGRAMMING

XU Qing-juan[1], JIAN Jin-bao[2]

(1.College of Mathematical and Statistics Sciences, Guangxi Teachers Education University, Nanning 530001, China)

(2. College of Science, Guangxi University for Nationalities, Nanning 530006, China)

Abstract: In this paper, two algorithm frameworks for semi-infinite programming (SIP) are discussed. Using discretization method and local reduction method, we present two algorithm frameworks for SIP. Under some mild assumptions, the algorithm framework based on discretization method possesses weak global convergence. Numerical experiments show that the proposed algorithm frameworks are effective.

Keywords: semi-infinite programming; discretization method; local reduction method; global convergence

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1 Introduction

In recent decades, semi infinite programming (SIP) attracted the attention and favor of many scholars at home and abroad, and the research results were abundant [1–4]. A simple form of SIP is given as follows:

$$\text{SIP} \qquad \min f(x) \quad \text{s.t. } g(x,\omega) \leq 0, \forall\, \omega \in \Omega = [a,b], \tag{1.1}$$

where $f: R^n \to R$ is continuously differentiable and $g: R^n \times [a,b] \to R$ is continuously differentiable with respect to x.

Discretization method is a common method, and local reduction method is an intrinsic method for solving SIP. Refs. [1–3] made a analysis in detail. In view of the wide application

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Biography: Xu Qingjuan (1979–), female, born at Chengwu, Shandong, associate professor, major in optimization methods and statistical computation.

Corresponding author: Jian Jinbao.

of discretization method in engineering, many scholars studied the discretized problem from SIP [5–11]. The discretized problem has the following form

$$\mathrm{SIP}_q \qquad \min f(x) \quad \text{s.t. } g(x,\omega) \le 0, \forall\, \omega \in \Omega_q, \tag{1.2}$$

where $\Omega_q = \{\omega \in \Omega \mid \omega = a + i \cdot \frac{b-a}{q}, i = 0, 1, 2, \cdots, q\}$, and q indicates the level of discretization. The interpretation of q can be seen in [10].

For better analysis of how to solve SIP through solving SIP_q, we attempt to present two algorithm frameworks based on discretization method and local reduction method, respectively. Of course, the two algorithm frameworks can be considered to be designed for the algorithms for SIP_q in [8–11]. In addition, under some necessary assumptions, the global convergence of the previous algorithm framework is proved. Finally, some preliminary numerical results are reported.

2 Algorithm Framework Based on Discretization Method

2.1 Description of Algorithm

In this section, inspired by the idea of discretization method [2], we present an algorithm framework for SIP on this basis of the algorithms for SIP_q in [8–11].

For clarity, we denote the algorithm for SIP_q in [8–11] as "Algorithm A", which is taken as an inner iteration of Algorithm below. Define the distance of Hausdorff between Ω and Ω_q as $\mathrm{dist}(\Omega_q, \Omega) = \max\limits_{\omega \in \Omega} \min\limits_{\hat{\omega} \in \Omega_q} \|\hat{\omega} - \omega\|$. As to discretization method, the sequence of discretized set $\{\Omega_{q_i}\}_{i \in N_0}$ satisfies the following conditions:

$$\Omega_{q_i} \subset \Omega,\ |\Omega_{q_i}| < \infty,\ i \in N_0,\ \lim_{i \to \infty} \mathrm{dist}(\Omega_{q_i}, \Omega) = 0.$$

Our algorithm framework based on discretization method is described as follows.

Algorithm 2.1 An algorithm framework based on discretization method for SIP.

Parameters $\{\tau_i\}_{i \in N_0}$ such that $0 < \tau_{i+1} < \tau_i, \forall i \in N_0$ and $\lim\limits_{i \to \infty} \tau_i = 0$, $q_0 \in N_0 \backslash \{0\}$, and initialized parameters of Algorithm A.

Data $x_0 = \bar{x}_{q_{-1}} \in R^n$, choose discretized set $\Omega_{q_0} \subset \Omega$ such that $|\Omega_{q_0}| < \infty$ and $\mathrm{dist}(\Omega_{q_0}, \Omega) \le \tau_0$.

