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SYMMETRY OF SOLUTIONS OF MONGE-AMPÈRE EQUATIONS IN THE DOMAIN OUTSIDE A BALL

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Abstract: We study the symmetry of solutions to a class of Monge-Ampère type equations in the domain outside a ball. By using a moving plane method and the transform introduced by Jian and Wang, we prove the radially symmetry of the solutions.

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1 Introduction

In this paper, we study the radially symmetry of solutions to the equation in the form of

$$\det D^2u(x) = (1+|Du(x)|^2)^p K(x,u(x),Du(x)), \tag{1.1}$$

where $p=\frac{n+2}{2}-\frac{1}{2\alpha}$ for some number α. This equation comes from the well-known Gauss curvature flow, $\nu=[\frac{\hat{K}}{K(x,u(x),Du(x))}]^\alpha$, where ν is the velocity along the normal direction of the moving hyper-surface in R^n and $\hat{K}$ is its Gauss curvature. In fact, (1.1) is exactly the equation of translating solutions to the Gauss curvature flow. See [1, 2] for the details. When K is a positive constant and $\alpha=\frac{1}{n+2}$, the classical results of Jorgens ($n=2$, [3]), Calabi ($n\le 5$, [4]) and Pogorelov ($n\ge 2$, [5]) asserted that any convex solution to (1.1) must be a quadratic polynomial. Obviously, it is radially symmetric under the following condition (1.2). When K is a positive constant and $\alpha\in(0,\frac{1}{2})$, there exists a radially symmetric solution to (1.1) as well as infinitely many smooth, non-radially symmetric convex solutions to (1.1). See Theorem 6 in [2] and Theorem 1.2 in [1], respectively.

When $K(x,u(x),Du(x))=[(xDu(x)-u(x)]^{n+2}$ and $p=0$, (1.1) is the equation of the well-known affine hyperbolic sphere, and its Bernstein property, which may implies the symmetry under the normal condition (1.2), was studied in [6]. Basing a transform introduced by Jian and Wang in [6], the authors in [7] studied the symmetry of solutions

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of (1.1) with $K(x, Du(x), u(x)) = [(xDu(x) - u(x)]^\beta F(u(x))$ in the entire space R^n. They proved that a C^4-strictly convex solution to (1.1) in R^n is the radially symmetric about the lowest point, say for example the origin, provided that

$$u(0) = 0, \quad Du(0) = 0, \quad D^2u(0) = I \ \text{ (the unit matrix)}, \tag{1.2}$$

$$u_{x_ix_jx_k}(0) = 0, \quad \forall i, j, k = 1, 2, \cdots, n \tag{1.3}$$

and

$$\lim_{x\to\infty} Du(x) = \infty. \tag{1.4}$$

Obviously, the strict convexity, (1.2) and (1.4) are natural assumptions, which implies that for any small $r > 0$,

$$u(x) > 0, \ x \cdot Du(x) - u(x) > b, \quad \forall x \in \partial B_r(0) \tag{1.5}$$

for some $b > 0$. But assumptions (1.3) and the C^4-smoothness for the solutions is too restrictive.

Recall that a C^2-function $h(x)$ is called strictly convex in Ω if its Hessian matrix $[D^2h(x)]$ is positive definite in Ω.

In this paper, we study the symmetry of solutions of (1.1) in the domain outside a ball. Interestingly, in this case we can remove assumption (1.3) and the smoothness of the solutions needs only C^2. Our main result is the following theorem.

Theorem 1.1 Suppose that $p, \beta \in R$ and

$$K(x, Du(x), u(x)) = [(xDu(x) - u(x)]^\beta F(u(x)) \tag{1.6}$$

with F satisfies

$$F \in C^1(0, \infty), \quad F(q) > 0, \quad \forall q \in (0, \infty) \tag{1.7}$$

and

$$\beta \le n + 1 - 2p. \tag{1.8}$$

If there exist positive constants c, r, b such that $u \in C^2(R^n \backslash B_r(0))$ satisfies (1.1) in $R^n \backslash \overline{B_r(0)}$, (1.4) and

$$u = c \ \text{ and } \ x \cdot Du(x) - c > b \ \text{ on } \ \partial B_r(0), \tag{1.9}$$

and if u is strictly convex, then u is radially symmetric in $R^n \backslash B_r(0)$ about the origin.

