

EXISTENCE OF SQUARE-MEAN s-ASYMPTOTICALLY ω -PERIODIC SOLUTIONS TO SOME STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: This paper is concerned with the existence of square-mean s-asymptotically ω -periodic mild solutions to some stochastic differential equations in a real separable Hilbert space. By using the new theorem of square-mean s-asymptotically ω -periodicity for stochastic process and Banach fixed point theorem, we obtain the existence and uniqueness of square-mean s-asymptotically ω -periodic mild solutions to the equations. To illustrate the abstract result, a concrete example is given.

Keywords: square-mean s-asymptotically ω -periodic; mild solution; stochastic differential equation; Hilbert space

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1 Introduction

Recently, some authors studied the theory of s-asymptotically ω -periodic functions with values in Banach spaces and applications to several problems. The concept of s-asymptotically ω -periodicity was introduced by Henríquez et al. [1]. In [2], Henríquez et al. established the existence and uniqueness theorems of s-asymptotically ω -periodic mild solutions to some classes of abstract neutral functional differential equations with infinite delay. In [3], it was proved that a scalar s-asymptotically ω -periodic function is asymptotically ω -periodic. Nicola et al. [4] provided two examples which show that the above assertion in [3] is false. Since then, it attracted the attention in many publications such as [5–11] and references therein.

Stochastic differential equations attracted great interest due to their applications in many characterizing problems in physics, biology, mechanics and so on. Taniguchi et al. [12]

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studied the existence, uniqueness and asymptotic behavior of mild solutions to stochastic partial functional differential equations with finite delay. Dorogovtsev et al. [13] proved the existence of periodic solutions to a stochastic equation in a Hilbert space. Bezandry et al. [14] established the existence of square-mean S^2 -almost periodic solutions of stochastic differential equations and nonautonomous stochastic evolution equations.

Motivated by these works, the main purpose of this paper is to introduce the notion of square-mean s-asymptotically ω -periodicity for stochastic processes and apply this new concept to investigate the existence and uniqueness of square-mean s-asymptotically ω -periodic solutions to the following stochastic differential equations

$$dx(t) = Ax(t)dt + f(t, x(t))dt + g(t, x(t))dW(t), \quad t \geq 0 \quad (1.1)$$

with

$$x(0) = x_0 \quad (1.2)$$

in a real separable Hilbert space, where $A : D(A) \subset L^2(P, H) \mapsto L^2(P, H)$ is the infinitesimal generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on $L^2(P, H)$, and $\{W(t)\}_{t \geq 0}$ is a Q -Wiener process. Here f, g are appropriate functions specified later and x_0 is F_0 measurable.

To the best of our knowledge, the concept of square-mean s-asymptotically ω -periodicity, the existence and uniqueness of square-mean s-asymptotically ω -periodic mild solutions to problems (1.1) and (1.2) in Hilbert space are untreated original problem, which constitutes one of the main motivations of this paper.

The paper is organized as follows: in Section 2, we introduce the notion of square-mean s-asymptotically ω -periodic stochastic process and study some of their basic properties. In Section 3, we give some sufficient conditions for the existence and uniqueness of a square-mean s-asymptotically ω -periodic mild solution to some nonlinear stochastic differential equations in a real separable Hilbert space. In Section 4, an example is given to illustrate our main results.

2 Preliminaries and Basic Results

In this section, we give some definitions and study some of their basic properties which will be used in the sequel. As in [15–17], we assume that $(H, \|\cdot\|, \langle \cdot, \cdot \rangle)$ and $(K, \|\cdot\|_K, \langle \cdot, \cdot \rangle_K)$ are two real separable Hilbert spaces. Let (Ω, F, P) be a complete probability space. The notation $L^2(P, H)$ stands for the space of all H -valued random variables x such that

$$E\|x\|^2 = \int_{\Omega} \|x\|^2 dP < \infty.$$

For $x \in L^2(P, H)$, let

$$\|x\|_2 = \left(\int_{\Omega} \|x\|^2 dP \right)^{\frac{1}{2}}.$$

Then it is routine to check that $L^2(P, H)$ is a Banach space equipped with the norm $\|x\|_2$.

