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WEIGHTED INEQUALITIES FOR MAXIMAL OPERATOR IN ORLICZ MARTINGALE CLASSES

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Abstract: In this paper, we study weighted inequalities for maximal operators in Orlicz martingale classes. By using the properties of weights, we obtain that inequalities of $a(\cdot)$ and $b(\cdot)$ imply weighted inequalities involving maximal operator in uniformly integral martingale classes. The converse is also considered, which extends the theory of class $L\log L$.

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1 Introduction

As is well known, for the real-valued local integrable functions f on $\mathbb{R}^n$, the classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|dy,$$

where Q is a non-degenerate cube with its sides paralleled to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

Suppose that $\Phi:[0,\infty)\to\mathbb{R}$ is nondecreasing and continuous with $\Phi(0)=0$ and $\lim\limits_{t\to+\infty}\Phi(t)=\infty$. Let

$$L^{\Phi}=\Big\{f:\int_{R^n}\Phi(\varepsilon|f|)dx<\infty \ \text{ for some } \varepsilon>0\Big\}.$$

Then L^{Φ} is called an Orlicz space. In Orlicz spaces, Kokilashvili and Krbec [1, Theorem 1.2.1] stated that the following statements are equivalent:

(a) there exists a positive constant C such that

$$\int_{\mathbb{R}^n}\Phi(Mf)dx\le C\int_{\mathbb{R}^n}\Phi(C|f|)dx,\ \ \forall f\in L^1_{\text{loc}}; \tag{1.1}$$

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(b) the function Φ^{α} is quasi-convex for some $\alpha \in (0,1)$.

When $\Phi(t) = \frac{t^p}{p}$, $p > 1$, L^{Φ} is the space L^p. The above result implies that M is bounded on L^p for $1 < p < \infty$ (see Grafakos [2, Theorem 2.1.6]) and unbounded on L^1(see Grafakos [2, Example 2.1.2]). In fact, M maps L^1 to $L^{1,\infty}$ (see Grafakos [2, Theorem 2.1.6]).

Let $\mathbb{T}$ be the group of real numbers modulo 2π and $f(x)$ a real-valued integrable function denoted on $\mathbb{T}$ with period 2π. The Hardy-Littlewood maximal function $M_{\mathbb{T}}f(x)$ is defined by $M_{\mathbb{T}}f(x) = \sup\limits_{x\in I} \frac{1}{|I|}\displaystyle\int_I |f(y)|dy$, where the supremum taken over all open $I \subseteq \mathbb{T}$ with $x \in I$.

Let $a(s)$ and $b(s)$ be two positive nondecreasing continuous functions defined on $[0,\infty)$ satisfying

$$\int_0^1 \frac{a(s)}{s}ds = K < +\infty, \tag{1.2}$$

$$\int_1^{\infty} \frac{a(s)}{s}ds = +\infty \tag{1.3}$$

and

$$\lim_{s\to\infty} b(s) = +\infty. \tag{1.4}$$

Then, we define

$$\Phi(t) = \int_0^t a(s)ds, \quad \Psi(t) = \int_0^t b(s)ds, \ \forall t \geq 0. \tag{1.5}$$

Under hypotheses (1.3)–(1.5), a series of inequalities involving $M_{\mathbb{T}}f$ and f were obtained. Kita [3, Theorem 2.1] gave necessary and sufficient conditions for $a(\cdot)$ and $b(\cdot)$ in order to guarantee that $M_{\mathbb{T}}f$ is in L^{Φ} whenever f is in L^{Ψ}. Conversely, Kita [4, Theorem 2.1] stated necessary and sufficient conditions for $a(\cdot)$ and $b(\cdot)$ in order to guarantee that a function f is in L^{Ψ} whenever the function $M_{\mathbb{T}}f$ is in L^{Φ}, which is a generalization of [5, Theorem 5.4] to functions in an Orlicz space L^{Ψ} and also a generalization of the result of [6, Theorem 3.1]. These are related to the class of $L\log L$. The class was introduced by Zygmund to give a sufficient condition on the local integrability of the Hardy-Littlewood maximal operator. The necessity of this condition was observed by Stein [7].

Under the additional hypothesis (1.2), Kita [8, Theorem 2.3, Theorem 2.5] extended the above results [3, Theorem 2.1] and [4, Theorem 2.1] to the cases on $\mathbb{R}^n$. Moreover, Kita [9, 10] obtained the related weighted theory on $\mathbb{R}^n$. Specially, Kita [10] also dealt with the iterated maximal operator.