Step 0 Set $i = 0$.

Step 1 Solve $\mathrm{SIP}(\Omega_{q_i})$ (i.e., SIP_{q_i}) by applying Algorithm A to obtain a KKT point $\bar{x}_{q_i}$.

Step 2 Choose discretized set $\Omega_{q_{i+1}} \subset \Omega$ such that $\Omega_{q_0} \subset \Omega_{q_{i+1}}$, $|\Omega_{q_{i+1}}| < \infty$ and $\mathrm{dist}(\Omega_{q_{i+1}}, \Omega) \le \tau_{i+1}$.

Step 3 Set $i = i + 1$, and go back to Step 1.

Discretization method is actually an outer approximation algorithm. The real solution is approximated by the exterior approximation of the feasible region of SIP. From a numerical viewpoint, only the conceptual discretization method is useful. The latest research on discretization method can be seen in refs. [12, 13].

2.2 Global Convergence

In this part, we will discuss the global convergence of Algorithm 2.1 under mild assumptions. For the sake of convenience, we denote

$$\psi(x) = \max\{0, \max_{\omega\in\Omega} g(x,w)\},\ \varphi_{q_i}(x) = \max\{0, \max_{\omega\in\Omega_{q_i}} g(x,w)\},$$
$$\Omega_{act}(x) = \{\omega\in\Omega|\ g(x,\omega)=0\},\ \Omega_{act,q_i}(x) = \{\omega\in\Omega_{q_i}|\ g(x,\omega)=0\}.$$

Definition 2.1 For $x_0 \in R^n$ and discretized set $\Omega_{q_i} \subset \Omega, i \in N_0$ in Algorithm 2.1, the level set is defined as

$$\begin{aligned} LV(x_0,\Omega_{q_0},\Omega) &= \{x\in R^n|\ \varphi_{q_0}(x)\le \psi(x^0)\},\\ LV(x_0,\Omega_{q_i},\Omega) &= \{x\in R^n|\ \varphi_{q_i}(x)\le \psi(x^0)\}. \end{aligned} \tag{2.1}$$

Assumption 2.1 Level set $LV(x^0,\Omega_{q_0},\Omega)$ is bounded, thus is compact.

Remark 1 The assumption that $\{x^k\}$ is bounded in refs. [8–11] can be replaced by Assumption 2.1.

Assumption 2.2 Functions $g(\cdot,\cdot)$ are Lipschitz continuous in the bounded set, i.e., there exist Lipschitz constants L_{g_x} and L_{g_ω} such that

$$\begin{aligned} |g(x_1,\omega)-g(x_2,\omega)| &\le L_{g_x}\|x_1-x_2\|,\ \forall\omega\in\Omega, x_1,x_2\in LV(x^0,\Omega_{q_0},\Omega);\\ |g(x,\omega_1)-g(x,\omega_2)| &\le L_{g_\omega}\|\omega_1-\omega_2\|,\ \forall x\in LV(x_0,\Omega_{q_0},\Omega),\omega_1,\omega_2\in\Omega. \end{aligned} \tag{2.2}$$

Assumption 2.3 Suppose that linearly independent constraint qualification (LICQ) is satisfied by problem SIP at any $\bar{x}\in\Omega_{act}$, i.e., the vectors $\{\nabla_x g(\bar{x},\omega),\omega\in\Omega_{act}(\bar{x})\}$ are linearly independent.

Lemma 2.1 (see [3]) Suppose that Assumption 2.3 holds, then the number of indices of $\Omega_{act}(\bar{x})$ is finite.

Definition 2.2 (see [3]) Suppose that $\bar{x}\in X, 0\ne|\Omega_{act}(\bar{x})|<\infty$, then $\bar{x}$ is the KKT point of SIP, if there exist $\bar{u}_1,\cdots,\bar{u}_\omega\ge 0,\omega\in\Omega_{act}(\bar{x})$ such that

$$\nabla f(\bar{x}) + \sum_{\omega\in\Omega_{act}(\bar{x})} \bar{u}_\omega \nabla_x g(\bar{x},\omega) = 0.$$

Lemma 2.2 Suppose that iteration point sequence$\{\bar{x}_{q_i}\}_{i\in N_0}$ is yielded by Algorithm 2.1, then there exists an accumulation point $\bar{x}$ of $\{\bar{x}_{q_i}\}_{i\in N_0}$.

