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EXISTENCE OF SOLUTIONS TO THE INITIAL VALUE PROBLEM OF SEMI-LINEAR GENERALIZED TRICOMI EQUATION

ZHANG Kang-qun, LI Yu-chen

(*Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China*)

Abstract: In this paper, we study the existence of solutions to the initial value problem of semi-linear generalized Tricomi equation with characteristic families coincided on $t=0$. Based on the inequality of $\dot{H}_q^{s_1}$-$\dot{H}_p^{s_0}$ estimates of two Fourier integral operators, we establish the local and global existence of solution for the semi-linear equation in hyperbolic half space. Meanwhile, we give the loss of regularities on degenerate domain and the decay rates when time tends to infinity in the weighted estimates.

Keywords: generalized Tricomi equation; semi-linear; weighted estimate; existence

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1 Introduction and Main Results

In this paper we study the local and global existence of solution semi-linear generalized Tricomi equation in $\mathbb{R}^+ \times \mathbb{R}^n$, $n \geq 2$,

$$\partial_t^2 u - t^m \triangle u = t^\alpha u|u|^\beta \tag{1.1}$$

with the initial value

$$u(t,x)|_{t=0} = \varphi_1(x), \partial_t u(t,x)|_{t=0} = \varphi_2(x), \tag{1.2}$$

where $\triangle = \sum\limits_{i=1}^{n} \partial_{x_i}^2$ is the Laplace operator, $m > 0$, $\beta > 0$ and some constant α.

Equation (1.1) is a semi-linear hyperbolic equation with variable coefficient and characteristic families coincide on $t=0$. There exist three fundamental features. First, variable coefficient with t makes the well known Duhamel's principle of wave equation do not work. Then, we have to estimate the solution to the linear equation with source term. Second, its multiple characteristics cause some loss of regularities on degenerate domain. This leads to

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Biography: Zhang Kangqun (1981–), male, born at Linyi, Shandong, associate professor, major in analysis of partial differential equations.

the difficulty of local solvability. Third, the unbounded speed of propagation is an obstacle in establishing the global solution for the needed uniform decay estimate.

For $m=1$ and $n=1$, the homogeneous linear equation is the classic Tricomi equation, which was extensively investigated with suitable boundary value conditions from various viewpoints, such as [1–3] and the references therein. For $n>1$, the local existence of solution to the equation $\partial_t^2 u - t^m \triangle u = f(t,x,u)$ with one initial datum $u(0,x)\in H^s(\mathbb{R}^n)$ $(s>\frac{n}{2})$ was established in mixed-type domain in [4, 5]. Meanwhile, the propagation of weakly singularity along characteristics was studied. The low regularity solution problem of the equation with two initial data was considered in [6–7] and the positive answers were obtained. In [6], the local existence of solution was established for the given discontinuous initial data and nonlinear source term with some restrictions. In [7], the global existence and nonexistence of solutions were considered in $L^q(\mathbb{R}^n)$ under some conditions and the decay of solutions when time trends to infinity was given. With respect to other specific cases, so far there were existence results of solution such as [8–10]. In this note, we focus on the local and global existence of weak solutions to the initial value problem of semi-linear generalized Tricomi equation under some general conditions compared with the result given in [7], and establish a uniform weighted estimate in homogeneous Sobolev space $\dot{H}^s_q(\mathbb{R}^n)$,

$$\dot{H}^s_q(\mathbb{R}^n)=\{f(x):\|f\|_{\dot{H}^s_q(\mathbb{R}^n)}=\|(-\triangle)^{\frac{s}{2}}f(x)\|_{L^q(\mathbb{R}^n)}=\|(|\xi|^s f^{\wedge}(\xi))^{\vee}(x)\|_{L^q(\mathbb{R}^n)}<\infty\}, \quad (1.3)$$

where $f^{\wedge}(\xi)=\displaystyle\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}f(x)dx$ and $g^{\vee}(x)$ is the corresponding inverse Fourier transformation.

Now we state our main results.