The space $L_2(K, H)$ stands for the space of all Hilbert-Schmidt operators acting from K into H , equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$. Let $K_0 = Q^{\frac{1}{2}}K$ and let $L_2^0 = L_2(K_0, H)$ with respect to the norm

$$\|\phi\|_{L_2^0}^2 = \|\phi Q^{\frac{1}{2}}\|_2^2 = \text{Tr}(\phi Q \phi^*).$$

In addition, for a symmetric nonnegative operator $Q \in L_2(P, H)$ with finite trace, we assume that $\{W(t), t \geq 0\}$ is a independent K -valued Q -Wiener process defined on $(\Omega, F, P, \{F_t\}_{t \geq 0})$.

Definition 2.1 A stochastic process $x : R \rightarrow L^2(P, H)$ is said to be continuous in the square-mean sense if

$$\lim_{t \rightarrow s} E\|x(t) - x(s)\|^2 = 0 \text{ for all } s \in R.$$

Definition 2.2 Let $x : [0, \infty) \rightarrow L^2(P, H)$ be continuous in the square-mean sense. x is said to be square-mean s -asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{t \rightarrow \infty} E\|x(t + \omega) - x(t)\|^2 = 0.$$

In this case, we say that ω is an asymptotic period of x and that x is square-mean s -asymptotically ω -periodic. The collection of all square-mean s -asymptotically ω -periodic stochastic process $x : [0, \infty) \rightarrow L^2(P, H)$ is denoted by $SA_\omega P(L^2(P, H))$.

Definition 2.3 A continuous function $f : [0, \infty) \times L^2(P, H) \rightarrow L^2(P, H)$, $(t, x) \rightarrow f(t, x)$, which is jointly continuous, is said to be square-mean uniformly s -asymptotically ω -periodic in $t \in [0, \infty)$ uniformly for all $x \in K$, where K is any bounded subset of $L^2(P, H)$, if for any $\omega > 0$, for each stochastic process $x : [0, \infty) \rightarrow K$,

$$\lim_{t \rightarrow \infty} E\|f(t + \omega, x) - f(t, x)\|^2 = 0,$$

and the set $\{E\|f(t, x)\|^2, t \geq 0, x \in K\}$ is bounded.

Theorem 2.4 $(SA_\omega P(L^2(P, H)), \|\cdot\|_\infty)$ is a Banach space with the norm given by

$$\|x\|_\infty = \sup_{t \in [0, \infty)} \|x(t)\|_2 = \sup_{t \in [0, \infty)} (E\|x(t)\|^2)^{\frac{1}{2}}.$$

Proof Let $\{x_n\} \subset SA_\omega P(L^2(P, H))$ be a Cauchy sequence with respect to $\|\cdot\|_\infty$, x_n converges to x with respect to $\|\cdot\|_2$, that is

$$\lim_{n \rightarrow \infty} \|x_n(t) - x(t)\|_2 = 0 \tag{2.1}$$

for all $t \geq 0$. So we need to prove that $x \in SA_\omega P(L^2(P, H))$.

Indeed, for $t \geq 0$, we write

$$x(t + \omega) - x(t) = x(t + \omega) - x_n(t + \omega) + x_n(t + \omega) - x_n(t) + x_n(t) - x(t),$$

Since $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for any real number a, b, c , then we get

$$\begin{aligned} E\|x(t + \omega) - x(t)\|^2 &\leq 3E\|x(t + \omega) - x_n(t + \omega)\|^2 + 3E\|x_n(t + \omega) - x_n(t)\|^2 \\ &\quad + 3E\|x_n(t) - x(t)\|^2. \end{aligned}$$

By (2.1) and stochastic continuity of x_n ,

$$\lim_{t \rightarrow \infty} E\|x(t + \omega) - x(t)\|^2 = 0.$$

That is $x(t)$ is square-mean s-asymptotically ω -periodic. The proof is completed.