By Theorem 1.1 and (1.5), it is easy to see that the main result in [7] still holds true if assumption (1.3) is replaced by the assumption that u is radially symmetric about the origin in $B_r(0)$ for some small $r > 0$.

The radial symmetry for the positive solutions of the equations $\Delta u + f(u) = 0$ for $x \in R^n$ was studied in [8] under the ground state conditions: $u(x) \to 0$ as $|x| \to \infty$. Afterwards, a lot of works were published along this direction, dealing with various symmetry problems for nonlinear elliptic equations. See, for example, [9–13] and the references in [7]. However, all those symmetric results, up to authors' knowledge, need the a priori ground state condition, which can not be satisfied for solutions to Monge-Ampère equations.

The proof of Theorem 1.1 is done in Section 3; in Section 2, some lemmas necessary for the proof are given.

2 Some Lemmas

In [6], Jian and Wang introduced a transform which reduces the behavior of u near ∞ to that of the new function near the origin and preserves the form of equation (1.1). This transform is

$$y=\frac{x}{u(x)},\quad v(y)=\frac{1}{u(x)},\tag{2.1}$$

where we assume $u\in C^2$ and $u>0$. Then

$$x=\frac{y}{v(y)},\quad u(x)=\frac{1}{v(y)}.$$

By calculations, we have

$$D_xu(x)=\frac{D_yv(y)}{y\cdot D_yv(y)-v(y)},\quad x\cdot D_xu(x)-u(x)=[y\cdot D_yv(y)-v(y)]^{-1},\tag{2.2}$$

$$u_{x_ix_j}=\frac{v}{y\cdot Dv-v}(\delta_{ik}-v_{y_i}b_k)(v_{y_ly_k})(\delta_{lj}-b_lv_{y_k}),\tag{2.3}$$

where $b_k=\frac{y_k}{y\cdot Dv-v}$, and

$$\det D_x^2u(x)=[\frac{v(y)}{y\cdot D_yv(y)-v(y)}]^{n+2}\det(D_y^2v(y)).\tag{2.4}$$

See (2.6)–(2.8) in [6] or (2.3)–(2.5) in [7].

Lemma 2.1 Suppose that a strictly convex function $u\in C^2(R^n\backslash B_r(0))$ satisfies (1.1) in $R^n\backslash\overline{B_r(0)}$, (1.4), (1.6) and (1.9). Let y and v be given by (2.1). Then

$$\{y=\frac{x}{u(x)}:x\in R^n\backslash B_r(0)\}=\overline{B_{\frac{r}{c}}(0)}\backslash\{0\},\quad \lim_{y\to0}v(y)=0\tag{2.5}$$

and $v\in C^2(\overline{B_{\frac{r}{c}}(0)}\backslash\{0\})$ is a strictly convex and positive function, satisfying the equation

$$\det D^2v=[\frac{y\cdot Dv-v}{v}]^{n+2-\beta}F(\frac{1}{v})[1+\frac{|Dv|^2}{(y\cdot Dv-v)^2}]^p,\quad \forall y\in B_{\frac{r}{c}}(0)\backslash\{0\},\tag{2.6}$$

and

$$v(y)>0,\quad 0<y\cdot Dv(y)-v(y)<\frac{1}{b},\forall y\in\overline{B_{\frac{r}{c}}}(0)\backslash\{0\}.\tag{2.7}$$

Proof Obviously, the convexity implies that $u(x)\ge c>0$ for all $x\in R^n\backslash B_r(0)$. To prove (2.7), we have, by (1.9) and the continuity, that $u(x)-x\cdot Du(x)<-b$ for x near $\partial B_r(0)$. This means the intersecting point, $(0,u(x)-x\cdot Du(x))$, between the supporting plane of the graph of u at $(x, u(x))$ with the x_{n+1}-axis is below the point $(0, -b)$. Since the convexity preserves this property as x is away from $\partial B_r(0)$, we obtain

$$u(x) - x \cdot Du(x) < -b, \ \ \forall x \in R^n \backslash B_r(0),$$

which, together with (2.2), implies (2.7).