Theorem 1.1 Assume $\frac{1}{p}+\frac{1}{q}=1$, $p=\frac{\beta+2}{\beta+1}$, $\frac{2}{m+2}\geq\frac{n\beta}{2+\beta}$ and $n-(\beta+1)p(\frac{2}{2+m}-\frac{n\beta}{2+\beta})=1$. Given initial data $\varphi_i(x)\in C_c^\infty(\mathbb{R}^n)$, $i=1,2$, then there exists a unique solution $u\in C([0,T];\dot{H}_q^{\frac{2}{m+2}-\frac{n\beta}{2+\beta}}(\mathbb{R}^n))\cap C^1([0,T]:\dot{H}_q^{-\frac{n\beta}{2+\beta}}(\mathbb{R}^n))$ to problem (1.1) with $\alpha=0$ and some positive T, which satisfies

$$\sup_{t\in[0,T]}(\|u(t,\cdot)\|_{\dot{H}_q^{\frac{2}{2+m}-\frac{n\beta}{2+\beta}}(\mathbb{R}^n)}+\|\partial_t u(t,\cdot)\|_{\dot{H}_q^{-\frac{n\beta}{2+\beta}}(\mathbb{R}^n)})\leq C(\|\varphi_1\|_{\dot{H}_p^{\frac{2}{m+2}}(\mathbb{R}^n)}+\|\varphi_2\|_{L^p(\mathbb{R}^n)}).$$

Theorem 1.2 Assume $\frac{1}{p}+\frac{1}{q}=1$, $p=\frac{\beta+2}{\beta+1}$, $\frac{m+4}{2(m+2)}\geq\frac{(n+1)\beta}{2(2+\beta)}$, $n-(\beta+1)p(\frac{2}{2+m}-\frac{n\beta}{2+\beta})=1$ and

$$(\beta+1)(\frac{m}{4}+\frac{(m+2)(n-1)\beta}{4(2+\beta)})-\alpha=1.$$

Given initial data $\varphi_i(x)\in C_c^\infty(\mathbb{R}^n)$, $i=1,2$ with

$$\|\varphi_1\|_{\dot{H}_p^{\frac{2}{m+2}}(\mathbb{R}^n)}+\|\varphi_2\|_{L^p(\mathbb{R}^n)}<\epsilon$$

for small ϵ, then there eixists a unique solution

$$u\in C((0,\infty);\ \dot{H}_q^{\frac{m+4}{2(m+2)}-\frac{(n+1)\beta}{2(2+\beta)}}(\mathbb{R}^n))\cap C^1((0,\infty);\ \dot{H}_q^{-\frac{m}{2(m+2)}-\frac{(n+1)\beta}{2(2+\beta)}}(\mathbb{R}^n))$$

to problem (1.1), which satisfies

$$\sup_{t\in(0,\infty)} t^{\frac{(m+2)(n-1)\beta}{4(2+\beta)}}\left(\|t^{\frac{m}{4}}u(t,\cdot)\|_{\dot H_q^{\frac{m+4}{2(2+m)}-\frac{(n+1)\beta}{2(2+\beta)}}(\mathbb{R}^n)}+\|t^{-\frac{m}{4}}\partial_t u(t,\cdot)\|_{\dot H_q^{-\frac{m}{2(2+m)}-\frac{(n+1)\beta}{2(2+\beta)}}(\mathbb{R}^n)}\right)$$
$$\le C(\|\varphi_1\|_{\dot H_p^{\frac{2}{m+2}}(\mathbb{R}^n)}+\|\varphi_2\|_{L^p(\mathbb{R}^n)}). \tag{1.4}$$

Remark 1.1 Generalized Tricomi equation has the property of "smooth effect" as we pointed out in [5]. In this note, by compared with the regularity given in Theorem 1.1 and Theorem 1.2 near $t=0$, we note that the solution has a higher regularity away from the degenerate domain.

Remark 1.2 In homogeneous Sobolev space $\dot H_q^s(\mathbb{R}^n)$, the weights in the uniform estimate (1.4) of global solution in time describe the degeneracy near $t=0$ and the decay as $t\to+\infty$. However unlike in inhomogeneous Sobolev space $H_q^s(\mathbb{R}^n)$ [5, 7], the author could not obtain those at the same time for the nonlinear problem.