Comparing with the results without weights on $\mathbb{R}^n$ and $\mathbb{T}$, Mei and Liu [11, 12] considered the martingale cases. In our paper, we prove weighted inequalities for maximal operators in Orlicz martingale classes. Let us introduce some preliminaries.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, and $(\mathcal{F}_n)_{n\geq 0}$ be an increasing sequence of sub-$\mathcal{F}$-fields of $\mathcal{F}$ with $\mathcal{F} = \bigvee \mathcal{F}_n$. If $f = (f_n)_{n\geq 0}$ is a martingale adapted to $(\mathcal{F}_n)_{n\geq 0}$, the maximal function (operator) Mf for martingale $f = (f_n)$ is defined by $Mf = \sup\limits_{n\geq 0}|f_n|$. In this paper, the limit of f_n is also denoted by f. Specifically, $\big(\Omega, (\mathcal{F}_n)_{n\geq 0}, \mathcal{F}, \nu\big)$ is said to satisfy the R-condition (or simply $\nu \in R$), if for all $n \geq 1$, $F_n \in \mathcal{F}_n$, there exists a $G_n \in \mathcal{F}_{n-1}$ such that $F_n \subset G_n$ and $\nu(G_n) \leq d \cdot \nu(F_n)$. In our paper, we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

A weight ω is a positive random variable with $E\omega < \infty$. For convenience, we assume $E\omega = 1$ in this paper. In addition, we say $\omega \in A_1$ (or B_1), if $\omega^{-1} \in L^\infty(d\mu)$ and there exists a constant $C \geq 1$ such that $\omega_n \leq C\omega$ (or $C\omega_n \geq \omega$). Moreover, for the weights as above, a function $f \in L^1(\omega d\mu)$ implies $f \in L^1(d\mu)$.

Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ satisfy (1.2)–(1.5). It is clear that Φ and Ψ are convex. Recall that $L^\Phi(d\mu) \cup L^\Psi(d\mu) \subset L^1(d\mu)$ when $\mu(\Omega) < \infty$.

Then we have the following Theorems 1.1 and 1.2, which are martingale versions of [8, Theorem 2.3]. Kita [8, Theorem 2.3] gave an assumption that the function $\frac{a(s)}{s}$ is bounded in a neighborhood of zero. However, we do not need the assumption in our Theorems 1.1 and 1.2.

Theorem 1.1 Let $\omega \in A_1$. Suppose that there exists a positive constant C_1 such that

$$\int_1^s \frac{a(t)}{t} dt \leq C_1 b(C_1 s), \ s \geq 1. \tag{1.6}$$

Then there exist positive constants C_2 and C_3 such that

$$\int_\Omega \Phi(Mf)\omega d\mu \leq C_2 \|f\|_{L^1(\omega d\mu)} + C_2 \int_\Omega \Psi(C_3|f|)\omega d\mu, \tag{1.7}$$

where C_2 and C_3 are constants independent of $f = (f_n)$.

Theorem 1.2 Let $(\Omega, \mathcal{F}, \omega)$ be non-atomic and $\omega \in R \cap B_1$. Suppose that $f \in L^\Psi(\omega d\mu)$ implies $Mf \in L^\Phi(\omega d\mu)$. Then (1.6) holds.

Following from Theorems 1.1 and 1.2, we easily have Corollaries 1.3 and 1.4, respectively.

Corollary 1.3 Let $\omega \in A_1$. Suppose that (1.6) is valid, then $Mf \in L^\Phi(\omega d\mu)$ for all $f \in L^\Psi(\omega d\mu)$.

Corollary 1.4 Let $(\Omega, \mathcal{F}, \omega)$ be non-atomic and $\omega \in R \cap B_1$. Suppose that (1.7) is valid, then we have (1.6).

We consider the converse inequality of maximal functions. Theorems 1.5 and 1.6 are martingale versions of [8, Theorem 2.5] and involve weights.

Theorem 1.5 Let $\omega \in B_1 \cap R$. Suppose that there exist positive constants C_6 and $s_0 > 1$ such that

$$\int_1^s \frac{a(t)}{t} dt \geq C_6 b(C_6 s), \ s \geq s_0. \tag{1.8}$$

Then there exist positive constants C_7 and C_8 such that

$$\int_\Omega \Psi(C_7|f|)\omega d\mu \leq C_8 \|f\|_{L^1(\omega d\mu)} + C_8 \int_\Omega \Phi(M(|f|))\omega d\mu, \tag{1.9}$$

where C_7 and C_8 are constants independent of $f = (f_n)$ and $\|f\|_{L^1(\omega d\mu)} \leq 1$.

Theorem 1.6 Let $(\Omega, \mathcal{F}, \omega)$ be non-atomic and $\omega \in A_1$. Suppose that (1.9) is valid. Then (1.8) holds.

