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SOBOLEV INEQUALITIES FOR MOEBIUS MEASURES ON THE UNIT CIRCLE

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Abstract: In this paper, we consider Moebius probability on the unit circle. By using the method in [1] and [2], we transfer the estimates on Moebius probability onto one-dimensional diffusion, and obtain two-sided estimates on optimal Poincaré constant, logarithmic Sobolev constant and Sobolev constant for Moebius measures on the unit circle.

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1 Introduction

Let $n \geq 2$ and let S^{n-1} be the unit sphere on $\mathbb{R}^n$ equipped with geodesic distance d and the uniform probability measure μ. For $x \in \mathbb{R}^n$ with $|x| < 1$, we consider the probability measure on S^{n-1} given by

$$d\mu_x^n(y) = \frac{(1-|x|^2)^{\frac{n-1}{2}}}{(1-(x,y))^{n-1}} d\mu(y), \quad y \in S^{n-1}.$$

It is the so-called Moebius measure we are working on. In fact, this probability is the image of μ under the Moebius transformation. The factor $\dfrac{(1-|x|^2)^{\frac{n-1}{2}}}{(1-(x,y))^{n-1}}$ is known as the invariant Poisson kernel $P(x,y)$: as a function of x, it is not harmonic but satisfies the equation $\tilde{\Delta} P(\cdot, y) = 0$, where $\tilde{\Delta}$ denotes the invariant Laplacian operator (the reader is referred to [3] for further information on this measure).

Let M be a connected complete Riemannian manifold with Riemannian metric d and ∇ is the gradient on M. Let $\mathcal{M}_1(M)$ be the space of all probabilities on M. Given any $\mu \in \mathcal{M}_1(M)$, we say that

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Biography: Lei Liangzhen (1979–), female, born at Qianjiang, HuBei, associate professor, major in large deviations of stochastic processes, concentration inequalities, statistics for stochastic processes.

1. μ satisfies a Poincaré inequality with a non-negative constant C if for any smooth function $f : M \to \mathbb{R}$, there exists a constant $C \geq 0$ such that

$$\mathrm{Var}_\mu(f) = \int_M f^2 d\mu - (\int_M f d\mu)^2 \leq C \int_M |\nabla f|^2 d\mu. \tag{1.1}$$

The optimal constant above is denoted by $C_{\mathrm{P}}(\mu)$.

2. μ satisfies a logarithmic Sobolev inequality with a constant $C \geq 0$ if for any smooth function $f : M \to \mathbb{R}$ with $\mu(f^2) = 1$,

$$\mathrm{Ent}_\mu(f^2) \leq 2C \int_M |\nabla f|^2 d\mu. \tag{1.2}$$

We denote by $C_{\mathrm{LS}}(\mu)$ the optimal logarithmic Sobolev constant.

3. μ satisfies a Sobolev inequality with exponent $p \geq 1$, if there exists one positive constant C such that for any $f : M \to \mathbb{R}$ smooth enough,

$$\frac{p}{2-p}\left(\int_M f^2 d\mu - (\int |f|^p d\mu_M)^{\frac{2}{p}}\right) \leq C \int_M |\nabla f|^2 d\mu, \tag{1.3}$$

In fact, the classical Poincaré inequality corresponds to the case $p = 1$ and the logarithmic Sobolev inequality turns out to the limit case when p tends to 2 since

$$\lim_{p \to 2^-} \frac{p(\int f^2 d\mu_x^n - (\int |f|^p d\mu_x^n)^{\frac{2}{p}})}{2-p} = \mathrm{Ent}_{\mu_x^n}(f^2),$$

where

$$\mathrm{Ent}_{\mu_x^n}(f^2) := \mu_x^n(f^2 \log f^2) - \mu_x^n(f^2) \log(\mu_x^n(f^2))$$

is the relative entropy of f^2 under μ_x^n. It was proved in [4] that

$$\Phi_p(\mu_x^n, f) = \frac{p(\int f^2 d\mu_x^n - (\int |f|^p d\mu_x^n)^{\frac{2}{p}})}{2-p}$$

is increasing on p for given f.

In this paper, we consider the Poincaré inequality, logarithmic Sobolev inequality and Sobolev inequality for Moebius measures on the unit circle.