Remark 1.3 In fact, the embedding theorem in homogeneous Sobolev space in page 119 [11, p.119, Theorem 2] and inequality (1.4) yield $u\in C((0,\infty);\ L^{\frac{qn}{n-(\frac{m+4}{2(2+m)}-\frac{(n+1)\beta}{2(2+\beta)})q}}(\mathbb{R}^n))$ such that

$$\sup_{t\in(0,\infty)} t^{\frac{m}{4}+\frac{(m+2)(n-1)\beta}{4(2+\beta)}}\|u(t,\cdot)\|_{L^{\frac{qn}{n-(\frac{m+4}{2(2+m)}-\frac{(n+1)\beta}{2(2+\beta)})q}}(\mathbb{R}^n)}\le C(\|\varphi_1\|_{\dot H_p^{\frac{2}{m+2}}(\mathbb{R}^n)}+\|\varphi_2\|_{L^p(\mathbb{R}^n)}).$$

Hence, for $\varphi_1(x)\in C_0^\infty(\mathbb{R}^n)$, $\varphi_2(x)\in C_0^\infty(\mathbb{R}^n)$, set $\frac{(n-1)\beta}{2(2+\beta)}\ge\frac{m}{2(m+2)}$, then in terms of Theorem 1.1, the conditions in Theorem 1.2 imply that $u\in C([0,\infty);\ L^{\frac{qn}{n-(\frac{m+4}{2(2+m)}-\frac{(n+1)\beta}{2(2+\beta)})q}}(\mathbb{R}^n))$.

This paper is organized as follows. In Section 2, for the later uses, we recite some preliminary results as our lemmas. In Section 3, by use of confluent hypergeometric functions, we derive some weighted homogeneous Sobolev regularity estimates for the corresponding inhomogeneous equation. Based on these estimates, we establish the local and global existence of solutions in Section 4 by constructing a contraction map.

2 Preliminaries

In this section, for reader's convenience, we will recall some fundamental results of confluent hypergeometric functions and useful estimates, which will be used in Section 3 below.

The confluent hypergeometric equation is $zw''(z)+(c-z)w'(z)-aw(z)=0$, where $z\in\mathbb{C}$, a and c are constants. When c is not an integer, (2.1) has one pair of linearly independent solutions

$$w_1(z)=\Phi(a,c;z),\quad w_2(z)=z^{1-c}\Phi(a-c+1,2-c;z), \tag{2.1}$$

where $\Phi(a,c;z)$ is Humbert's symbol and w_i is called the confluent hypergeometric function.

The Wronskian determinant for the system w_1 and w_2 is

$$W(w_1,w_2)=(1-c)z^{-c}e^z. \tag{2.2}$$

Next, we list some basic properties of the confluent hypergeometric functions which can be found in [12] .

Lemma 2.1 1) For $-\pi < \arg z < \pi$ and large $|z|$, then

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(c-a)}(e^{i\pi\epsilon}z^{-1})^a \sum_{n=0}^{M} \frac{(a)_n(a-c+1)_n}{n!}(-z)^{-n} + O\big(|z|^{-a-M-1}\big)$$
$$+ \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c} \sum_{n=0}^{N} \frac{(c-a)_n(1-a)_n}{n!} z^{-n} + O\big(|e^z z^{a-c-N-1}|\big), \tag{2.3}$$

where $\epsilon = 1$ if $\mathrm{Im} z > 0$, $\epsilon = -1$ if $\mathrm{Im} z < 0$, and $M, N = 0, 1, 2, 3, \cdots$.

2)

$$\frac{d}{dz}\Phi(a,c;z) = \frac{a}{c}\Phi(a+1,c+1;z). \tag{2.4}$$

Then by (2.3) in Lemma 2.1, we obtain

$$\left|\Phi(\frac{m}{2(m+2)}, \frac{m}{m+2}; |\xi|)\right| = C(|\xi|)^{-\frac{m}{2(m+2)}}\left(1 + O\big(|\xi|^{-1}\big)\right), |\xi| \gg 1 \tag{2.5}$$

and

$$\left|\Phi(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; |\xi|)\right| = C(|\xi|)^{-\frac{m+4}{2(m+2)}}\left(1 + O\big(|\xi|^{-1}\big)\right), |\xi| \gg 1. \tag{2.6}$$

Next, we list Theorem 1.11 in [13] and Lemma 4 in [14] as our Lemma 2.2 and Lemma 2.3.

Lemma 2.2 Let f be a measurable function such that, with $1 < r < +\infty$, we have for some constant C,

$$\mathrm{meas}\{\xi : |f(\xi)| \geq k\} \leq Ck^{-r}, \tag{2.7}$$

then $f \in M_p^q$ if $1 < p \leq 2 \leq q < +\infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$.