We easily obtain Corollary 1.7 by Theorems 1.5. In addition, following the proof of Theorem 1.6, we have Corollary 1.8.

Corollary 1.7 Let $\omega \in B_1 \cap R$. Suppose that (1.8) is valid. Then $f \in L^{\Psi}(\omega d\mu)$ for all f satisfying $Mf \in L^{\Phi}(\omega d\mu)$.

Corollary 1.8 Let $(\Omega, \mathcal{F}, \omega)$ be non-atomic and $\omega \in A_1$. Suppose that $Mf \in L^{\Phi}(\omega d\mu)$ implies $f \in L^{\Psi}(\omega d\mu)$. Then (1.8) holds.

2 Proofs of Theorems

First, we give some lemmas which will be used in our proofs several times.

Lemma 2.1 Suppose that ν is an finite measure on $(\Omega, \mathcal{F})$, then for each function $f \in L^1(d\nu)$, we have

$$\int_{\{|f|>t\}} |f| d\nu = t \cdot \nu\{|f| > t\} + \int_t^{\infty} \nu\{|f| > s\} ds, \ t > 0. \tag{2.1}$$

Proof The proof follows in the same way as the proof of [9, Lemma 3.1].

Lemma 2.2 Let $\omega \in A_1$. Then there exists a positive constant C_0 such that

$$\omega\{Mf > \lambda\} \leq \frac{C_0}{\lambda} \int_{\{|f|>\frac{\lambda}{2}\}} |f| \omega d\mu \tag{2.2}$$

for all $\lambda > 0$ and $f \in L^1(\omega d\mu)$.

Proof By the assumption that $\omega \in A_1$, there exists a positive constant C such that

$$\omega\{Mg > \lambda\} \leq \frac{C}{\lambda} \|g\|_{L^1(\omega)}, \ \ g \in L^1(\omega)$$

in view of [13, Theorem 6.6.2]. For $\lambda > 0$, we set $f_\lambda = f \cdot \chi_{\{|f| \leq \frac{\lambda}{2}\}}$ and $f^\lambda = f - f_\lambda$. Then $f^\lambda \in L^1(\omega)$ and $Mf_\lambda = \sup\limits_{n \geq 0} |E_n(f_\lambda)| \leq \sup\limits_{n \geq 0} E_n(|f_\lambda|) \leq \frac{\lambda}{2}$. It follows that

$$\omega\{Mf > \lambda\} \leq \omega\{Mf_\lambda > \frac{\lambda}{2}\} + \omega\{Mf^\lambda > \frac{\lambda}{2}\} \leq \frac{C_0}{\lambda} \int_{\{|f|>\frac{\lambda}{2}\}} |f| \omega d\mu,$$

where $C_0 = 2C$.

Lemma 2.3 Let $\omega \in A_1$. Then there exists a positive constant C such that

$$\omega\{Mf > t\} \leq \frac{C}{t} \int_{\frac{t}{2}}^{\infty} \omega\{|f| > s\} ds \tag{2.3}$$

for all $t > 0$ and $f \in L^{\Psi}(\omega)$.

Proof Fix $f \in L^{\Psi}(\omega d\mu)$. For $t > 0$, set $f_t = (|f| \wedge \frac{t}{2}) \cdot \operatorname{sign} f$ and $f^t = f - f_t$. Trivially, we have $f^t \in L^1(\omega)$. By Lemma 2.2, we also have

$$\omega\{Mf^t > \lambda\} \leq \frac{C_0}{\lambda} \int_{\{|f^t|>\frac{\lambda}{2}\}} |f^t| \omega d\mu. \tag{2.4}$$

Let $\lambda = \frac{t}{2}$ in (2.4). It follows that

$$\omega\{Mf^t > \frac{t}{2}\} \le \frac{2C_0}{t}\int_\Omega |f^t|\omega d\mu. \tag{2.5}$$

Hence,

$$\begin{aligned}
\omega\{Mf > t\} &\le \omega\{Mf^t > \frac{t}{2}\} + \omega\{Mf_t > \frac{t}{2}\} = \omega\{Mf^t > \frac{t}{2}\} \\
&\le \frac{2C_0}{t}\int_\Omega |f^t|\omega d\mu = \frac{2C_0}{t}\int_{\{|f|>\frac{t}{2}\}} |f - \frac{t}{2}\cdot \operatorname{sign} f|\omega d\mu \\
&= \frac{2C_0}{t}\int_{\{|f|>\frac{t}{2}\}} \Big||f| - \frac{t}{2}\Big|\omega d\mu = \frac{2C_0}{t}\int_{\{|f|>\frac{t}{2}\}} (|f| - \frac{t}{2})\omega d\mu.
\end{aligned}$$

Combining with (2.1), we have (2.3) is valid with $C = 2C_0$.