Theorem 2.5 Let $f : [0, \infty) \times L^2(P, H) \rightarrow L^2(P, H)$ be square-mean uniformly s-asymptotically ω -periodic and assume that $f(t, \cdot)$ is asymptotically uniformly continuous on each bounded subset $K \subset L^2(P, H)$ uniformly for $t \in [0, \infty)$, that is for all $\epsilon > 0$, there exist $L_\epsilon \geq 0$ and $\delta_\epsilon \geq 0$, such that $E\|f(t, x) - f(t, y)\|^2 < \epsilon$ for all $t \geq L_\epsilon$ and all $x, y \in K$ with $E\|x - y\|^2 < \delta_\epsilon$. Then for any square-mean s-asymptotically ω -periodic process $x : [0, \infty) \rightarrow L^2(P, H)$, the stochastic process $F : [0, \infty) \rightarrow L^2(P, H)$ given by $F(\cdot) = f(\cdot, x(\cdot))$ is square-mean s-asymptotically ω -periodic.

Proof For $x \in SA_\omega P(L^2(P, H))$, we have

$$\lim_{t \rightarrow \infty} E\|x(t + \omega) - x(t)\|^2 = 0 \quad (2.2)$$

for $t \geq 0$. For f is square-mean uniformly s-asymptotically ω -periodic, by Definition 2.3, there exists a bounded subset $K \subset L^2(P, H)$, such that $x \in K$ for $t \geq 0$ and we get

$$\lim_{t \rightarrow \infty} E\|f(t + \omega, x) - f(t, x)\|^2 = 0 \quad (2.3)$$

for $t \geq 0$ and $x \in K$.

Note that for $t \geq 0$,

$$F(t + \omega) - F(t) = f(t + \omega, x(t + \omega)) - f(t + \omega, x(t)) + f(t + \omega, x(t)) - f(t, x(t)).$$

Since $(a + b)^2 \leq 2(a^2 + b^2)$ for any real number a, b , so we obtain

$$\begin{aligned} E\|F(t + \omega) - F(t)\|^2 &\leq 2E\|f(t + \omega, x(t + \omega)) - f(t + \omega, x(t))\|^2 \\ &\quad + 2E\|f(t + \omega, x(t)) - f(t, x(t))\|^2. \end{aligned}$$

By (2.2) and asymptotically uniform continuity of $f(t, x)$ in $x \in K$, we have

$$\lim_{t \rightarrow \infty} E\|f(t + \omega, x(t + \omega)) - f(t + \omega, x(t))\|^2 = 0. \quad (2.4)$$

By (2.3), we get

$$\lim_{t \rightarrow \infty} E\|f(t + \omega, x(t)) - f(t, x(t))\|^2 = 0. \quad (2.5)$$

For $t \geq 0$, we can deduce from (2.4) and (2.5) that, $\lim_{t \rightarrow \infty} E\|F(t + \omega) - F(t)\|^2 = 0$, which prove that $F(t)$ is square-mean s-asymptotically ω -periodic. The proof is completed.

3 Main Results and Proofs

In this section, we suppose that the following assumptions hold:

(H1) $A : D(A) \subset L^2(P, H) \rightarrow L^2(P, H)$ is the infinitesimal generator of an exponentially stable C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on $L^2(P, H)$; that is, there exists constants $M > 0$ and $\delta > 0$ such that $\|T(t)\| \leq Me^{-\delta t}$ for $t \geq 0$.

(H2) The function $f : [0, \infty) \times L^2(P, H) \rightarrow L^2(P, H)$ satisfies the following conditions:

(1) f is square-mean uniformly s -asymptotically ω -periodic and $f(\cdot, x)$ is asymptotically uniformly continuous in every bounded subset $K \subset L^2(P, H)$ uniformly for $t \in [0, \infty)$;

(2) there exists a constant $L_f > 0$ such that

$$E\|f(t, x) - f(t, y)\|^2 \leq L_f E\|x - y\|^2$$

for all $x, y \in K$ and $t \geq 0$.

(H3) The function $g : [0, \infty) \times L^2(P, H) \rightarrow L^2(P, L_2^0)$ satisfies the following conditions:

(1) g is square-mean uniformly s -asymptotically ω -periodic and $g(\cdot, x)$ is asymptotically uniformly continuous in every bounded subset $K \subset L^2(P, H)$ uniformly for $t \in [0, \infty)$;

(2) there exists a constant $L_g > 0$ such that

$$E\|g(t, x) - g(t, y)\|_{L_2^0}^2 \leq L_g E\|x - y\|^2$$

for all $x, y \in K$ and $t \geq 0$.