Since $-b < 0$, the position of the supporting plane of the graph of u shows the graph $\{(x, u(x)) : x \text{ near } \partial B_r(0)\}$ is above the cone $\{(x, c\frac{|x|}{r}) : x \text{ near } \partial B_r(0)\}$, so the whole graph $\{(x, u(x)) : x \in R^n \backslash \overline{B_r(0)}\}$ is above the whole cone $\{(x, c\frac{|x|}{r}) : x \in R^n \backslash \overline{B_r(0)}\}$, i.e.,

$$u(x) > c\frac{|x|}{r}, \ \ \forall x \in R^n \backslash \overline{B_r(0)},$$

which implies

$$|y| = \frac{|x|}{u(x)} < \frac{r}{c}, \ \ \forall x \in R^n \backslash \overline{B_r(0)}.$$

Obviously,

$$|y| = \frac{|x|}{u(x)} = \frac{r}{c}, \ \ \forall x \in \partial B_r(0)$$

and

$$\lim_{x \to \infty} \frac{x}{u(x)} = 0$$

by (1.4) and (1.9). Hence, we obtain (2.5).

Now, by (1.1), (1.6), (2.2) and (2.3), we see that $v \in C^2(\overline{B_{\frac{r}{c}}}(0) \backslash \{0\})$ is a strictly convex and positive function, satisfying (2.6).

Lemma 2.2 Let $w \in C^2(\Omega)$ be a nonnegative solution to

$$\sum_{i,j=1}^{n} a^{ij}(x)\partial_{ij} w + \sum_{i=1}^{n} b^i(x)\partial_i w + C(x)w \le 0, \forall x \in \Omega, \tag{2.8}$$

where Ω is an open set in R^n, $a^{ij}, b^i, C(x) \in L^\infty(\Omega')$ and the matrix $[a^{ij}(x)]$ is positive-definite in Ω' for any compact set $\Omega' \subset \bar{\Omega}$. Then either $w \equiv 0$ in Ω or $w(x) > 0$ for all $x \in \Omega$. Moreover, if $w \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$ and $w(x_0) > 0$ for some $x_0 \in \Omega$ and $w(\bar{x}) = 0$ for some $\bar{x} \in \partial\Omega$ which is smooth near $\bar{x}$, then $\frac{\partial w}{\partial \nu}(\bar{x}) < 0$ where ν is the unit outer normal of $\partial\Omega$.

The proof of Lemma 2.2 can be found in [14].

Lemma 2.3 Suppose that diam $(\Omega) \le d$, $a^{ij}, b^i, C(x) \in L^\infty(\Omega)$ and the matrix $[a^{ij}(x)]$ is positive in $\bar{\Omega}$. Let $w \in C^2(\Omega)$ satisfies (2.8) and $\underline{\lim}_{x \to \partial\Omega} w(x) \ge 0$. There exists a $\delta > 0$ depending only on n, d and the bound of the coefficients such that $w(x) \ge 0$ in Ω provided that the measure $|\Omega| < \delta$.

This lemma is exactly Proposition 1.1 in [9].

3 Proof of Theorem 1.1

Assume $u \in C^2(R^n) \backslash B_r(0)$ is strictly convex, satisfies (1.1) in $R^n \backslash \overline{B_r(0)}$, (1.4) and (1.9) with K satisfying (1.6)–(1.8). Replacing u by $\frac{u}{c}$, we may assume $c = 1$ in Theorem 1.1. Let

y and $v(y)$ be given by (2.1). It is enough to prove that v is radially symmetric about the origin in $B_r(0)\backslash\{0\}$.