Proof If $x\in LV(x_0,\Omega_{q_i},\Omega)$, according to $\Omega_{q_0}\subset\Omega_{q_i},\forall i\in N_0\backslash\{0\}$, one can see that $\varphi_{q_0}(x)\le\varphi_{q_i}(x_0)\le\psi(x_0)$. Thus, it follows that $LV(x_0,\Omega_{q_i})\subset LV(x_0,\Omega_{q_0},\Omega)$. Further, we can conclude that set $LV(x_0,\Omega_{q_i},\Omega)$ is a closed subset of compact set $LV(x_0,\Omega_{q_0},\Omega)$, and so it is compact. For iteration point sequence $\{\bar{x}_{q_i}\}_{i\in N_0}$, one has $\bar{x}_{q_i}\in LV(x_0,\Omega_{q_i},\Omega)$, thus, there exists an accumulation point $\bar{x}$ of $\{\bar{x}_{q_i}\}_{i\in N_0}$, and the proof is finished.

Lemma 2.3 Suppose that iteration point sequence $\{\bar{x}_{q_i}\}_{i\in N_0}$ is generated by Algorithm 2.1, and the subset $\{\bar{x}_{q_i}\}_{i\in I}, I\subseteq N_0, |I|=\infty$ converges to $\bar{x}$. Then $\{\varphi_{q_i}(\bar{x}_{q_i})\}_{i\in I}$ is convergent, and $\lim\limits_{i\to\infty,i\in I}\varphi_{q_i}(\bar{x}_{q_i})=\psi(\bar{x})$.

Proof First, we prove that $0 \leq \psi(x) - \varphi_{q_i}(x) \leq L_{g_\omega}\operatorname{dist}(\Omega_{q_i}, \Omega), \forall x \in LV(x^0, \Omega_{q_0}, \Omega), i \in N_0$.

If $\psi(x) = 0$, then the formula above hold obviously.

If $\psi(x) > 0$, choosing $\omega_g \in \Omega_g(x)$, then there exists $\omega_{q_i} \in \Omega_{q_i}$ such that $\|\omega_g - \omega_{q_i}\| \leq L_{g_\omega}\operatorname{dist}(\Omega_{q_i}, \Omega)$. Furthermore, it follows that

$$0 \leq \psi(x) - \varphi_{q_i}(x) \leq g(x, \omega_g) - g(x, \omega_{q_i}) \leq L_{g_\omega}\|\omega_g - \omega_{q_i}\| \leq L_{g_\omega}\operatorname{dist}(\Omega_{q_i}, \Omega). \tag{2.3}$$

Next, we prove that $\{\varphi_{q_i}(\bar{x}_{q_i})\}_{i\in I}$ is convergent.

From (2.3), one has $\varphi_{q_i}(\bar{x}_{q_i}) \leq \psi(\bar{x}_{q_i})$. Further,

$$\varphi(\bar{x}_{q_i}) = \varphi(\bar{x}_{q_i}) - \psi(\bar{x}_{q_i}) + \psi(\bar{x}_{q_i}) \geq -L_{g_\omega}\operatorname{dist}(\Omega_{q_i}, \Omega) + \psi(\bar{x}_{q_i}) \geq -L_{g_\omega}\tau_i + \psi(\bar{x}_{q_i}).$$

By Algorithm 2.1, one can see that $\lim\limits_{i\to\infty} \tau_i = 0$. Applying (2.3), we can conclude that $\lim\limits_{i\to\infty, i\in I} \varphi(\bar{x}_{q_i}) = \psi(\bar{x})$. Thus, the proof of this lemma is finished.

Lemma 2.4 Suppose that the stated assumption of Lemma 2.3 holds. Then, for $\bar{\omega} \in \Omega_{act}(\bar{x})$, there exists an iteration point sequence $\{\omega_{q_i}\}_{i\in I}$ such that $\omega_{q_i} \in \Omega_{act,q_i}(\bar{x}_{q_i})$, and $\omega_{q_i} \to \bar{\omega}$ holds for $i \in I$ large enough.