Definition 3.1 Assume that x_0 is F_0 measurable. An F_t -progressively measurable stochastic process $x(t)$ is called a mild solution to problems (1.1) and (1.2) if it satisfies the corresponding stochastic integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)g(s, x(s))dW(s) \quad (3.1)$$

for all $t \in [0, \infty)$.

Throughout the rest of the paper we denote by Γ_1, Γ_2 , the nonlinear integral operators defined by

$$\begin{aligned} (\Gamma_1 x)(t) &= \int_0^t T(t-s)f(s, x(s))ds, \\ (\Gamma_2 x)(t) &= \int_0^t T(t-s)g(s, x(s))dW(s). \end{aligned}$$

Lemma 3.2 Assume that (H1) and (H2) are satisfied. If $x : [0, \infty) \rightarrow L^2(P, H)$ is square-mean s -asymptotically ω -periodic, then the function $\Gamma_1 x$ is square-mean s -asymptotically ω -periodic.

Proof Let $x \in SA_\omega P(L^2(P, H))$. By (H2)(1) and Theorem 2.5, the function $s \rightarrow f(s, x(s))$ belongs to $SA_\omega P(L^2(P, H))$. Since $F(\cdot) = f(\cdot, x(\cdot)) \in SA_\omega P(L^2(P, H))$, there exists $\omega > 0$, such that

$$\lim_{t \rightarrow \infty} E\|F(t+\omega) - F(t)\|^2 = 0 \quad (3.2)$$

for $t \geq 0$. Furthermore, for each $\epsilon > 0$, there exists a positive constant L_ϵ such that $E\|F(t + \omega) - F(t)\|^2 < \epsilon$ for every $t \geq L_\epsilon$. Under these conditions, for $t \geq L_\epsilon$, we can write

$$\begin{aligned} & E\|(\Gamma_1 x)(t + \omega) - (\Gamma_1 x)(t)\|^2 \\ &= E\left\| \int_0^\omega T(t + \omega - s)F(s)ds + \int_0^{L_\epsilon} T(t - s)[F(s + \omega) - F(s)]ds \right. \\ &\quad \left. + \int_{L_\epsilon}^t T(t - s)[F(s + \omega) - F(s)]ds \right\|^2. \end{aligned}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & E\|(\Gamma_1 x)(t + \omega) - (\Gamma_1 x)(t)\|^2 \\ &\leq 3E\left\| \int_0^\omega T(t + \omega - s)F(s)ds \right\|^2 + 3E\left\| \int_0^{L_\epsilon} T(t - s)[F(s + \omega) - F(s)]ds \right\|^2 \\ &\quad + 3E\left\| \int_{L_\epsilon}^t T(t - s)[F(s + \omega) - F(s)]ds \right\|^2 \\ &\leq 3M^2E\left(\int_0^\omega e^{-\delta(t+\omega-s)}\|F(s)\|ds \right)^2 + 3M^22^2E\left(\int_0^{L_\epsilon} e^{-\delta(t-s)}\|F(s)\|ds \right)^2 \\ &\quad + 3\epsilon^2E\left(\int_{L_\epsilon}^t e^{-\delta(t-s)}ds \right)^2 \\ &\leq 3M^2e^{-2\delta t}\left(\int_0^\omega e^{-\delta(\omega-s)}ds \right)\left(\int_0^\omega e^{-\delta(\omega-s)}E\|F(s)\|^2ds \right) \\ &\quad + 12M^2e^{-2\delta(t-L_\epsilon)}\left(\int_0^{L_\epsilon} e^{-\delta(L_\epsilon-s)}ds \right)\left(\int_0^{L_\epsilon} e^{-\delta(L_\epsilon-s)}E\|F(s)\|^2ds \right) \\ &\quad + 3\epsilon^2\left(\int_0^\infty e^{-\delta s}ds \right)^2 \\ &\leq 3M^2e^{-2\delta t}\left(\int_0^\omega e^{-\delta(\omega-s)}ds \right)^2 \sup_{t \geq 0} E\|F(t)\|^2 \\ &\quad + 12M^2e^{-2\delta(t-L_\epsilon)}\left(\int_0^{L_\epsilon} e^{-\delta(L_\epsilon-s)}ds \right)^2 \sup_{t \geq 0} E\|F(t)\|^2 + 3\epsilon^2\left(\int_0^\infty e^{-\delta s}ds \right)^2 \\ &\leq 3\frac{M^2}{\delta^2}e^{-2\delta t}E\|F(t)\|^2 + 12\frac{M^2}{\delta^2}e^{-2\delta(t-L_\epsilon)} \sup_{t \geq 0} E\|F(t)\|^2 + 3\epsilon^2\frac{1}{\delta^2} \end{aligned}$$