It follows Lemma 2.1 that Then $v \in C^2(\overline{B_r(0)}\backslash\{0\})$ is a strictly convex function, satisfying

$$\det D^2 v(x) = G(v)(x), \ \forall x \in B_r(0)\backslash\{0\}, \tag{3.1}$$

where

$$\begin{aligned} G(v)(x) &:= [I(v)(x)]^{n+2-\beta}\frac{F(v^{-1}(x))}{v^{n+2}(x)}[1+\frac{|Dv(x)|^2}{(I(v)(x))^2}]^p, \\ I(v)(x) &:= x\cdot Dv(x) - v(x) \end{aligned} \tag{3.2}$$

and

$$v(x) > 0, \ \ 0 < x\cdot Dv(x) - v(x) < \frac{1}{b}, \quad \forall x \in \overline{B_r}(0)\backslash\{0\} \tag{3.3}$$

and

$$v = 1 \ \text{ on } \ \partial B_r(0), \ \lim_{x\to 0} v(x) = 0. \tag{3.4}$$

We will use the moving planes method to prove that v is radially symmetric with respect to the origin. This needs to show that v is symmetric in any direction with respect to the origin. Since equation (3.1) is invariant under orthonogal transforms, it is sufficient to do this in one direction. Without loss of generality, we will do it in e_1-direction. In a word, to show Theorem 1.1, it is enough to prove that

$$v(-x_1, x_2, \cdots, x_n) = v(x_1, x_2, \cdots, x_n), \ \ \forall x = (x_1, x_2, \cdots, x_n) \in B_r(0)\backslash\{0\}. \tag{3.5}$$

Use $(x_1, x') := (x_1, x_2, \cdots, x_n)$ to denote a point x in R^n. For any $\lambda \in R$, denote

$$\begin{aligned} &B_r^\lambda := \{x = (x_1, x_2, \cdots, x_n) \in B_r(0) : x_1 > \lambda\}, \\ &T_r^\lambda := \{x = (x_1, x_2, \cdots, x_n) \in \overline{B_r(0)} : x_1 = \lambda\}, \\ &x^\lambda = (2\lambda - x_1, x'), \ \ v_\lambda(x) = v(x^\lambda), \\ &w_\lambda(x) = v(x) - v_\lambda(x) \end{aligned}$$

and

$$G_\lambda(v)(x) := [I_\lambda(v)(x)]^{n+2-\beta}\frac{F(v_\lambda^{-1}(x))}{v_\lambda^{n+2}(x)}[1+\frac{|Dv_\lambda(x)|^2}{(I(v_\lambda)(x))^2}]^p,$$

where

$$I_\lambda(v)(x) := \begin{cases} x_1 v_{x_1}(x) + \sum\limits_{i=2}^n x_i (v_\lambda)_{x_i}(x) - v_\lambda(x), & \text{if } \ v_{x_1}(x^\lambda) \ge 0, \\ I(v_\lambda)(x) = \sum\limits_{i=1}^n x_i (v_\lambda)_{x_i}(x) - v_\lambda(x), & \text{if } \ v_{x_1}(x^\lambda) < 0. \end{cases}$$

As in [7], we introduce the following differential operator

$$L_\lambda(v)(x) := \det D^2 v(x) - \det D^2 v_\lambda(x) + G_\lambda(v)(x) - G(v)(x).$$

By assumption (1.7) and a mean value theorem, we have the following obvious result.

Lemma 3.1 Let r be any positive constant. If $v \in C^2(\overline{B_r(0)}\backslash\{0\})$ is bounded, positive and strictly convex, then for any compact Ω in $\overline{B_r(0)}\backslash\{0, 0^\lambda\}$, there exist a constant $C_1 > 0$ independent of $\lambda \in (0, \infty)$ (but depending on Ω) and piecewise continuous functions $\{a_\lambda^{ij}(x)\}$, $\{b_\lambda^i(x)\}$, $C_\lambda(x)$ (all depending on the v and its derivatives up to second order in Ω), such that

$$L_\lambda(v)(x) = a_\lambda^{ij}(x)\partial_{ij}w_\lambda(x) + b_\lambda^i(x)\partial_i w_\lambda(x) + C_\lambda(x)w_\lambda(x), \forall x \in \Omega \cap \{x_1 > \lambda\}$$

and

$$C_1^{-1}I \le (a_\lambda^{ij}(x)) \le C_1 I,\ |b_\lambda^i(x)| + |C_\lambda(x)| \le C_1,\quad \forall x \in \Omega \cap \{x_1 > \lambda\}.$$

We will complete the proof of Theorem 1.1 after proving the following three claims.