Proof of Theorem 1.1 For $C_3 > 0$ (which will be determined later), fix $f \in L^\Psi(\omega d\mu)$ such that $\int_\Omega \Psi(C_3|f|)\omega d\mu < \infty$. With the constant C in Lemma 2.3, it follows from (1.2), (1.3) and (1.6) that

$$\begin{aligned}
\int_\Omega \Phi(Mf)\omega d\mu &= \int_0^\infty \omega\{Mf > t\}a(t)dt \\
&\le C\int_0^\infty \frac{a(t)}{t}\Big(\int_{\frac{t}{2}}^\infty \omega\{|f| > s\}ds\Big)dt \\
&= C\int_0^\infty \omega\{|f| > s\}\Big(\int_0^{2s} \frac{a(t)}{t}dt\Big)ds \\
&= C\int_0^\infty \omega\{|f| > s\}\Big(\int_0^1 \frac{a(t)}{t}dt + \int_1^{2s} \frac{a(t)}{t}dt\Big)ds \\
&\le KC\|f\|_{L^1(\omega d\mu)} + CC_1\int_0^\infty \omega\{|f| > s\}b(2C_1 s)ds \\
&= KC\|f\|_{L^1(\omega d\mu)} + \frac{C}{2}\int_\Omega \Psi(2C_1|f|)\omega d\mu.
\end{aligned}$$

Let $C_2 = (KC) \vee \frac{C}{2}$ and $C_3 = 2C_1$. The we have (1.7).

In order to give the proof of Theorem 1.2, we need the following lemmas.

Lemma 2.4 Let $\hat{E}_n(\cdot)$ be the conditional expectation with respect to $(\Omega, (\mathcal{F}_n)_{n\ge 0}, \mathcal{F}, \omega d\mu)$ and $\hat{M}(\cdot) = \sup_{n\ge 0}\hat{E}_n(\cdot)$. Suppose $\omega \in R$. Then

$$\int_{\{|f|>\lambda\}} |f|\omega d\mu \le d\lambda \cdot \omega\{\hat{M}(|f|) > \lambda\},\ f \in L^1(\omega d\mu),\ \lambda \ge \|f\|_{L^1(\omega d\mu)}. \tag{2.6}$$

Proof For $f \in L^1(\omega d\mu)$, the sequence $\big(\hat{E}_n(|f|)\big)_{n\ge 0}$ is a uniformly integral martingale with respect to $(\Omega, (\mathcal{F}_n)_{n\ge 0}, \mathcal{F}, \omega d\mu)$. In view of [13, Theorem 1.3.2.9], we have $\lim_{n\to\infty}\hat{E}_n(|f|) = |f|$ a.e.. Then it follows that $|f| \le \hat{M}(|f|)$.

Because of $\omega \in R$, we have

$$\int_{\{\hat{M}f>\lambda\}} |f|\omega d\mu \leq d\lambda \cdot \omega\{\hat{M}f > \lambda\},\ f \in L^1(\omega d\mu),\ \lambda \geq \|f\|_{L^1(\omega d\mu)},$$

where we use [13, Theorem 7.1.2]. Combining this with $|f| \leq \hat{M}(|f|)$, we have

$$\int_{\{|f|>\lambda\}} |f|\omega d\mu \leq \int_{\{\hat{M}f>\lambda\}} |f|\omega d\mu \leq d\lambda \cdot \omega\{\hat{M}(|f|) > \lambda\},\ f \in L^1(\omega d\mu),\ \lambda \geq \|f\|_{L^1(\omega d\mu)}.$$

Lemma 2.5 Let ω be a weight. Then

$$\hat{E}(f \cdot \omega^{-1}|\mathcal{F}_n) \cdot E(\omega|\mathcal{F}_n) = E(f|\mathcal{F}_n),\ f \in L^1(d\mu).$$

Proof For all $A \in \mathcal{F}_n$, we have

$$\begin{aligned} \int_A f d\mu &= \int_A f \cdot \omega^{-1}\omega d\mu = \int_A \hat{E}(f \cdot \omega^{-1}|\mathcal{F}_n) \cdot \omega d\mu \\ &= \int_A \hat{E}(f \cdot \omega^{-1}|\mathcal{F}_n) \cdot E(\omega|\mathcal{F}_n) d\mu. \end{aligned}$$

It follows that $\hat{E}(f \cdot \omega^{-1}|\mathcal{F}_n) \cdot E(\omega|\mathcal{F}_n) = E(f|\mathcal{F}_n)$.