Proof From Lemma 2.1, we have $|\Omega_{act}(\bar{x})| < \infty$. For any $\bar{\omega} \in \Omega_{act}(\bar{x})$, one obtains $g(\bar{x}, \bar{\omega}) = 0$. Let $\omega_{q_i} \in \Omega_{q_i}$, we can conclude that

$$\begin{aligned}
\|g(\bar{x}_{q_i}, \omega_{q_i}) - g(\bar{x}, \bar{\omega})\| &\leq L_{g_\omega}\|\omega_{q_i} - \omega\| + L_{g_x}\|\bar{x}_{q_i} - \bar{x}\| \\
&\leq L_{g_\omega}\operatorname{dist}(\Omega_{q_i}, \Omega) + L_{g_x}\|\bar{x}_{q_i} - \bar{x}\| \\
&\leq L_{g_\omega}\tau_i + L_{g_x}\|\bar{x}_{q_i} - \bar{x}\|.
\end{aligned}$$

In view of $\lim\limits_{i\to\infty} \tau_i = 0$ and $\lim\limits_{i\to\infty, i\in I} \bar{x}_{q_i} \to \bar{x}$, one can conclude that $\lim\limits_{i\to\infty, i\in I} g(\bar{x}_{q_i}, \omega_{q_i}) = 0$. Thus, the conclusion is at hand.

Theorem 2.1 Suppose that Assumptions 2.1–2.3 hold, and $\{\bar{x}_{q_i}\}_{i\in N_0}$ is yielded by Algorithm 2.1, there exists an accumulation point $\bar{x}$ of $\{\bar{x}_{q_i}\}_{i\in N_0}$ which is a KKT point for SIP (1.1). In such sense, Algorithm 2.1 is said to possess weak global convergence.

Proof By Lemma 2.2, one knows that there exists an accumulation point $\bar{x}$ of $\{\bar{x}_{q_i}\}_{i\in N_0}$. Choose an subset $\{\bar{x}_{q_i}\}_{i\in I}, I \subseteq N_0, |I| = \infty$ that converges to $\bar{x}$. From the structure of Algorithm 2.1, we can see that $\bar{x}_{q_i}$ is the KKT point of SIP_{q_i}. Further, taking into account the theorem of Caratheodory, for $s = n+1, i \in I$ and $\omega_{q_i} \in \Omega_{act,q_i}(\bar{x}_{q_i})$, one can conclude that

$$\left\{\begin{aligned}
&\nabla f(\bar{x}_{q_i}) + \sum_{j=1}^{s} \lambda_i^j \nabla_x g(\bar{x}_{q_i}, \omega_{q_i}^j) = 0, && (2.4)\\
&\lambda_i^j \cdot g(\bar{x}_{q_i}, \omega_{q_i}^j) = 0,\ j = 1, 2, \cdots, s, && (2.5)\\
&\lambda_i^j \geq 0, && (2.6)\\
&\varphi_{q_i}(\bar{x}_{q_i}) = 0. && (2.7)
\end{aligned}\right.$$

In view of [10, Lemma 3.4] and [11, Lemma 3.5], we can conclude that $\{\lambda_i^j\}_{i\in I}, j = 1,2,\cdots,s$ are bounded. Thus, there exists a subset that converges to $\bar{\lambda}^j, j = 1,2,\cdots,s$. Without loss of generality, we regard the subset as original sequence. Denote $J_{act}(\bar{x})$ as an index set of $\Omega_{act}(\bar{x})$. By Lemma 2.4, one knows that, for $\bar{\omega}^j \in \Omega_{act}(\bar{x}), j \in J_{act}(\bar{x})$, there exists a sequence $\{\omega_{q_i}^j\}_{i\in I}$ such that $\omega_{q_i}^j \in \Omega_{act,q_i}(\bar{x}_{q_i})$, and $\omega_{q_i}^j \to \bar{\omega}^j, j \in J_{act}(\bar{x})$ for $i \in I$ large enough. Furthermore, through putting in order, one can get that, the first $l(= |J_{act}(\bar{x})|)$ indices of $S(= 1,2,\cdots,s)$ correspond to $\omega_{q_i}^j$ of $\Omega_{act,q_i}(\bar{x}_{q_i})$, and $\omega_{q_i}^j \to \bar{\omega}^j, j = 1,2,\cdots,l$. Moreover, note that $|S| < \infty$ and Ω are compact, we know that sequence or its subsequence $\{\omega_{q_i}^j\}_{i\in I}, j = l+1,\cdots,s$ converges to $\bar{\omega}^j \in \Omega, j = l+1,\cdots,s$. So, passing to the limit for $i \in I$ and $i \to \infty$ in (2.4)–(2.7), combining with Lemma 2.3, we have