Claim 1 Here exists a $\bar{\lambda} \in (\frac{r}{2}, r)$ such that

$$w_\lambda > 0 \text{ in } B_r^\lambda \text{ and } \frac{\partial v}{\partial x_1} > 0 \text{ on } T_r^\lambda,\quad \forall \lambda \in (\bar{\lambda}, r).$$

To show it, we see, by (3.1), (1.7) and (3.3), that

$$\Delta v \ge n[\det(D^2 v)]^{\frac{1}{n}} > 0 \text{ in } B_r(0)\backslash\{0\}.$$

So, the usual strong maximum principle and Hopf's boundary point lemma in [4] implies

$$0 < v < 1 \text{ in } B_r(0)\backslash\{0\} \tag{3.6}$$

and

$$\frac{\partial v}{\partial x_1} > 0 \text{ on } \partial B_r(0) \bigcap \{x_1 > 0\}, \tag{3.7}$$

where we have used the fact that the angel between the vector $(x_1, x') \in \partial B_r(0)$ and the out normal of $\partial B_r(0)$ at the point $x = (x_1, x')$ is less than $\frac{\pi}{2}$ for $x_1 > 0$.

By continuity and (3.7), we see that that there is a $\lambda_1 \in (\frac{r}{2}, r)$ such that

$$\frac{\partial v}{\partial x_1} > 0 \ \text{ on } \ T_r^\lambda,\quad \forall \lambda \in (\lambda_1, r). \tag{3.8}$$

By (3.4) and (3.6), we see that

$$w_\lambda > 0 \ \text{ on } \ \partial B_r^\lambda \backslash \{(x_1, x') \in B_r : x_1 = \lambda\}, \forall \lambda \in (0, r). \tag{3.9}$$

To complete the proof, we notice, for $\lambda > 0$ and $\lambda < x_1 < 2\lambda$, that

$$(2\lambda - x_1)v_{x_1}(x^\lambda) < \begin{cases} x_1 v_{x_1}(x), & \text{if } v_{x_1}(x^\lambda) \ge 0, \\ -x_1 v_{x_1}(x^\lambda) = x_1 (v_\lambda)_{x_1}(x), & \text{if } v_{x_1}(x^\lambda) < 0, \end{cases}$$

where we have used the fact $v_{x_1}(x^\lambda) < v_{x_1}(x)$ by the convexity. It follows that

$$\begin{aligned} I(v)(x^\lambda) &= x^\lambda \cdot Dv(x^\lambda) - v(x^\lambda) \\ &= (2\lambda - x_1)v_{x_1}(x^\lambda) + \sum_{i=2}^{n} x_i v_i(x^\lambda) - v(x^\lambda) \\ &= (2\lambda - x_1)v_{x_1}(x^\lambda) + \sum_{i=2}^{n} x_i (v_\lambda)_{x_i}(x) - v_\lambda(x) \\ &< I_\lambda(v)(x). \end{aligned} \tag{3.10}$$

This, together with assumptions (1.8) , implies

$$G(v)(x^\lambda) > G_\lambda(v)(x), \quad \forall \lambda > 0, \quad \forall x \in B_r^\lambda \backslash (B_r^{2\lambda} \cup \{0^\lambda\}). \tag{3.11}$$

Noting $\det D^2 v(x^\lambda) = \det D^2 v_\lambda(x)$ and using (3.11), we have, by equation (3.1), that

$$\begin{aligned} 0 &= \det D^2 v(x) - G(v)(x) - \det D^2 v(x^\lambda) + G(v)(x^\lambda) \\ &> L_\lambda(v)(x), \quad \forall \lambda > 0, \quad \forall x \in B_r^\lambda \backslash (B_r^{2\lambda} \cup \{0^\lambda\}). \end{aligned} \tag{3.12}$$