Reforming Lemma 2.5, we have

$$\hat{E}(g|\mathcal{F}_n) \cdot E(\omega|\mathcal{F}_n) = E(g\omega|\mathcal{F}_n),\ g \in L^1(\omega d\mu). \tag{2.7}$$

Lemma 2.6 Suppose $\omega \in B_1 \cap R$. Then there exist constants C_4 and C_5 such that

$$\int_{\{|f|>C_4\lambda\}} |f|\omega d\mu \leq C_5\lambda \cdot \omega\{M(|f|) > \lambda\},\ \ f \in L^1(\omega d\mu),\ \lambda \geq \frac{1}{C}\|f\|_{L^1(\omega d\mu)}, \tag{2.8}$$

where the constant C is the one in the definition of B_1.

Proof Fix $f \in L^1(\omega d\mu)$ and $\lambda \geq \frac{1}{C}\|f\|_{L^1(\omega d\mu)}$. Since $\omega \in B_1$, there exists a constant $C \geq 1$ such that $C \cdot \omega_n \geq \omega$. Combining this with (2.7), we have

$$\hat{E}_n(|f|) = \omega_n^{-1}E_n(|f|\omega) = E_n(|f|\omega_n^{-1}\omega) \leq E_n(|f|\omega_n^{-1}\omega) \leq CE_n(|f|).$$

Thus

$$\hat{M}(|f|) \leq CM(|f|). \tag{2.9}$$

Since $C\lambda \geq \|f\|_{L^1(\omega d\mu)}$, we have

$$\int_{\{|f|>C\lambda\}} |f|\omega d\mu \leq Cd \cdot \lambda \cdot \omega\{CM(|f|) > C\lambda\} \tag{2.10}$$

in view of Lemma 2.4. Thus, (2.8) is valid with $C_4 = C$ and $C_5 = Cd$.

Proof of Theorem 1.2 Assume for contradiction that (1.6) does not hold. Then there exists a sequence of positive numbers $s_k \geq 1$ such that

$$\int_1^{s_k} \frac{a(t)}{t} dt > 2^k b(2^k s_k),\ k \geq 1. \tag{2.11}$$

Set $\alpha_k = \frac{1}{\alpha 2^k \varPsi(2^k s_k)}$. It is clear that we can choose α big enough to make $\sum\limits_{k=1}^{\infty} \alpha_k \leq 1$ and $\frac{1}{\alpha} \leq \varPsi(1)$. By the assumption that $(\Omega, \mathcal{F}, \omega)$ is non-atomic, we have a family of measurable sets $Q_k \in \mathcal{F}$ such that

$$\omega(Q_k) = \alpha_k \text{ and } Q_{k_1} \cap Q_{k_2} = \emptyset,\ k_1 \neq k_2. \tag{2.12}$$

Set $f = \sum\limits_{k=1}^{\infty} 2^k s_k \cdot \chi_{Q_k}$, then

$$\begin{aligned} \int_\Omega \varPsi(|f|)\omega d\mu &= \sum_{k=1}^{\infty} \int_{Q_k} \varPsi(|f|)\omega d\mu \\ &= \sum_{k=1}^{\infty} \varPsi(2^k s_k)\omega(Q_k) \\ &= \sum_{k=1}^{\infty} \varPsi(2^k s_k) \frac{1}{\alpha 2^k \varPsi(2^k s_k)} \\ &= \frac{1}{\alpha} \leq \varPsi(1). \end{aligned}$$

Thus $f \in L^{\varPsi}(\omega d\mu)$. By Jensen's inequality, we have $f \in L^1(\omega d\mu)$ and $\|f\|_{L^1(\omega d\mu)} \leq 1$.

On the other side, we claim that the function Mf is not in $L^{\varPhi}(\omega d\mu)$. Then we get a contradiction. To show the claim in the following way: fix a constant $\varepsilon \in (0, 1]$ for εf, with C, C_4 and C_5 in Lemma 2.6, we have

$$\frac{\varepsilon}{\lambda} \int_{\{\varepsilon f > C_4 \lambda\}} f \omega d\mu \leq C_5 \cdot \omega\{\varepsilon Mf > \lambda\},\ \lambda \geq \frac{\varepsilon}{C}\|f\|_{L^1(\omega d\mu)}. \tag{2.13}$$