In [3], Schechtman and Schmuckenschläger proved that μ_x^n with any $|x| < 1$ has a uniform Gaussian concentration property, which is similar to the one of μ_0^n. In [5], they obtained logarithmic Sobolev and Poincaré inequalities for harmonic measures on unit sphere S^{n-1} for $n \geq 3$ and in [2] they had similar results for harmonic measures when $n = 2$. And then in [1], they obtained Sobolev inequalities for harmonic measures when $n \geq 2$.

Following the idea in [5], they obtained in [6] similar results for Moebius measures on unit sphere for $n \geq 3$. In this paper, we will work on the Moebius measures on unit circle

$$\mu_x(dy) = \frac{\sqrt{1-|x|^2}}{1-(x,y)} \mu(dy), \quad y \in S \tag{1.4}$$

with $x \in \mathbb{R}^2, |x| < 1$ and μ the uniform probability on the unit circle.

The main result of this paper is the following.

Theorem 1 Let μ_x be the Moebius measure on the unit circle. We have

a) the optimal Poincaré constant $C_{\mathrm{P}}(\mu_x)$ satisfies

$$1 \le C_{\mathrm{P}}(\mu_x) \le \frac{2\sqrt{1+|x|}}{\sqrt{1+|x|}+\sqrt{1-|x|}} \le 2;$$

b) the optimal logarithmic Sobolev constant $C_{\mathrm{LS}}(\mu_x)$ satisfies

$$\frac{\sqrt{3}}{18}\pi \log(1+\frac{\pi}{2\sqrt{1-|x|}}) \le C_{\mathrm{LS}}(\mu_x) \le 8\pi \log(1+\frac{e^2\pi}{\sqrt{1-|x|}}) + \log 4;$$

c) the optimal Sobolev constant $C_p(\mu_x)$ satisfies

$$\frac{p(1-(1+\frac{1}{2\sqrt{1-|x|}})^{\frac{p-2}{p}})}{3(2-p)} \le C_p(\mu_x) \le \frac{8\pi p}{2-p}\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{\sqrt{1-|x|}}\right)^{\frac{p-2}{p}}\right) + \log 4$$

for $1 < p < 2$.

2 Proof of the Estimate on $C_{\mathrm{P}}(\mu_x)$

We first present a crucial lemma, which combines a particular case of Lemma 1.1 in [2] and a lemma in [1].

Lemma 2.1 Define

$$\nu_a(d\theta) = \frac{\sqrt{1-a^2}}{\pi}\frac{1}{1-a\cos\theta}d\theta,\ \theta \in [0,\pi] \tag{2.1}$$

for $0 < a < 1$. We have, respectively,

(1) the corresponding Poincaré constant satisfies

$$C_{\mathrm{P}}(\nu_{|x|}) \le C_{\mathrm{P}}(\mu_x) \le \max\{C_{\mathrm{P}}(\nu_{|x|}), \frac{1}{\lambda^{DD}(\nu_{|x|})}\};$$

(2) similarly, the optimal logarithmic Sobolev constants satisfy

$$C_{\mathrm{LS}}(\nu_{|x|}) \le C_{\mathrm{LS}}(\mu_x) \le C_{\mathrm{LS}}(\nu_{|x|}) + \frac{1}{\lambda^{DD}(\nu_{|x|})},$$

here $\lambda^{DD}(\nu_{|x|})$ is defined as

$$\lambda^{DD}(\nu_{|x|}) := \inf\left\{\frac{\displaystyle\int (f')^2 d\nu_{|x|}}{\nu(f^2)} : f(0) = f(\pi) = 0, \quad f \quad \text{non constant}\right\};$$

(3) the optimal Sobolev constant satisfies

$$C_p(\nu_{|x|}) \le C_p(\mu_x) \le C_p(\nu_{|x|}) + \frac{1}{\lambda^{DD}(\nu_{|x|})}.$$

Define the diffusion operator $\mathcal{L}_a$ as

$$\mathcal{L}_a f(\theta) = f''(\theta) - \frac{a\sin\theta}{1-a\cos\theta} f'(\theta)$$

for any smooth function $f:[0,\pi]\to\mathbb{R}$. The corresponding Dirichlet form is

$$\mathcal{E}_a(f,f) = \int_0^\pi (f^{'})^2 d\nu_a = \int_0^\pi f(-\mathcal{L}_a f) d\nu_a.$$

The optimal Poincaré constant $C_{\mathrm{P}}(\nu_a) = \frac{1}{\lambda_1(\nu_a)}$, where $\lambda_1(\nu_a)$ has classic variational formula

$$\lambda_1(\nu_a) = \left\{ \frac{\mathcal{E}_a(f,f)}{\mathrm{Var}_a(f)},\ f \text{ non constant} \right\}.$$