for $t \geq 0$. For F is bounded, then we immediately obtain that

$$\lim_{t \rightarrow \infty} E\|(\Gamma_1 x)(t + \omega) - (\Gamma_1 x)(t)\|^2 = 0$$

for $t \geq 0$. Thus we conclude that $\Gamma_1 x$ is square-mean s-asymptotically ω -periodic. This completes the proof.

Lemma 3.3 Assume that (H1) and (H3) hold. If $x : [0, \infty) \rightarrow L^2(P, H)$ is square-mean s-asymptotically ω -periodic, then the function $\Gamma_2 x$ is square-mean s-asymptotically ω -periodic.

Proof Let $x \in SA_\omega P(L^2(P, H))$. By (H3) (1) and Theorem 2.5, the function $s \rightarrow g(s, x(s))$ belongs to $SA_\omega P(L^2(P, L_2^0))$. Since $G(\cdot) = g(\cdot, x(\cdot)) \in SA_\omega P(L^2(P, L_2^0))$, there exists $\omega > 0$, such that

$$\lim_{t \rightarrow \infty} E \|G(t + \omega) - G(t)\|_{L_2^0}^2 = 0 \quad (3.3)$$

for $t \geq 0$. Furthermore, for each $\epsilon > 0$, there exists a positive constant L_ϵ such that $E \|G(t + \omega) - G(t)\|_{L_2^0}^2 < \epsilon$ for every $t \geq L_\epsilon$. Let $\widetilde{W}(\sigma) = W(\sigma + \omega) - W(\omega)$ for each $\sigma \geq 0$. Note that \widetilde{W} is also a Brownian motion and has the same distribution as W . Under these conditions, we can write

$$\begin{aligned} & E \|(\Gamma_2 x)(t + \omega) - (\Gamma_2 x)(t)\|^2 \\ = & E \left\| \int_0^\omega T(t + \omega - \sigma) G(\sigma) d\widetilde{W}(\sigma) + \int_0^{L_\epsilon} T(t - \sigma) [G(\sigma + \omega) - G(\sigma)] d\widetilde{W}(\sigma) \right. \\ & \left. + \int_{L_\epsilon}^t T(t - \sigma) [G(\sigma + \omega) - G(\sigma)] d\widetilde{W}(\sigma) \right\|^2. \end{aligned}$$