Lemma 2.3 Let P be real, C^∞ in neighborhood of the support of $v \in C_0^\infty$. Assume that the rank of $H_p(y) = \partial^2_{y_k y_l}P(y)$ is at least ρ on the support of v. Then for some integer M,

$$\|(e^{itP}v)^\wedge\|_\infty \leq C(1+|t|)^{-\frac{\rho}{2}} \sum_{|\alpha|\leq M} \|D^\alpha v\|_1, \tag{2.8}$$

where C depends on bounds of the derivatives of P on supp(v) and on a lower bound of the maximum of the absolute values of the minors of order ρ of H_P on supp(v), and on supp(v).

3 Weighted Estimates of Solutions to the Linear Equation

In this section, we first take partial Fourier-transformation on $x \in \mathbb{R}^n$, then the linear equation $\partial_t^2 v - t^m \triangle v = 0$ becomes

$$y''(t,\xi) + t^m|\xi|^2 y(t,\xi) = 0, \tag{3.1}$$

where $y(t,\xi) = \int_{\mathbb{R}^n} v(t,x)e^{-ix\cdot\xi}dx$ with $\xi \in \mathbb{R}^n$ and $y''(t,\xi) \equiv \partial_t^2 y(t,\xi)$.

Set $z=\frac{4i}{2+m}t^{\frac{2+m}{2}}|\xi|$ and $w(z)=y(\frac{z}{2i})e^{\frac{z}{2}}$, then if $t\neq 0$ and $|\xi|\neq 0$, the equation (3.1) can be written as

$$zw''(z)+(\frac{m}{2+m}-z)w'(z)-\frac{m}{2(2+m)}w(z)=0. \tag{3.2}$$

Thus (3.2) is a special case of the confluent hypergeometric equation with $c=\frac{m}{2+m}$ and $a=\frac{m}{2(2+m)}$.

Next, we recite the result in the reference [15].

Lemma 3.1 The functions

$$V_1(t,|\xi|)=e^{-\frac{z}{2}}\Phi(\frac{m}{2(2+m)},\frac{m}{2+m};z),\quad V_2(t,|\xi|)=te^{-\frac{z}{2}}\Phi(\frac{4+m}{2(2+m)},\frac{4+m}{2+m};z)$$

on $\mathbb{R}_t\times\mathbb{R}^n_\xi$, where $z=\frac{4i}{2+m}t^{\frac{2+m}{2}}|\xi|$, form the fundamental solution system of (3.2), which satisfy

$$V_1(0,|\xi|)=1,\ \ \partial_tV_1(0,|\xi|)=0;\ \ V_2(0,|\xi|)=0,\ \ \partial_tV_2(0,|\xi|)=1.$$

Unlike the method of using maximum principle to solve the degenerate parabolic equation [16], here the hypergeometric functions play an important role. According to Lemma 2.1, it is easy to verify that $(V_1(1,|\xi|))^\vee$ and $(V_2(1,|\xi|))^\vee$ are two Fourier integral operators with symbols of order $-\frac{m}{2(m+2)}$ or $-\frac{m+4}{2(m+2)}$. Then they are bounded in $L^p(\mathbb{R}^n)$. Moreover, the following results were obtained by taking a similar procedure in Theorem 3.1 [7] in terms of some micro-local representations.

Lemma 3.2 For $\phi\in\dot{H}^{s_0}_p(\mathbb{R}^n)$, $n>1$, set $1<p<2$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|\big(V_1(t,|\xi|)\phi^\wedge(\xi)\big)^\vee\|_{\dot{H}^{s_0+s}_q(\mathbb{R}^n)}\leq Ct^{-\frac{(2+m)(s+n(\frac{1}{p}-\frac{1}{q}))}{2}}\|\phi\|_{\dot{H}^{s_0}_p(\mathbb{R}^n)} \tag{3.3}$$

with $-n(\frac{1}{p}-\frac{1}{q})\leq s\leq\frac{m}{2(2+m)}-\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})$ and

$$\|\big(\partial_tV_1(t,|\xi|)\phi^\wedge(\xi)\big)^\vee\|_{\dot{H}^{s_0+s}_q(\mathbb{R}^n)}\leq Ct^{-1-\frac{(2+m)(s+n(\frac{1}{p}-\frac{1}{q}))}{2}}\|\phi\|_{\dot{H}^{s_0}_p(\mathbb{R}^n)} \tag{3.4}$$

with $-1-n(\frac{1}{p}-\frac{1}{q})\leq s\leq-\frac{m+4}{2(2+m)}-\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})$;