$$\begin{cases} \nabla f(\bar{x}) + \displaystyle\sum_{j=1}^{s} \bar{\lambda}^j \nabla_x g(\bar{x}, \bar{\omega}^j) = 0, & (2.8) \\ \bar{\lambda}^j \cdot g(\bar{x}, \bar{\omega}^j) = 0,\ j = 1,2,\cdots,s, & (2.9) \\ \bar{\lambda}^j \geq 0, & (2.10) \\ \psi(\bar{x}) = 0. & (2.11) \end{cases}$$

From (2.11), one knows that $\bar{x} \in X$. Further, combining with Definition 2.2, according to (2.8)–(2.10), we can conclude that $\bar{x}$ is the KKT point of SIP, and the proof is finished.

3 Algorithm Framework Based on Local Reduction Method

Local reduction method originates from ref. [14], which studies how to convert SIP local reduction to optimization problem with finite constraints. Conditions for the establishment of local reduction lemma and related conclusions can be seen in [1, 15]. The essence of local reduction lemma is that, under certain conditions, the original SIP problem is locally equivalent to an implicit finite constrained programming in the optimal solution.

Note that the algorithms for SIP_q [8–11] can obtain an approximate solution of SIP. Inspired by local reduction method [1, 15], we present a two phase algorithm framework for SIP. In the first phase, we apply algorithms for SIP_q to obtain an approximate solution of SIP. Then, taking the approximate solution as a initial point, we switch to the second stage, i.e., solve the local reduction problem of SIP, which is based on the idea of local reduction. As to the iterative method solving for SIP, a sufficient condition for the local reduction is given below.

Assumption 3.1 (see [15]) Suppose that iteration point sequence $\{x^k\}_{k\in N_0}$ is yielded by some iteration method for SIP. For any iteration point $x^k,\ k \in N_0$, problem

$$(P(x^k)) \quad \max_{\omega\in\Omega} g(x^k, \omega)$$

is regular, i.e.,

(i) any critical point of problem $P(x^k)$ is non-degenerative;

(ii) LICQ is satisfied by problem $P(x^k)$ at any $\omega \in \Omega$.

Remark 2 Originally, local reduction lemma (see [1]) and the assumptions need to solve global maximum points of $P(x^k)$. However, it is difficult to solve the global solution. In practice, we would like to solve local maximum points of $P(x^k)$. Some scholars improve the assumptions of local reduction lemma, in which we need to solve the local maximum points. Assumption 3.1 implies the updated assumptions hold [15], i.e., it is a sufficient condition for the local reduction lemma. Moreover, it is also the basis of Lemma 3.1 below.

Lemma 3.1 (see [15]) Suppose that Assumption 3.1 holds. Then there exists an neighborhood $U(x^k)$ of x^k such that, for any $x \in U(x^k)$, problem SIP (1.1) is equivalent to the following local reduction problem

$$\mathrm{SIP}^l_{red}(x^k) \qquad \begin{array}{ll} \min & f(x), \\ \text{s.t.} & G^j(x) \triangleq g(x, \omega_l^j(x)) \le 0,\ j \in J_l(x^k), \\ & x \in U(x^k). \end{array} \tag{3.1}$$

The lemma above is the corollary of local reduction lemma, which is the basis of Algorithm 3.1 below.