Note that $1 \geq \frac{\varepsilon}{C}\|f\|_{L^1(\omega d\mu)}$, we have

$$\begin{aligned} \int_\Omega \varPhi(\varepsilon \cdot Mf)\omega d\mu &= \int_0^\infty \omega\{\varepsilon \cdot Mf > \lambda\} a(\lambda) d\lambda \\ &\geq \int_1^\infty \omega\{\varepsilon \cdot Mf > \lambda\} a(\lambda) d\lambda \\ &\geq \int_1^\infty \Big(\frac{\varepsilon}{C_5 \lambda} \int_{\{\varepsilon \cdot |f| > C_4 \lambda\}} |f| \omega d\mu\Big) a(\lambda) d\lambda \\ &= \frac{\varepsilon}{C_5} \int_\Omega |f| \Big(\int_1^{\frac{\varepsilon \cdot |f|}{C_4}} \frac{a(\lambda)}{\lambda} d\lambda\Big) \omega d\mu \\ &\geq \frac{\varepsilon}{C_5} \sum_{k=1}^{\infty} \int_{Q_k} 2^k s_k \Big(\int_1^{\frac{\varepsilon \cdot 2^k s_k}{C_4}} \frac{a(\lambda)}{\lambda} d\lambda\Big) \omega d\mu \end{aligned}$$

For the above ε, we can choose $k(\varepsilon)$ such that $\frac{\varepsilon \cdot 2^k}{C_4} > 1,\ k \geq k(\varepsilon)$. Put

$$L = \frac{\varepsilon}{C_5} \sum_{k=1}^{k(\varepsilon)-1} \int_{Q_k} 2^k s_k \Big(\int_1^{\frac{\varepsilon \cdot 2^k s_k}{C_4}} \frac{a(\lambda)}{\lambda} d\lambda\Big) \omega d\mu,$$

then L is a real number. Therefore, it follows from (2.11) and (2.12) that

$$\begin{aligned}
&\int_\Omega \varPhi(\varepsilon\cdot Mf)\omega d\mu \ge L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}\int_{Q_k}2^ks_k\Big(\int_1^{\frac{\varepsilon\cdot 2^ks_k}{C_4}}\frac{a(\lambda)}{\lambda}d\lambda\Big)\omega d\mu\\
\ge{}& L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}\int_{Q_k}2^ks_k\Big(\int_1^{s_k}\frac{a(\lambda)}{\lambda}d\lambda\Big)\omega d\mu \ge L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}\int_{Q_k}2^ks_k\big(2^kb(2^ks_k)\big)\omega d\mu\\
={}& L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}2^k2^ks_kb(2^ks_k)\omega(Q_k)=L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}2^k2^ks_kb(2^ks_k)\frac{1}{\alpha 2^k\varPsi(2^ks_k)}\\
\ge{}& L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}2^k2^ks_kb(2^ks_k)\frac{1}{\alpha 2^k(2^ks_k)b(2^ks_k)}=L+\frac{\varepsilon}{C_5}\sum_{k=k(\varepsilon)}^{\infty}\frac{1}{\alpha}=\infty.
\end{aligned}$$

The claim is proved and the proof is finished.

Proof of Theorem 1.5 For $C_7=\frac{C_6}{C}$, fix $f\in L^{\varPhi}(\omega d\mu)$ such that $\int_\Omega \varPhi(M(|f|))\omega d\mu<\infty$ and $\|f\|_{L^1(\omega d\mu)}\le 1$, where the constant C is the one in the definition of B_1. Then

$$\begin{aligned}
\int_\Omega \varPsi(\frac{C_6}{C}|f|)\omega d\mu &= \frac{C_6}{C}\int_0^\infty \omega\{|f|>t\}b(\frac{C_6}{C}t)dt\\
&= C_6\int_0^\infty \omega\{|\frac{f}{C}|>t\}b(C_6t)dt\\
&= C_6\int_0^{s_0} \omega\{|\frac{f}{C}|>t\}b(C_6t)dt+C_6\int_{s_0}^\infty \omega\{|\frac{f}{C}|>t\}b(C_6t)dt.
\end{aligned}$$

Denote

$$I_0=C_6\int_0^{s_0}\omega\{|\frac{f}{C}|>t\}b(C_6t)dt \text{ and } I_1=C_6\int_{s_0}^\infty\omega\{\frac{f}{C}|>t\}b(C_6t)dt.$$

Then, for I_0, we have

$$\begin{aligned}
I_0 &\le C_6b(C_6s_0)\int_0^{s_0}\omega\{|\frac{f}{C}|>t\}dt\le C_6b(C_6s_0)\int_0^{\infty}\omega\{|\frac{f}{C}|>t\}dt\\
&= \frac{1}{C}\cdot C_6b(C_6s_0)\|f\|_{L^1(\omega d\mu)}.
\end{aligned}$$

It follows from (1.8) that

$$\begin{aligned}
I_1 &\le \int_{s_0}^\infty \omega\{|\frac{f}{C}|>t\}\Big(\int_1^t\frac{a(s)}{s}ds\Big)dt\le\int_1^\infty \omega\{|\frac{f}{C}|>t\}\Big(\int_1^t\frac{a(s)}{s}ds\Big)dt\\
&= \int_1^\infty\frac{a(s)}{s}\Big(\int_s^\infty\omega\{|\frac{f}{C}|>t\}dt\Big)ds.
\end{aligned}$$