Put $f(\theta) = 1 - a\cos\theta$. We get

$$\nu_a(f) = \nu_a(f^2) = \sqrt{1-a^2}$$

and

$$\begin{aligned}\mathcal{E}_a(f,f) &= \frac{\sqrt{1-a^2}}{\pi}\int_0^\pi \frac{a^2\sin^2\theta}{1-a\cos\theta} d\theta \\ &= \frac{\sqrt{1-a^2}}{\pi}\int_0^\pi \big(\frac{a^2-1}{1-a\cos\theta} + 1 - a\cos\theta\big) d\theta \\ &= \sqrt{1-a^2} - (1-a^2).\end{aligned}$$

So by the variational formula

$$\lambda_1(\nu_a) \le \frac{\mathcal{E}_a(f,f)}{\mathrm{Var}_a(f)} = 1.$$

Therefore we have $C_{\mathrm{P}}(\nu_a) \ge 1$. Now we work on the upper bound for $C_{\mathrm{P}}(\nu_a)$. The variational formula for $\lambda_1(\nu_a)$ by Chen in [7] could be understood as

$$C_{\mathrm{P}}(\nu_a) = \inf_{f\in\mathcal{F}} \sup_{\theta\in[0,\pi]} \left[\frac{1-a\cos\theta}{f'(\theta)} \int_\theta^\pi \frac{f(y)}{1-a\cos y} dy \right],$$

where

$$\mathcal{F} = \{ f : f' > 0\,, x\in[0,\pi];\ \nu_a(f) = 0 \}.$$

Set $\rho(\theta) = 1 - a\cos\theta - \sqrt{1-a^2}$, it is easy to see that ρ is strictly increasing on $[0,\pi]$ and

$\nu_a(\rho) = 0$. Thus

$$\begin{aligned}
\lambda_1(\nu_a)^{-1} &\le \sup_{\theta\in(0,\pi)} \frac{(1-a\cos\theta)}{a\sin\theta}\int_\theta^\pi \frac{1-a\cos y-\sqrt{1-a^2}}{1-a\cos y}dy\\
&= \sup_{\theta\in(0,\pi)} \frac{2(1-a\cos\theta)}{a\sin\theta}\left(\arctan(\cot\frac{\theta}{2}) - \arctan\left(\sqrt{\frac{1-a}{1+a}}\cot\frac{\theta}{2}\right)\right)\\
&\le \sup_{\theta\in(0,\pi)} \frac{2(1-a\cos\theta)}{a\sin\theta}\frac{(1-\sqrt{\frac{1-a}{1+a}})\cot\frac{\theta}{2}}{1+\left(\sqrt{\frac{1-a}{1+a}}\cot\frac{\theta}{2}\right)^2} \qquad (2.2)\\
&= \frac{1+a}{a}(1-\sqrt{\frac{1-a}{1+a}})\\
&= \frac{2}{1+\sqrt{\frac{1-a}{1+a}}},
\end{aligned}$$

where the first equality comes true by the fact that for $\alpha \in [0,\pi]$,

$$\int_\alpha^\pi \frac{d\theta}{1-a\cos\theta} = \frac{2}{\sqrt{1-a^2}}\arctan(\sqrt{\frac{1-a}{1+a}}\cot\frac{\alpha}{2}). \qquad (2.3)$$

And the last but second equality holds by the fact

$$\frac{2\cot(\frac{\theta}{2})}{1+\left(\sqrt{\frac{1-a}{1+a}}\cot\frac{\theta}{2}\right)^2} = \frac{(1+a)\sin\theta}{(1+a)\sin^2\frac{\theta}{2}+(1-a)\cos^2\frac{\theta}{2}} = \frac{(1+a)\sin\theta}{1-a\cos\theta}.$$