$$\|\big(V_2(t,|\xi|)\phi^\wedge(\xi)\big)^\vee\|_{\dot{H}^{s_0+s}_q(\mathbb{R}^n)}\leq Ct^{1-\frac{(2+m)(s+n(\frac{1}{p}-\frac{1}{q}))}{2}}\|\phi\|_{\dot{H}^{s_0}_p(\mathbb{R})} \tag{3.5}$$

with $-n(\frac{1}{p}-\frac{1}{q})\leq s\leq\frac{m+4}{2(2+m)}-\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})$ and

$$\|\big(\partial_tV_2(t,|\xi|)\phi^\wedge(\xi)\big)^\vee\|_{\dot{H}^{s_0+s}_q(\mathbb{R}^n)}\leq Ct^{-\frac{(2+m)(s+n(\frac{1}{p}-\frac{1}{q}))}{2}}\|\phi\|_{\dot{H}^{s_0}_p(\mathbb{R})} \tag{3.6}$$

with $-n(\frac{1}{p}-\frac{1}{q})\leq s\leq-\frac{m}{2(2+m)}-\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})$.

Remark 3.1 Inequalities (3.3)–(3.6) take effect for $p<2<q$ and $n\geq 2$.

Next, we consider the initial value problems of homogeneous equation

$$\begin{cases}\partial_t^2u_0-t^m\triangle u_0=0 & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^n,\\ u_0(0,x)=\varphi_1(x), & \partial_tu_0(0,x)=\varphi_2(x).\end{cases} \tag{3.7}$$

As direct results of Lemma 3.1 and Lemma 3.2, we obtain the representation of distributional solution

$$u_0^\wedge(t,\xi)=V_1(t,|\xi|)\varphi_1^\wedge(\xi)+V_2(t,|\xi|)\varphi_2^\wedge(\xi), \tag{3.8}$$

and the following estimates.

Lemma 3.3 Assume $\varphi_1(x)\in\dot{H}_p^{s_0+\frac{2}{m+2}}(\mathbb{R}^n)$, $\varphi_2(x)\in\dot{H}_p^{s_0}(\mathbb{R}^n)$, then $u_0(t,x)=\big(V_1(t,|\xi|)v_1^\wedge(\xi)+V_2(t,|\xi|)v_2^\wedge(\xi)\big)^\vee$ solves (3.7) and satisfies

$$\begin{aligned}
&\|t^{\frac{m}{4}}u_0(t,\cdot)\|_{\dot{H}_q^{s_0+\frac{m+4}{2(2+m)}-\frac{n+1}{2}(\frac1p-\frac1q)}(\mathbb{R}^n)}+\|t^{-\frac{m}{4}}\partial_t u_0(t,\cdot)\|_{\dot{H}_q^{s_0-\frac{m}{2(2+m)}-\frac{n+1}{2}((\frac1p-\frac1q)}(\mathbb{R}^n)}\\
\le\ & Ct^{\frac{(2+m)(1-n)}{4}(\frac1p-\frac1q)}(\|\varphi_1\|_{\dot{H}_p^{s_0+\frac{2}{2+m}}(\mathbb{R}^n)}+\|\varphi_2\|_{\dot{H}_p^{s_0}(\mathbb{R}^n)}); \qquad (3.9)\\
&\|u_0(t,\cdot)\|_{\dot{H}_q^{s_0+\frac{2}{2+m}-n(\frac1p-\frac1q)}(\mathbb{R}^n)}+\|\partial_t u_0(t,\cdot)\|_{\dot{H}_q^{s_0-n(\frac1p-\frac1q)}(\mathbb{R}^n)}\\
\le\ & C(\|\varphi_1\|_{\dot{H}_p^{s_0+\frac{2}{2+m}}(\mathbb{R}^n)}+\|\varphi_2\|_{\dot{H}_p^{s_0}(\mathbb{R}^n)}) \qquad (3.10)
\end{aligned}$$

for $1<p<2$, $\frac1p+\frac1q=1$.

Proof In terms of the representation of $u_0(t,x)$ defined in (3.8), by use of the inequalities (3.3)–(3.6) with the value of s fixed on the right endpoint number separately, it is easy to verify that (3.9) holds. And (3.10) can be derived by choosing the value of s with the corresponding left endpoint number.