Algorithm 3.1 A two phase algorithm framework based on local reduction method for SIP

Phase 1 (Approximate phase) Choose a proper positive integer q (depending on the length of $[a, b]$), and discretize Ω into Ω_q. For any $x \in R^n$, applying Algorithm A (see [8–11]) to solve SIP_q and obtain an approximate solution $\bar{x}_q$.

Phase 2 (see [1, 15]) (Global phase)

Step 0 $x^0 = \bar{x}_q$. Set $k = 0$;

Step 1 Solve $P(x^k)$ to obtain all local maximum points $\omega_l^j, j \in J_l(x^k)$;

Step 2 Set $i = 0, X_i^k = x^k$;

Step 3 Applying some iterative methods such as SQP to solve $\mathrm{SIP}^l_{red}(x^k)$. Suppose the i_k iterations are performed, and let initial point be X_0^k, then the inner iteration points are in turn $X_i^k, i = 1, 2, \cdots, i_k$. If $i \in \{1, 2, \cdots, i_{k-1}\}$, then local maximum points of $P(X_i^k)$ are made local correction, and yield ω^j_{l,k_i}, $j \in J_l(X_i^k)$;

Step 4 Set $x^{k+1} = X^k_{i_k}, k = k + 1$, and go back to Step 1.

Although the hypothesis of local reduction method is strong, local reduction is intrinsic method for SIP. In recent years, it is still concerned and studied, such as ref. [16].

4 Numerical Experiments

In this section, some preliminary numerical results are reported. All the numerical experiments are implemented on MATLAB 2016a on a 64-bit PC with an Intel Core i7-4790 CPU and 32GB of RAM. The tested problem `P1` from [17], and `P2` through `P3` are taken from [18], which have the following form

$$\text{minimize} \quad \max_{\omega \in \Omega} |g(x, \omega)|$$

with g and Ω as follows:

P1 : $g(x,\omega) = (1-\omega^2) - (0.5x^2 - 2x\omega),\ \Omega = [-1,1],\ x^0 = 1,\ f(x^*) = 1.$

P2 : $g(x,\omega) = \omega^2 - (x_1\omega + x_2\exp(\omega)),\ \Omega = [0,2],\ x^0 = (1,1),\ f(x^*) = 0.53825.$

P3 : $g(x,\omega) = \frac{1}{1+\omega} - (x_1\omega + x_2\exp(\omega)),\ \Omega = [-0.5,0.5],\ x^0 = (1,1),\ f(x^*) = 0.08716.$

For the record, x^0 is the initial point used the same as that of algorithms [6, 7], and $f(x^*)$ is the objective function value given in refs. [15,17,18]. Moreover, the tested problems above can be equivalent to inequality constrained SIP such as (1.1), and can be solved by our algorithm framework.

From the viewpoint of discretization method, for the closed interval $\Omega = [a,b]$ of variation ω, it can be discretized into the following set by

$$\Omega_q = \{a, a + \frac{b-a}{q}, a + \frac{2(b-a)}{q}, \dots, a + \frac{(q-1)(b-a)}{q}, b\},$$

where q reflects the discretization level of SIP (1.1).

During the test experiments, the following parameters are used for all tested problems:

$$\alpha = 0.2,\ \beta = 0.6,\ \tau = 2.4,\ \nu = 0.55,\ \gamma_0 = 3,\ \gamma_\omega = 0.5\gamma_0,\ \forall\ \omega \in \Omega;$$
$$\sigma_1 = \sigma_2 = 100,\ \eta_0 = 0.4,\ M = 5,\quad c_0 = 0.01,\quad \xi = \zeta = 0.5,\ \delta = 1\mathrm{E}-4.$$

In addition, for a given discretization level q, the stopping criterion is $\|d^k\| \le 1 \times 10^{-4}$ or $|z_k| \le 10^{-4}$, which is the same as of refs. [7, 8].

To test the validity of Algorithm 2.1, in view of the equivalence of $\lim\limits_{i\to\infty} \tau_i = 0$ and $\lim\limits_{i\to\infty} q_i = \infty$, without loss of generality, we can assume that $q_{i+1} = 2q_i$. The algorithm [8] is selected as Algorithm A, and the partial iterative results of Algorithm 2.1 for P1–P3 are reported in Table 1.