Step 1 Lower bound for $\lambda^{DD}(\nu_a)$. Choose f as $f(\theta) = \sin\theta$ for $\theta \in [0,\pi]$. Clearly, f satisfies

$$f(0) = f(\pi) = 0; f'(x)|_{0<x<\pi/2} > 0; f'(x)|_{\pi/2<x<\pi} < 0.$$

So by Theorem 1.1 in [8],

$$\begin{aligned}
\frac{1}{\lambda^{DD}(\nu_a)} &\le \sup_{x\in(0,\pi/2)} \frac{1}{\sin x}\int_0^x (1-a\cos y)dy \int_y^{\pi/2}\frac{\sin u}{1-a\cos u}du\\
&\quad \vee \sup_{x\in(\pi/2,\pi)} \frac{1}{\sin x}\int_x^\pi (1-a\cos y)dy\int_{\pi/2}^y \frac{\sin u}{1-a\cos u}du\\
&\le \sup_{x\in(0,\pi),x\ne\pi/2} \frac{1-a\cos x}{\cos x}\int_x^{\pi/2}\frac{\sin u}{1-a\cos u}du \qquad (2.4)\\
&= \sup_{x\in(0,\pi),x\ne\pi/2} \frac{1-a\cos x}{a\cos x}\ln\frac{1}{1-a\cos x}\\
&= \sup_{|t|\le a}(1-\frac{1}{t})\ln(1-t) = (1+\frac{1}{a})\ln(1+a).
\end{aligned}$$

In fact, for $0 \le a \le 1$,

$$(1+\frac{1}{a})\ln(1+a) \le \frac{2}{1+\sqrt{\frac{1-a}{1+a}}}.$$

Now, combining the upper and lower bound for $\lambda_1(\nu_a)$ as well as the lower bound for $\lambda^{DD}(\nu_a)$, we have by Lemma 2.1,

$$\frac{1}{2} \le \frac{\sqrt{1+|x|}+\sqrt{1-|x|}}{2\sqrt{1+|x|}} \le \lambda_1(\mu_x) \le 1.$$

The part a) of Theorem 1 follows.

3 Proof of Log-Sobolev Inequality

By (2.3), it is clear that the median of ν_a is $\theta_a = \arccos a$. Define

$$S^-_{\mathrm{LS}}(a,r) := \sup_{\alpha\in(0,\theta_a)} \int_0^\alpha \frac{d\theta}{1-a\cos\theta}\log\Big(1+\frac{r\pi}{\int_0^\alpha \frac{\sqrt{1-a^2}}{1-a\cos\theta}d\theta}\Big)\cdot\int_\alpha^{\theta_a}(1-a\cos\theta)d\theta$$

and

$$S^+_{\mathrm{LS}}(a,r) := \sup_{\alpha\in(\theta_a,\pi)} \int_\alpha^\pi \frac{d\theta}{1-a\cos\theta}\log\left(1+\frac{r\pi}{\int_\alpha^\pi \frac{\sqrt{1-a^2}}{1-a\cos\theta}d\theta}\right)\cdot\int_{\theta_a}^\alpha(1-a\cos\theta)d\theta$$

with $r=\frac{1}{2}$ or $r=e^2$. It is trivial to check that

$$\frac{\sin\alpha}{1-a\cos\alpha} \le \int_\alpha^\pi \frac{1}{1-a\cos\theta}d\theta \le \frac{2}{(1+a)\sin\frac{\alpha}{2}} \tag{3.1}$$

and

$$\int_\beta^\alpha \frac{1}{1-a\cos\theta}d\theta \le \frac{\pi}{\sqrt{1-a^2}} \tag{3.2}$$

for any $0<\beta<\alpha<\pi$.

Step 1 Upper bound for $S^+_{\mathrm{LS}}(a,e^2)$ and $S^-_{\mathrm{LS}}(a,e^2)$. For given $b>0$, $x\log(1+b/x)$ is increasing on $x>0$. We have by (3.1),

$$\begin{aligned} S^+_{\mathrm{LS}}(a,e^2) &\le \frac{2}{1+a}\sup_{\alpha\in(\theta_a,\pi)}\frac{1}{\sin\frac{\alpha}{2}}\log\left(1+\frac{e^2\pi\sqrt{1+a}\sin\frac{\alpha}{2}}{2\sqrt{1-a}}\right)(\alpha-a\sin\alpha)\\ &\le \frac{2}{1+a}\sup_{\alpha\in(\theta_a,\pi)}\log\left(1+\frac{e^2\pi}{\sqrt{2(1-a)}}\right)\frac{\alpha-a\sin\alpha}{\sin\frac{\alpha}{2}}\\ &\le 2\pi\log\left(1+\frac{e^2\pi}{\sqrt{1-a}}\right), \end{aligned} \tag{3.3}$$

where the last inequality is true since $\frac{2}{\pi}\le\frac{\sin x}{x}\le 1$ for any $x\in(0,\frac{\pi}{2})$.