Last, we consider the inhomogeneous problem

$$\begin{cases}\partial_t^2 v-t^m\triangle v=f(t,x) & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^n,\\ v(0,x)=0,\quad \partial_t v(0,x)=0.\end{cases} \tag{3.11}$$

In terms of Lemma 3.1 and a direct computation with (2.2), the distribution solution of (3.11) can be expressed as

$$v^\wedge(t,\xi)\equiv(Ef)^\wedge(t,\xi)=\int_0^t(V_2(t,|\xi|)V_1(\tau,|\xi|)-V_1(t,|\xi|)V_2(\tau,|\xi|))f^\wedge(\tau,\xi)d\tau. \tag{3.12}$$

Then we conclude

Lemma 3.4 If $f(t,x)\in C([0,\infty),\dot{H}_p^{s_0}(\mathbb{R}^n))$, then

$$\begin{aligned}
&\|t^{\frac{m}{4}}Ef(t,\cdot)\|_{\dot{H}_q^{s_0+\frac{m+4}{2(2+m)}-\frac{n+1}{2}(\frac1p-\frac1q)}(\mathbb{R}^n)}+\|t^{-\frac{m}{4}}\partial_t Ef(t,\cdot)\|_{\dot{H}_q^{s_0-\frac{m}{2(2+m)}-\frac{n+1}{2}(\frac1p-\frac1q)}(\mathbb{R}^n)}\\
\le\ & Ct^{\frac{(m+2)(1-n)}{4}(\frac1p-\frac1q)}\int_0^t\|f(\tau,\cdot)\|_{\dot{H}_p^{s_0}(\mathbb{R}^n)}d\tau; \qquad (3.13)\\
&\|Ef(t,\cdot)\|_{\dot{H}_q^{s_0+\frac{2}{2+m}-n(\frac1p-\frac1q)}(\mathbb{R}^n)}+\|\partial_t Ef(t,\cdot)\|_{\dot{H}_q^{s_0-n(\frac1p-\frac1q)}(\mathbb{R}^n)}\\
\le\ & C\int_0^t\|f(\tau,\cdot)\|_{\dot{H}_p^{s_0}(\mathbb{R}^n)}d\tau \qquad (3.14)
\end{aligned}$$

for $1<p<2$, $\frac1p+\frac1q=1$.

Remark 3.2 Uniform weighted estimates of (3.9) and (3.13) give the decay rates of the solution at infinity in time. Meanwhile (3.10) and (3.14) imply the local existence with low regularity.

Proof By a direct computation, we obtain

$$(\partial_t Ef)^\wedge(t,\xi)=\int_0^t(\partial_t V_2(t,|\xi|)V_1(\tau,|\xi|)-\partial_t V_1(t,|\xi|)V_2(\tau,|\xi|))f^\wedge(\tau,\xi)d\tau, \tag{3.15}$$

then it is easy to verify that

$$\begin{aligned}\|(-\triangle)^{\frac{s}{2}}Ef(t,\cdot)\|_{L^q(\mathbb{R}^n)}\le&C\int_0^t\|(-\triangle)^{\frac{s}{2}}V_2(t,|\xi|)V_1(\tau,|\xi|)f^\wedge(\tau,\xi)\|_{L^q(\mathbb{R}^n)}\\&+\|(-\triangle)^{\frac{s}{2}}V_1(t,|\xi|)V_2(\tau,|\xi|)f^\wedge(\tau,\xi)\|_{L^q(\mathbb{R}^n)}d\tau\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\|(-\triangle)^{\frac{s}{2}}\partial_t Ef(t,\cdot)\|_{L^q(\mathbb{R}^n)}\le&C\int_0^t\|(-\triangle)^{\frac{s}{2}}\partial_t V_2(t,|\xi|)V_1(\tau,|\xi|)f^\wedge(\tau,\xi)\|_{L^q(\mathbb{R}^n)}\\&+\|(-\triangle)^{\frac{s}{2}}\partial_t V_1(t,|\xi|)V_2(\tau,|\xi|)f^\wedge(\tau,\xi)\|_{L^q(\mathbb{R}^n)}d\tau.\end{aligned} \tag{3.17}$$

In terms of (2.4) in Lemma 2.1 or (3.11)–(3.12) as given in [12], we obtain

$$\partial_t V_1(t,\xi)=i(\frac{2+m}{4i})^{\frac{2}{2+m}}z^{\frac{m}{2+m}}|\xi|^{\frac{2}{2+m}}e^{-\frac{z}{2}}(\Phi(\frac{4+3m}{2(2+m)},\frac{2(1+m)}{2+m};z)-\Phi(\frac{m}{2(2+m)},\frac{m}{2+m};z))$$

and

$$\partial_t V_2(t,\xi)=e^{-\frac{z}{2}}(\Phi(\frac{4+m}{2(2+m)},\frac{2}{2+m};z)+\frac{(2+m)z}{4}e^{-\frac{z}{2}}\Phi(\frac{4+m}{2(2+m)},\frac{4+m}{2+m};z)).$$

Then for the similarity of the procedure on proof of above estimate, so we only consider the term $V_2(t,|\xi|)V_1(\tau,|\xi|)f^\wedge(\tau,\xi)$ with $\tau\in(0,t)$.