It follows from (2.1), (2.6) and (2.9) that

$$\begin{aligned} I_1 &\le \int_1^\infty \frac{a(s)}{s}\Big(\int_{\{|\frac{f}{C}|>s\}}|\frac{f}{C}|\omega d\mu\Big)ds \le \int_1^\infty \frac{a(s)}{s}\Big(d\cdot s\cdot\omega\{\hat{M}(|\frac{f}{C}|)>s\}\Big)ds \\ &= d\cdot\int_1^\infty a(s)\cdot\omega\{\hat{M}(|\frac{f}{C}|)>s\}ds \le d\cdot\int_1^\infty a(s)\cdot\omega\{C\cdot M(\frac{|f|}{C})>s\}ds \\ &\le d\cdot\int_\Omega \varPhi(M(|f|))\omega d\mu, \end{aligned}$$

where we have used $\|\frac{f}{C}\|_{L^1(\omega d\mu)} \le \|f\|_{L^1(\omega d\mu)} \le 1$. Therefore, (1.9) is valid with $C_7 = \frac{C_6}{C}$ and $C_8 = \frac{C_6 b(C_6 s_0)}{C} \vee d$.

Proof of Theorem 1.6 We proceed by contradiction and assume that (1.8) dose not hold. Then we get a sequence $s_k > 1$ satisfying

$$\frac{4}{3} < s_1 < s_2 < \cdots < s_k < s_{k+1} < \cdots \text{ and } \lim_{k\to\infty} s_k = \infty; \tag{2.14}$$

$$\int_1^{s_k} \frac{a(t)}{t}dt < \frac{1}{2^k}b(\frac{1}{2^k}s_k),\ k\ge 1; \tag{2.15}$$

$$s_{k+1} > 4s_k,\ k \ge 1; \tag{2.16}$$

$$b(\frac{s_k}{2^k}) \ge k2^k,\ k\ge 1. \tag{2.17}$$

Let $\alpha_k = \frac{1}{\frac{s_k}{2^k}b(\frac{s_k}{2^k})}$, then

$$\sum_{k=1}^\infty \alpha_k = \sum_{k=1}^\infty \frac{1}{\frac{s_k}{2^k}b(\frac{s_k}{2^k})} \le \sum_{k=1}^\infty \frac{1}{\frac{s_k}{2^k}k2^k} = \sum_{k=1}^\infty \frac{1}{ks_k} \le \sum_{k=1}^\infty \frac{1}{s_k} \le \frac{1}{s_1}\sum_{k=1}^\infty \frac{1}{4^{k-1}} \le 1.$$

By the assumption that $(\Omega, \mathcal{F}, \omega)$ is non-atomic, we have a family of measurable sets $Q_k \in \mathcal{F}$ such that

$$\omega(Q_k) = \alpha_k \text{ and } Q_{k_1}\cap Q_{k_2} = \emptyset,\ k_1 \ne k_2. \tag{2.18}$$

Set $f = \sum\limits_{k=1}^\infty \frac{ks_k}{2^k}\cdot\chi_{Q_k}$, then

$$\int_\Omega |f|\omega d\mu = \sum_{k=1}^\infty \int_{Q_k}|f|\omega d\mu = \sum_{k=1}^\infty \frac{ks_k}{2^k}\omega(Q_k) = \sum_{k=1}^\infty \frac{ks_k}{2^k}\frac{1}{\frac{s_k}{2^k}b(\frac{s_k}{2^k})} \le \sum_{k=1}^\infty \frac{1}{2^k} = 1.$$

Thus $f \in L^1(\omega d\mu)$ and $\|f\|_{L^1(\omega d\mu)} \le 1$. Moreover, we also have $\int_\Omega \varPhi(Mf)\omega d\mu < \infty$. In fact, for the sequence s_k, we have

$$\frac{(k+1)s_{k+1}}{2^{k+1}} \ge \frac{(k+1)4s_k}{2^{k+1}} > \frac{4ks_k}{2^{k+1}} \ge 2\frac{ks_k}{2^k} > \frac{ks_k}{2^k},\ k\ge 1.$$

It is clear that $\frac{s_{k+1}}{2^{k+1}} > \frac{s_k}{2^k}$, thus

$$\omega(Q_{k+1}) = \frac{1}{\frac{s_{k+1}}{2^{k+1}}b(\frac{s_{k+1}}{2^{k+1}})} \le \frac{1}{\frac{4s_k}{2^{k+1}}b(\frac{s_k}{2^k})} \le \frac{1}{\frac{2s_k}{2^k}b(\frac{s_k}{2^k})} = \frac{1}{2}\omega(Q_k),\ k\ge 1.$$