By using the Cauchy-Schwarz inequality and Lemma 7.2 in [15], we have

$$\begin{aligned} & E \|(\Gamma_2 x)(t + \omega) - (\Gamma_2 x)(t)\|^2 \\ \leq & 3E \left\| \int_0^\omega T(t + \omega - \sigma) G(\sigma) d\widetilde{W}(\sigma) \right\|^2 \\ & + 3E \left\| \int_0^{L_\epsilon} T(t - \sigma) [G(\sigma + \omega) - G(\sigma)] d\widetilde{W}(\sigma) \right\|^2 \\ & + 3E \left\| \int_{L_\epsilon}^t T(t - \sigma) [G(\sigma + \omega) - G(\sigma)] d\widetilde{W}(\sigma) \right\|^2 \\ \leq & 3\text{Tr}Q \cdot \left(\int_0^\omega \|T(t + \omega - \sigma)\|^2 E \|G(\sigma)\|_{L_2^0}^2 d\sigma \right) \\ & + 3\text{Tr}Q \cdot \left(\int_0^{L_\epsilon} \|T(t - \sigma)\|^2 E \|G(\sigma + \omega) - G(\sigma)\|_{L_2^0}^2 d\sigma \right) \\ & + 3\text{Tr}Q \cdot \left(\int_{L_\epsilon}^t \|T(t - \sigma)\|^2 E \|G(\sigma + \omega) - G(\sigma)\|_{L_2^0}^2 d\sigma \right) \\ \leq & 3\text{Tr}Q \cdot M^2 e^{-2\delta t} \left(\int_0^\omega e^{-2\delta(\omega - \sigma)} d\sigma \right) \sup_{t \geq 0} E \|G(t)\|_{L_2^0}^2 \\ & + 12\text{Tr}Q \cdot M^2 e^{-2\delta(t - L_\epsilon)} \left(\int_0^{L_\epsilon} e^{-2\delta(L_\epsilon - \sigma)} d\sigma \right) \sup_{t \geq 0} E \|G(t)\|_{L_2^0}^2 \\ & + 3\text{Tr}Q \cdot \epsilon^2 \left(\int_0^\infty e^{-2\delta\sigma} d\sigma \right) \\ \leq & 3\text{Tr}Q \cdot M^2 e^{-2\delta t} \left(\int_0^\omega e^{-2\delta(\omega - \sigma)} d\sigma \right) \sup_{t \geq 0} E \|G(t)\|_{L_2^0}^2 \\ & + 12\text{Tr}Q \cdot M^2 e^{-2\delta(t - L_\epsilon)} \left(\int_0^{L_\epsilon} e^{-2\delta(L_\epsilon - \sigma)} d\sigma \right) \sup_{t \geq 0} E \|G(t)\|_{L_2^0}^2 \end{aligned}$$

$$\begin{aligned}
 & +3\text{Tr}Q \cdot \epsilon^2 \left(\int_0^\infty e^{-2\delta\sigma} d\sigma \right) \\
 \leq & 3\text{Tr}Q \cdot \left(\frac{M^2}{2\delta} e^{-2\delta t} \sup_{t \geq 0} E\|G(t)\|_{L_2^0}^2 + 2\frac{M^2}{\delta} e^{-2\delta(t-L_c)} \sup_{t \geq 0} E\|G(t)\|_{L_2^0}^2 + \epsilon^2 \frac{1}{2\delta} \right)
 \end{aligned}$$

for $t \geq 0$. For G is bounded, then we immediately obtain that

$$\lim_{t \rightarrow \infty} E\|(\Gamma_2x)(t + \omega) - (\Gamma_2x)(t)\|^2 = 0$$

for $t \geq 0$. Thus we conclude that Γ_2x is square-mean s-asymptotically ω -periodic. This completes the proof.

Theorem 3.4 Assume that assumptions (H1)–(H3) hold. Then the stochastic differential equations (1.1) and (1.2) have a unique square-mean s-asymptotically ω -periodic mild solution whenever Θ is small enough, that is $\Theta = 2\frac{M^2L_f}{\delta^2} + \text{Tr}Q \cdot \frac{M^2L_g}{\delta} < 1$.

Proof Define

$$\begin{aligned}
 (\Gamma x)(t) &= T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)g(s, x(s))dW(s) \\
 &= T(t)x_0 + (\Gamma_1x)(t) + (\Gamma_2x)(t).
 \end{aligned}$$

From previous assumptions and the properties of $\{T(t)\}_{t \geq 0}$, one can easily see that $T(t)x_0 \rightarrow 0$ as $t \rightarrow \infty$, then the function $T(t)x_0 \in SA_\omega P(L^2(P, H))$. By Lemmas 3.2 and 3.3, $\Gamma_i, i = 1, 2$ maps $SA_\omega P(L^2(P, H))$ into itself. To complete the proof, it suffices to prove that Γ has a fixed point. Clearly

$$\begin{aligned}
 & \|(\Gamma x)(t) - (\Gamma y)(t)\| = \|(\Gamma_1x)(t) - (\Gamma_1y)(t) + (\Gamma_2x)(t) - (\Gamma_2y)(t)\| \\
 \leq & \|(\Gamma_1x)(t) - (\Gamma_1y)(t)\| + \|(\Gamma_2x)(t) - (\Gamma_2y)(t)\| \\
 \leq & M \int_0^t e^{-\delta(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds + \left\| \int_0^t T(t-s)[g(s, x(s)) - g(s, y(s))]dW(s) \right\|.
 \end{aligned}$$