Observing that the measure of B_r^λ can be smaller than any positive constant if $r - \lambda_1 > 0$ is small. Applying Lemmas 3.1 and 2.3 to (3.12), we obtain a constant $\bar{\lambda} \in (\lambda_1, r)$ such that $w_\lambda \geq 0$ in $B_r^\lambda \backslash (B_r^{2\lambda} \bigcup \{0^\lambda\})$ for all $\lambda \in [\bar{\lambda}, r)$. Since λ is arbitrary in $[\bar{\lambda}, r)$, we have $w_\lambda \geq 0$ in B_r^λ. Furthermore, Lemma 2.2 implies

$$w_\lambda > 0 \text{ in } B_r^\lambda, \quad \forall \lambda \in [\bar{\lambda}, r),$$

which, together with (3.7) and continuity, complete the proof of Claim 1.

Let

$$Q = \{\lambda \in (0, r) : w_\lambda > 0 \text{ in } B_r^\lambda \text{ and } \frac{\partial v}{\partial x_1} > 0 \text{ on } T_r^\lambda\}.$$

Then Claim 1 means that $[\bar{\lambda}, r) \subset Q$.

Claim 2 The set Q is open in $(0, r)$. Suppose this Claim is false, i.e., there exist a $\lambda' \in Q$ and a number sequence λ_k and a sequence of points $\{x^k\} \subset B_r^\lambda$ such that

$$\lim_{k \to \infty} \lambda_k = \lambda' > 0 \text{ and } w_{\lambda_k}(x^k) \leq 0, \quad k = 1, 2, \cdots. \tag{3.13}$$

Otherwise, we have a sequence $y_k \in T_r^{\lambda_k}$ such that $\frac{\partial v}{\partial x_1}(y_k) \leq 0$, which implies immediately that $\frac{\partial v}{\partial x_1}(y) \leq 0$ for some $y \in T_r^{\lambda'}$, contradicting with $\lambda' \in Q$.

Next, we want to get a contradiction by (3.13). First, by the boundedness of $\{x_k\}$, we can choose a subsequence such that

$$x^k \to x^0 \text{ as } k \to \infty. \tag{3.14}$$

Then

$$w_{\lambda'}(x^0) \leq 0 \text{ and thus } x^0 \in T_r^{\lambda'}.$$

Because $\lambda' \in Q$, we have

$$\frac{\partial v}{\partial x_1}(x^0) > 0. \tag{3.15}$$

Sine (3.13) means that $v(x^k) \le v((x^k)^{\lambda_k})$, and

$$(x_k)_1^{\lambda_k} = 2\lambda_k - x_1^k < \lambda_k < x_1^k,$$

we see that

$$\frac{\partial v}{\partial x_1}(\xi^k) \le 0$$

for some ξ^k in the segment connecting x^k and $(x^k)^{\lambda_k}$ for $k = 1, 2, \cdots$. Moreover, $\xi^k \to x^0$ by (3.14) and the fact $x^0 \in T_r^{\lambda'}$. Then, $\frac{\partial v}{\partial x_1}(x^0) \le 0$, contradicting (3.15). This proves Claim 2.

Claim 3 Let (λ_0, r) be the connected component of Q in $(0, r)$ containing $[\bar{\lambda}, r)$. Then $\lambda_0 = 0$.

By the definition of λ_0, we have

$$w_{\lambda_0} \ge 0 \text{ in } B_r^{\lambda_0} \tag{3.16}$$

and

$$w_\lambda > 0 \text{ in } \quad B_r^\lambda, \quad \forall \lambda \in (\lambda_0, r).$$