Let $\beta_k = \frac{ks_k}{2^k}$, $k \geq 1$ and $\beta_0 = 0$. Obviously, $\beta_k \uparrow \infty$. For $k \geq 1$, when $s \in [\beta_{k-1}, \beta_k)$, we have

$$\omega\{|f| > s\} = \sum_{i=0}^{\infty} \omega(Q_{k+i}) \leq \omega(Q_k) \sum_{i=0}^{\infty} \frac{1}{2^i} = 2\omega(Q_k)$$

and

$$\begin{aligned} &\int_1^{2s} \frac{a(t)}{t} dt \leq 0 \leq \frac{1}{2^k} b(\frac{s_k}{2^k}), \text{ if } 2s \leq 1; \\ &\int_1^{2s} \frac{a(t)}{t} dt \leq \int_1^{2\beta_k} \frac{a(t)}{t} dt \leq \int_1^{s_k} \frac{a(t)}{t} dt < \frac{1}{2^k} b(\frac{s_k}{2^k}), \text{ if } 2s > 1. \end{aligned}$$

Combining (2.3), (1.2) and (1.3), we have

$$\begin{aligned} & \int_\Omega \Phi(Mf) \omega d\mu = \int_0^\infty \omega\{Mf > t\} a(t) dt \\ \leq \; & \int_0^\infty \Big(\frac{C}{t} \int_{\frac{t}{2}}^\infty \omega\{|f| > s\} ds \Big) a(t) dt \\ = \; & C \int_0^\infty \omega\{|f| > s\} \Big(\int_0^{2s} \frac{a(t)}{t} dt \Big) ds \\ = \; & C \int_0^\infty \omega\{|f| > s\} \Big(\int_0^1 \frac{a(t)}{t} dt + \int_1^{2s} \frac{a(t)}{t} dt \Big) ds \\ = \; & KC + C \sum_{k=1}^\infty \int_{\beta_{k-1}}^{\beta_k} \omega\{|f| > s\} \Big(\int_1^{2s} \frac{a(t)}{t} dt \Big) ds \\ \leq \; & KC + 2C \sum_{k=1}^\infty \omega(Q_k) \Big(\frac{1}{2^k} b(\frac{s_k}{2^k}) \Big) \beta_k \\ = \; & KC + 2C \sum_{k=1}^\infty \frac{1}{\frac{s_k}{2^k} b(\frac{s_k}{2^k})} \Big(\frac{1}{2^k} b(\frac{s_k}{2^k}) \Big) \frac{ks_k}{2^k} \\ = \; & KC + 2C \sum_{k=1}^\infty \frac{k}{2^k} < \infty. \end{aligned}$$

On the other hand, we have

$$\int_\Omega \Psi(C_7 \cdot |f|) \omega d\mu = \sum_{k=1}^\infty \Psi(C_7 \cdot \frac{ks_k}{2^k}) \omega(Q_k) = \sum_{k=1}^\infty \Psi(C_7 \cdot \frac{ks_k}{2^k}) \frac{1}{\frac{s_k}{2^k} b(\frac{s_k}{2^k})}.$$

For the above C_7, we can choose a constant $k(C_7) > 0$ such that $k \cdot C_7 \geq 2$, $k > k(C_7)$. Therefore,

$$\begin{aligned} \int_\Omega \Psi(C_7 \cdot |f|) \omega d\mu \;\; & \geq \sum_{k=k(C_7)}^\infty \Psi(C_7 \cdot \frac{ks_k}{2^k}) \frac{1}{\frac{s_k}{2^k} b(\frac{s_k}{2^k})} \\ & \geq \sum_{k=k(C_7)}^\infty \int_{\frac{s_k}{2^k}}^{2\frac{s_k}{2^k}} \frac{b(t)}{\frac{s_k}{2^k} b(\frac{s_k}{2^k})} dt \geq \sum_{k=k(C_7)}^\infty 1 = \infty. \end{aligned}$$

The result contradicts assumption (1.9). This completes the proof.

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Orlicz鞅类中极大算子的加权不等式

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摘要: 本文研究了极大算子的加权不等式. 利用权的性质, 证明了$a(\cdot)$和$b(\cdot)$的不等式蕴含极大算子的加权不等式, 也证明了相应的逆命题. 本文的结果将$L\log L$的相关理论拓展到了加权的Orlicz鞅类中.

关键词: 鞅空间; 极大算子; 加权不等式; 正则条件

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