Since $(a + b)^2 \leq 2(a^2 + b^2)$ for any real number a, b , we get

$$\begin{aligned}
 E\|(\Gamma x)(t) - (\Gamma y)(t)\|^2 &\leq 2M^2E \left(\int_0^t e^{-\delta(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds \right)^2 \\
 &\quad + 2E \left(\left\| \int_0^t T(t-s)[g(s, x(s)) - g(s, y(s))]dW(s) \right\| \right)^2.
 \end{aligned}$$

We evaluate the first term of the right-hand side as follows:

$$\begin{aligned}
 & E \left(\int_0^t e^{-\delta(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds \right)^2 \\
 \leq & E \left[\left(\int_0^t e^{-\delta(t-s)} ds \right) \left(\int_0^t e^{-\delta(t-s)} \|f(s, x(s)) - f(s, y(s))\|^2 ds \right) \right] \\
 \leq & \left(\int_0^t e^{-\delta(t-s)} ds \right) \left(\int_0^t e^{-\delta(t-s)} E\|f(s, x(s)) - f(s, y(s))\|^2 ds \right) \\
 \leq & L_f \left(\int_0^t e^{-\delta(t-s)} ds \right) \left(\int_0^t e^{-\delta(t-s)} E\|x(s) - y(s)\|^2 ds \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq L_f \left(\int_0^t e^{-\delta(t-s)} ds \right)^2 \sup_{t \geq 0} E \|x(t) - y(t)\|^2 \\
&\leq L_f \left(\int_0^\infty e^{-\delta(t-s)} ds \right)^2 \sup_{t \geq 0} E \|x(t) - y(t)\|^2 \\
&\leq \frac{L_f}{\delta^2} \sup_{t \geq 0} E \|x(t) - y(t)\|^2.
\end{aligned}$$

As to the second term, we use again an estimate on the Ito integral established in [15] to obtain

$$\begin{aligned}
&E \left(\left\| \int_0^t T(t-s) [g(s, x(s)) - g(s, y(s))] dW(s) \right\|^2 \right) \\
&\leq E \left[\int_0^t \|T(t-s) [g(s, x(s)) - g(s, y(s))]\|^2 ds \right] \\
&\leq \text{Tr}Q \cdot E \left[\int_0^t \|T(t-s)\|^2 \|g(s, x(s)) - g(s, y(s))\|_{L_2^0}^2 ds \right] \\
&\leq \text{Tr}Q \cdot M^2 \int_0^t e^{-2\delta(t-s)} E \|g(s, x(s)) - g(s, y(s))\|_{L_2^0}^2 ds \\
&\leq \text{Tr}Q \cdot M^2 L_g \left(\int_0^t e^{-2\delta(t-s)} ds \right) \sup_{t \geq 0} E \|x(t) - y(t)\|^2 \\
&\leq \text{Tr}Q \cdot M^2 L_g \left(\int_0^\infty e^{-2\delta(t-s)} ds \right) \sup_{t \geq 0} E \|x(t) - y(t)\|^2 \\
&\leq \text{Tr}Q \cdot \frac{M^2 L_g}{2\delta} \sup_{t \geq 0} E \|x(t) - y(t)\|^2.
\end{aligned}$$

So we have

$$E \|(\Gamma x)(t) - (\Gamma y)(t)\|^2 \leq \left(2 \frac{M^2 L_f}{\delta^2} + \text{Tr}Q \cdot \frac{M^2 L_g}{\delta} \right) \sup_{t \geq 0} E \|x(t) - y(t)\|^2,$$

that is

$$\|(\Gamma x)(t) - (\Gamma y)(t)\|_2^2 \leq \Theta \sup_{t \geq 0} \|x(t) - y(t)\|_2^2. \quad (3.4)$$

Note that

$$\sup_{t \geq 0} \|x(t) - y(t)\|_2^2 \leq \left(\sup_{t \geq 0} \|x(t) - y(t)\|_2 \right)^2. \quad (3.5)$$

Hence, by (3.4) and (3.5), for $t \geq 0$, we obtain $\|(\Gamma x)(t) - (\Gamma y)(t)\|_2 \leq \sqrt{\Theta} \|x(t) - y(t)\|_\infty$.