Observing $w_\lambda = 0$ on T_r^λ, by Lemmas 3.1 and 2.2, we see that $\frac{\partial w_\lambda}{\partial x_1} > 0$ and so $\frac{\partial v}{\partial x_1} > 0$ on each T_r^λ for all $\lambda \in (\lambda_0, r)$. That is

$$\frac{\partial v}{\partial x_1} > 0 \text{ in } B_r^{\lambda_0}. \tag{3.17}$$

Suppose the contrary $\lambda_0 > 0$. We conclude that

$$w_{\lambda_0} > 0 \text{ in } B_r^{\lambda_0}. \tag{3.18}$$

Where (3.18) false, $w_{\lambda_0}(\bar{x}) = 0$ for some $\bar{x} \in B_r^{\lambda_0}$, which is a minimum point of w_{λ_0} in $B_r^{\lambda_0}$ by (3.16). Then $\frac{\partial w_{\lambda_0}}{\partial x_1}(\bar{x}) = 0$, which, together with (3.17), implies

$$\frac{\partial v}{\partial x_1}(\bar{x}^{\lambda_0}) = -\frac{\partial v}{\partial x_1}(\bar{x}) < 0.$$

Set

$$\Omega_0^{\lambda_0} = \{x : x \in B_r(0)\backslash\{0\} : x_1 < \lambda_0 \text{ and } \frac{\partial v}{\partial x_1}(x) < 0\}.$$

Let Ω_0 be the symmetric set of $\Omega_0^{\lambda_0}$ with respect to the plane $x_1 = \lambda_0$. Then Ω_0 is an open set and $\bar{x}$ is its interior point, and

$$\frac{\partial v}{\partial x_1}(x^{\lambda_0}) < 0, \quad \forall x \in \Omega_0.$$

Recalling the definition of $I_{\lambda_0}(v)(x)$, we have, by the assumption $\lambda_0 > 0$, that

$$\begin{aligned} I(v)(x^{\lambda_0}) &= x^{\lambda_0} \cdot Dv(x^{\lambda_0}) - v(x^{\lambda_0}) \\ &< -x_1 v_{x_1}(x^{\lambda_0}) + \sum_{i=2}^{n} x_i v_{x_i}(x^{\lambda_0}) - v(x^{\lambda_0}) \\ &= x_1 (v_{\lambda_0})_{x_1}(x) + \sum_{i=2}^{n} x_i (v_{\lambda_0})_{x_i}(x) - v_{\lambda_0}(x) \\ &= I_{\lambda_0}(v)(x) \end{aligned}$$

for all $x \in \Omega_0$.

Hence, (3.10)–(3.12) hold for $\lambda = \lambda_0$ and all $x \in \Omega_0$, i.e.,

$$L_{\lambda_0}(v)(x) < 0, \quad \forall x \in \Omega_0. \tag{3.19}$$

Using Lemmas 2.2 and 3.1, we obtain that $w_{\lambda_0} \equiv 0$ in a ball $B_\delta(\bar{x})$ contained in Ω_0. Therefore $v \equiv v_{\lambda_0}$ and so $L_{\lambda_0}(v) \equiv 0$ in $B_\delta(\bar{x})$, contradicting (3.19). This proves (3.18).

With (3.18) in hands, we use Lemmas 2.2 and 3.1 again to obtain

$$\frac{\partial w_{\lambda_0}}{\partial x_1} > 0 \ \text{ on } \ T_r^{\lambda_0},$$

which implies

$$\frac{\partial v}{\partial x_1} > 0 \ \text{ on } \ T_r^{\lambda_0}.$$

This and (3.18) mean $\lambda_0 \in Q$, contradicting the definition of λ_0. In this way, we have proved Claim 3.

Now we complete the proof of Theorem 1.1. Since $\lambda_0 = 0$, by Claim 3 and (3.16), we have

$$v(x_1, x') \geq v(-x_1, x'), \forall x = (x_1, x') \in B_r(0)\,, x_1 > 0.$$

The opposite inequality is also true, because $V(x) := v(-x_1, x')$ is a solution to (3.1) in $B_r(0) \setminus \{0\}$ and the same conditions as v holds for V. This proves (3.5) and thus Theorem 1.1.

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外球区域Monge-Ampère方程解的对称性

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摘要: 本文研究了外球区域中一类Monge-Ampère方程解的对称性. 利用移动平面法和简-汪引进的一类变换, 证明了解是旋转对称的.

关键词: 解的对称性; Monge-Ampère方程; 移动平面法; 极值原理

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