Similarly, by (3.2) and the fact $\sin\theta_a=\sqrt{1-a^2}$, we have

$$\begin{aligned} S^-_{\mathrm{LS}}(a,e^2) &\le \sup_{\alpha\in(0,\theta_a)}(\theta_a-a\sin\theta_a)\times\frac{\pi}{\sqrt{1-a^2}}\log\left(1+e^2\sqrt{1-a^2}\right)\\ &\le \frac{\pi\theta_a}{\sin\theta_a}\log(1+e^2)\le\frac{\pi^2}{2}\log(1+e^2). \end{aligned} \tag{3.4}$$

Step 2 Lower bound for $S_{\rm LS}^+(a,\frac{1}{2})$. Choosing $\alpha=\frac{2\pi}{3}$, we have by (3.1) that

$$\int_\alpha^\pi \frac{1}{1-a\cos\theta}d\theta \ge \frac{\sqrt{3}}{2+a} \ge \frac{\sqrt{3}}{3}$$

and

$$\int_{\theta_a}^\alpha (1-a\cos\theta)d\theta \ge \int_{\pi/2}^{2\pi/3}(1-a\cos\theta)d\theta \ge \frac{\pi}{6}.$$

Therefore from the monotonicity of $x\log(1+b/x)$ for $x>0$ when $b>0$, it holds

$$S_{\rm LS}^+(a,\frac{1}{2}) \ge \frac{\sqrt{3}\pi}{18}\log(1+\frac{\sqrt{3}\pi}{2\sqrt{1-a^2}}) \ge \frac{\sqrt{3}\pi}{18}\log(1+\frac{\pi}{2\sqrt{1-a}}). \tag{3.5}$$

Barthe-Roberto's characterization for logarithmic Sobolev constants tells (see [9])

$$\max\{S_{\rm LS}^-(a,\frac{1}{2}),S_{\rm LS}^+(a,\frac{1}{2})\} \le C_{\rm LS}(\nu_a) \le 4\max\{S_{\rm LS}^-(a,e^2),S_{\rm LS}^+(a,e^2)\}. \tag{3.6}$$

Therefore, it follows from (3.3), (3.4) and (3.5) that

$$\frac{\sqrt{3}\pi}{18}\log(1+\frac{\pi}{2\sqrt{1-a}}) \le C_{\rm LS}(\nu_a) \le 8\pi\log\left(1+\frac{e^2\pi}{\sqrt{1-a}}\right).$$

By (2.4), we have

$$\frac{1}{\lambda^{DD}(\nu_a)} \le (1+\frac{1}{a})\ln(1+a) \le \log 4.$$

Thereby by Lemma 2.1, we get

$$\frac{\sqrt{3}}{18}\pi\log(1+\frac{\pi}{2\sqrt{1-a}}) \le C_{\rm LS}(\mu_x) \le 8\pi\log(1+\frac{e^2\pi}{\sqrt{1-a}})+\log 4,$$

which completes the proof of b) of Theorem 1.

4 Proof of the Estimate on $C_p(\mu_x)$.

Define

$$B_+(a,p) := \sup_{x\in(\theta_a,\pi)}\int_x^\pi \frac{1}{1-a\cos\theta}d\theta\int_{\theta_a}^x(1-a\cos\theta)d\theta \times\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{\sqrt{1-a^2}\int_x^\pi\frac{1}{1-a\cos\theta}d\theta}\right)^{\frac{p-2}{p}}\right),$$

$$B_-(a,p) := \sup_{x\in(0,\theta_a)}\int_0^x \frac{d\theta}{1-a\cos\theta}\int_x^{\theta_a}(1-a\cos\theta)d\theta \times\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{\sqrt{1-a^2}\int_0^x\frac{1}{1-a\cos\theta}}\right)^{\frac{p-2}{p}}\right).$$