Therefore, we get $\|(\Gamma x)(t) - (\Gamma y)(t)\|_\infty \leq \sqrt{\Theta} \|x(t) - y(t)\|_\infty$, which implies that Γ is a contraction by $\Theta < 1$. So by the Banach fixed point theorem, we conclude that there exists a unique fixed point $x(\cdot)$ for $\Gamma \in SA_\omega P(L^2(P, H))$, such that $\Gamma x = x$, that is

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)g(s, x(s))dW(s)$$

for $t \geq 0$. This completes the proof.

4 Example

To complete this work, we consider the existence and uniqueness of square-mean s-asymptotically ω -periodic solutions to the stochastic partial differential equation given by the system

$$dX(t, x) = \frac{\partial^2}{\partial x^2} X(t, x) dt + a(t)f(X(t, x))dt + b(t)g(X(t, x))dW(t), \quad t \geq 0, \quad x \in [0, \pi] \quad (4.1)$$

with

$$X(t, 0) = X(t, \pi) = 0, \quad t \geq 0 \quad (4.2)$$

and

$$X(0, x) = X_0(x), \quad x \in [0, \pi], \quad (4.3)$$

where W is a Q -Wiener process with $\text{Tr}Q < \infty$, $a(t) = \sin(\ln(t+1))$, $b(t) = \cos(\sqrt{t})$ and f, g are appropriate functions.

Take $H = L^2([0, \pi])$ equipped with its natural topology. The stochastic partial differential equation (4.1) with conditions (4.2) and (4.3) can be written as the following form

$$du(t) = (Au(t))dt + F(t, u)dt + G(t, u)dW(t), \quad t \geq 0$$

and $u(0) = u_0$, where $u(t) = X(t, x)$ and A is the operator defined by

$$Au = u'', \quad \forall u \in D(A) = H_0^1([0, \pi]) \cap H^2([0, \pi]).$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ and $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

Since $a'(t) = \frac{\cos(\ln(t+1))}{t+1}$, we have $\lim_{t \rightarrow \infty} a'(t) = 0$. For any $\omega > 0$, $a(t + \omega) - a(t) = a'(t + \theta\omega)\omega$, where $0 < \theta < 1$, which implies that $\lim_{t \rightarrow \infty} |a(t + \omega) - a(t)| = 0$. That is, a is s-asymptotically ω -periodic, for any $\omega > 0$. Similarly, we also have $\cos(\sqrt{t})$ is s-asymptotically ω -periodic.

Assume that there exist constants $l_f > 0$ and $l_g > 0$ such that

$$E\|f(x) - f(y)\|^2 \leq l_f E\|x - y\|^2, \quad E\|g(x) - g(y)\|_{L_2^2}^2 \leq l_g E\|x - y\|^2$$

for all $x, y \in L^2(P, H)$.

Therefore, by Theorem 3.4, the stochastic partial differential equation (4.1) with conditions (4.2) and (4.3) has a unique square-mean s-asymptotically ω -periodic solution whenever l_f and l_g are small enough.

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一类随机微分方程均方 s 渐进 ω 周期解的存在性

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摘要: 本文研究了在可分的实Hilbert空间中一类随机微分方程均方 s 渐进 ω 周期温和解的存在性问题. 利用均方 s 渐进 ω 周期随机过程理论及Banach不动点定理, 获得了此类方程均方 s 渐进 ω 周期温和解的存在及唯一性结果. 最后给出相关例子来验证理论结果.

关键词: 均方 s 渐进 ω 周期; 温和解; 随机微分方程; Hilbert 空间

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