Table 1: Numerical results of Algorithm 2.1 for P1–P3

	i	q_i	Ni	Nf	Ng	$\Sigma\lvert\Omega_k\rvert$	$\lvert\bar{\Omega}\rvert$	$\lvert\Omega_*\rvert$	$\bar{z}_{q_i}$	$\bar{x}_{q_i}$	$f(\bar{x}_{q_i})$
P1	1	1	30	183	283	66	2.2	4	-7.3117e-05	-0.000000	0.00022045
	2	2	3	18	35	1	0.33	1	-3.1840e-05	0	1.00009600
	1	1	10	64	164	24	2.4	4	-3.3706e-06	(1.999999,0.000000)	0.00001016
	2	2	11	101	231	13	1.18	3	-1.1921e-05	(0.306094,0.403837)	0.40387300
P2	3	4	12	126	313	13	1.08	3	-8.3264e-07	(-0.171766,0.517761)	0.51776423
	4	8	8	49	309	16	2	4	-7.1111e-06	(0.107312,0.440563)	0.53004487
	5	16	10	101	765	11	1.1	3	-1.6021e-06	(-0.171766,0.517761)	0.53655642
	6	32	9	101	1471	11	1.22	4	-1.5011e-06	(0.107312,0.440563)	0.53769342
	7	64	6	52	867	14	2.33	7	-6.7147e-07	(0.107312,0.440563)	0.53824736
	1	1	6	22	36	14	2.33	4	-7.6310e-07	(1.154700,-1.098612)	0.00000230
	2	2	8	78	144	9	1.13	3	-2.5102e-05	(0.306094,0.403837)	0.07150425
P3	3	4	9	106	218	9	1	3	-2.9179e-06	(1.053596,-1.193414)	0.08654223
	4	8	5	42	122	6	1.2	3	-1.3682e-06	(0.107312,0.440563)	0.08653756
	5	16	5	42	202	8	1.6	4	-9.8282e-07	(-0.171766,0.517761)	0.08677081
	6	32	6	59	459	10	1.67	6	-1.5011e-06	(1.052859,-1.194162)	0.08717568

Table 2: Numerical results for problems with different discretization level q

	q	Ni	Nf	Ng	$\Sigma\|\Omega_k\|$	$\|\bar{\Omega}\|$	$\|\Omega_*\|$	z_*	$f(x^*)$
	10	9	38	280	23	2.56	4	-3.02028989e-06	0.99500910
	20	8	38	509	21	2.63	5	-5.56185121e-05	0.99891768
	50	10	41	1362	40	4	8	-1.23599387e-06	0.99980374
	100	11	49	3438	70	6.36	16	-7.32838656e-05	1.00035978
	200	14	54	8087	150	10.71	27	-3.37661879e-05	1.00013933
P1	500	10	39	12308	143	14.3	47	-3.78111079e-05	1.00019245
	1000	9	39	24419	96	10.67	76	-3.27012630e-05	1.00010035
	2000	9	39	45068	471	52.33	165	-2.16547854e-05	1.00013857
	5000	13	51	169614	3191	245.46	699	-8.15322392e-05	1.00037159
	10000	14	75	505960	7502	535.86	1887	-6.07285192e-05	1.00021315
	10	11	78	908	33	3	6	-2.66886963e-05	0.53759353
	20	13	87	1898	50	3.85	8	-2.28686508e-06	0.53790604
	50	13	81	3874	56	4.31	11	-2.87105889e-05	0.53820906
	100	14	84	7996	92	6.57	19	-7.95567085e-06	0.53821973
	200	16	92	17560	181	11.31	33	-1.74243194e-06	0.53823741
P2	500	15	88	40595	270	18	70	-2.34649150e-06	0.53825019
	1000	15	89	86316	513	34.2	138	-3.44802908e-06	0.53825539
	2000	11	54	114124	475	43.18	250	-6.40386123e-06	0.53826450
	5000	11	54	285112	1129	102.64	622	-6.11781531e-06	0.53826375
	10000	11	54	570114	2207	200.64	1238	-5.99926998e-06	0.53826343
	10	8	34	259	22	2.75	5	-7.56306662e-06	0.08705465
	20	8	34	528	26	3.25	7	-1.09723332e-05	0.08706499
	50	8	35	1241	33	4.13	11	-1.24830335e-05	0.08718970
	100	8	34	2266	54	6.75	20	-1.10952696e-05	0.08718554
	200	8	33	4115	106	13.25	36	-5.19684618e-06	0.08717465
P3	500	8	33	10233	236	29.5	84	-5.15388882e-06	0.08717517
	1000	8	36	22770	291	36.38	159	-1.68912606e-05	0.08721079
	2000	8	36	45500	570	71.25	317	-1.56011544e-05	0.08720955
	5000	8	36	113682	1402	175.25	790	-1.66461270e-05	0.08721317
	10000	9	38	268916	3919	435.44	1581	-3.94962106e-06	0.08718320