It is easy to check that both $x(1-(1+\frac{C}{x})^{\frac{p-2}{p}})$ and $1-(1+x)^{(p-2)/p}$ are increasing on $\mathbb{R}_+$ for $C>0$. Recalling the estimates

$$\frac{\sin\alpha}{1-a\cos\alpha}\leq\int_\alpha^\pi\frac{1}{1-a\cos\theta}d\theta\leq\frac{2}{(1+a)\sin\frac{\alpha}{2}}\tag{4.1}$$

and

$$\int_\beta^\alpha\frac{1}{1-a\cos\theta}d\theta\leq\frac{\pi}{\sqrt{1-a^2}}\tag{4.2}$$

for any $0<\beta<\alpha<\pi$. We get

$$\begin{aligned}B_+(a,p)&\leq\sup_{x\in(\theta_a,\pi)}\frac{2x}{(1+a)\sin\frac{x}{2}}\left(1-\left(1+\frac{(p-1)^{\frac{p}{p-2}}}{\frac{\sqrt{1-a^2}}{\pi}\frac{2}{(1+a)\sin\frac{x}{2}}}\right)^{\frac{p-2}{p}}\right)\\&\leq\frac{2\pi}{1+a}\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{2}\sqrt{\frac{1+a}{1-a}}\right)^{\frac{p-2}{p}}\right)\\&\leq2\pi\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{\sqrt{1-a}}\right)^{\frac{p-2}{p}}\right),\end{aligned}\tag{4.3}$$

where the second inequality holds by $\frac{x}{\sin x}\leq\frac{\pi}{2}$, $\forall x\in(0,\pi/2)$.

Similarly, we get

$$\begin{aligned}B_-(a,p)&\leq\sup_{x\in(0,\theta_a)}\frac{\theta_a\pi}{\sqrt{1-a^2}}\left(1-\left(1+(p-1)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\right)\\&=\sup_{x\in(0,\theta_a)}\frac{\pi\theta_a}{\sin\theta_a}\left(1-\left(1+(p-1)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\right)\leq\frac{\pi^2}{2}.\end{aligned}\tag{4.4}$$

Finally, Barthe-Roberto's characterization for Sobolev constant guarantees that

$$C_p(\nu_a)\leq\frac{4p}{2-p}\max\{B_+(a,p),B_-(a,p)\}\leq\frac{8\pi p}{2-p}\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{\sqrt{1-a}}\right)^{\frac{p-2}{p}}\right).\tag{4.5}$$

Set $f=\frac{1-a\cos\theta}{\sqrt{1-a^2}}$, then

$$\nu_a(f)=1,\ \nu_a(f^2)=\frac{1}{\sqrt{1-a^2}},\ \mathcal{E}_a(f,f)=\frac{1}{\sqrt{1-a^2}}-1.$$

By Lemma 2.4 in [1], we know

$$\begin{aligned}C_p(\nu_a)&\geq\frac{p}{2-p}\frac{\nu_a(f^2)-\nu_a(f^p)^{\frac{2}{p}}}{\mathcal{E}_a(f,f)}\geq\frac{p}{3(2-p)}\frac{\nu_a(f^2)}{\mathcal{E}_a(f,f)}(1-\nu_a(f^2)^{\frac{p-2}{p}})\\&=\frac{p}{3(2-p)}\frac{1}{1-\sqrt{1-a^2}}\left(1-\left(1+\frac{1-\sqrt{1-a^2}}{\sqrt{1-a^2}}\right)^{\frac{p-2}{p}}\right)\\&\geq\frac{p}{3(2-p)}\left(1-(1+\frac{1}{2\sqrt{1-|x|}})^{\frac{p-2}{p}}\right),\end{aligned}\tag{4.6}$$

where the last inequality holds by the facts that $r(1-(1+\frac{C}{r})^{\frac{p-2}{p}})$ is increasing and $\frac{1}{1-\sqrt{1-a^2}} \geq 1$. Combining (4.5), (4.6), $\frac{1}{\lambda^{DD}(\nu_a)} \leq \log 4$ and Lemma 2.1 together, we have

$$\frac{p(1-(1+\frac{1}{2\sqrt{1-|x|}})^{\frac{p-2}{p}})}{3(2-p)} \leq C_p(\mu_x) \leq \frac{8\pi p}{2-p}\left(1-\left(1+\frac{\pi(p-1)^{\frac{p}{p-2}}}{\sqrt{1-|x|}}\right)^{\frac{p-2}{p}}\right)+\log 4,$$

which completes the proof of theorem.

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圈上Moebius测度的Sobolev不等式

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摘要: 本文考虑单位圈上的Moebius测度的问题. 利用文献[1]和[2]中的方法, 把圈上的Moebius测度的估计转化为对一维扩散过程的相关估计上, 得到了单位圈上的Moebius测度的Poincaré不等式, 对数Sobolev 不等式和Sobolev 不等式的最佳常数的双边估计.

关键词: Moebius 测度; Sobolev 不等式; Poincaré 不等式; 对数Sobolev 不等式

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