In Table 1, the column i is ith iteration; q_i indicates the discretization level at ith iteration. At ith iteration, SIP_{q_i} is solved, which is an inner loop. Assume that k iterations are executed in the inner loop. The columns Ni ($=k$) and Nf are the number of iterations and objective function evaluations, respectively; Ng is the number of constraint function $g(x,\omega)$ evaluations for a given x and ω; $\Sigma|\Omega_k|$ is the sum over all iterations of the size of Ω_k; $|\bar{\Omega}|$ means the average size of Ω_k, i.e., $|\bar{\Omega}| = \Sigma|\Omega_k|/\mathrm{Ni}$; $|\Omega_*|$ is the size of Ω_k at the end of ith iteration; $\bar{z}_{q_i}$ is the value of z_k at the end of the ith iteration. Finally, $f(\bar{x}_{q_i})$ is the objective function value at the end of ith iteration. From the column of $|\bar{\Omega}|$, we find that the average number of constraints per iteration is small, which can reduce the computational cost of Algorithm 2.1. Compared with previous numerical results, our algorithm framework based on discretization method is effective.

In addition, in view of ref. [15] reported the theory and the numerical experimentation in detail on Phase 2 of Algorithm 3.1, we just need to further illustrate the efficiency of Algorithm A in Phase 1 of Algorithm 3.1. For this purpose, we select $q = 10, 20, 50, 100, 200, 500, 1000,\ 2000, 5000, 10000$, respectively. For a given discretization level q, we apply Algorithm A (may as well take Algorithm A of [8] as an example) to solve SIP_q. The computational results are reported in Table 2.

In Table 2, the column q indicates the given discretization level. The meanings of Ni, Nf, Ng, $\Sigma|\Omega_k|$, $|\bar{\Omega}|$, $|\Omega_*|$ are similar to those above, but all of them are generated by solving SIP_q. Moreover, z_* is the value of z_k at the final iterate, and $f(x^*)$ is the objective function value at the final iterate point.

Comparing the results of Table 2 with previous numerical results [6, 7, 15, 17, 18], we find that choosing an appropriately large q can reduce calculation cost greatly, and the solution of SIP_q is usually a good approximate solution of SIP, which is exactly required in Phase 1 of Algorithm 3.1. Thus, Phase 1 of Algorithm 3.1 can be better implemented.

5 Conclusions

In this paper, based on discretization method and local reduction, we present two algorithm frameworks for SIP, and solve problem how to solve SIP by the proposed algorithm in [8–11]. The two algorithm frameworks not only improve the theory of algorithms for [8–11], but also play an important role to achieve the real solution of SIP. Finally, some preliminary numerical results are reported, which show the proposed two algorithm frameworks are effective.

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半无限规划基于离散化方法和局部约化的两个算法框架

徐庆娟[1], 简金宝[2]

(1.广西师范学院数学与统计科学学院, 广西 南宁 530001)

(2. 广西民族大学理学院, 广西 南宁 530006)

摘要: 本文研究了求解半无限规划的两个算法框架. 利用离散化方法和局部约化方法, 提出了两个求解半无限规划的算法框架. 在温和的条件下, 证明了基于离散化方法的算法框架具有弱全局收敛性. 数值试验表明所提出的算法框架是有效的.

关键词: 半无限规划; 离散化方法; 局部约化; 全局收敛性

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