Since $V_i^\vee(t,|\xi|)$, $i=1,2$ is the Fourier integral operator with the symbol of order $-\frac{m}{m+2}$ or $-\frac{m+4}{2(m+2)}$. Then we obtain

$$\|V_i^\vee(\tau,|\xi|)f(\tau,x)\|_{L^p(\mathbb{R}^n)}\le Ct^{i-1}\|f(\tau,x)\|_{L^p(\mathbb{R}^n)},\quad i=1,2. \tag{3.18}$$

Hence, by use of (17) in Lemma 3.2, we derive

$$\begin{aligned}&\|(-\triangle)^{\frac{s}{2}}\big(V_2(t,|\xi|)V_1(\tau,|\xi|)f^\wedge(\tau,\xi)\big)^\vee\|_{L^q(\mathbb{R}^n)}\\\le&Ct^{1-\frac{(2+m)(s+n(\frac{1}{p}-\frac{1}{q}))}{2}}\|V_1^\vee(\tau,|\xi|)f(\tau,x)\|_{L^p(\mathbb{R}^n)}\\\le&Ct^{1-\frac{(2+m)(s+n(\frac{1}{p}-\frac{1}{q}))}{2}}\|f(\tau,x)\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{3.19}$$

Choose $s=\frac{m+4}{2(m+2)}-\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})$, inequality (3.19) becomes

$$\begin{aligned}&\|\big(V_2(t,|\xi|)V_1(\tau,|\xi|)f^\wedge(\tau,\xi)\big)^\vee\|_{\dot H_q^{\frac{m+4}{2(m+2)}-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)}\\\le&Ct^{-\frac{m}{4}-\frac{(m+2)(n-1)}{4}(\frac{1}{p}-\frac{1}{q})}\|f(\tau,x)\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{3.20}$$

On the other hand, by a similar method, we obtain

$$\begin{aligned}&\|\big(V_1(t,|\xi|)V_2(\tau,|\xi|)f^\wedge(\tau,\xi)\big)^\vee\|_{\dot H_q^{\frac{m+4}{2(m+2)}-\frac{n+1}{2}(\frac1p-\frac1q)}(\mathbb{R}^n)}\\ \le& Ct^{-\frac m4-\frac{(m+2)(n-1)}{4}(\frac1p-\frac1q)}\|f(\tau,x)\|_{L^p(\mathbb{R}^n)}.\end{aligned}\tag{3.21}$$

Combining (3.17) with inequalities (3.20)–(3.21), we conclude

$$\|t^{\frac m4}Ef(t,\cdot)\|_{\dot H_q^{\frac{m+4}{2(2+m)}-\frac{n+1}{2}(\frac1p-\frac1q)}(\mathbb{R}^n)}\le Ct^{\frac{(m+2)(1-n)}{4}(\frac1p-\frac1q)}\int_0^t\|f(\tau,\cdot)\|_{L^p(\mathbb{R}^n)}d\tau.\tag{3.22}$$

Then, by taking a similar procedure, we obtain (3.13)–(3.14).

4 Existence of Solutions to the Semi-Linear Problem

In this section, we construct a contraction map on a defined closed set separately, then use fixed point theorem to prove the local and global existence of solution to initial value problem (1.1).

Proof of Theorem 1.1 Define a closed set

$$\begin{aligned}S_M=&\{u\in C([0,T];\dot H_q^{\frac{2}{2+m}-n(\frac1p-\frac1q)}(\mathbb{R}^n))\cap C^1([0,T];\dot H_q^{-n(\frac1p-\frac1q)}(\mathbb{R}^n)):\\ &\|u\|_M=\sup_{t\in[0,T]}(\|u\|_{\dot H_q^{\frac{2}{2+m}-n(\frac1p-\frac1q)}(\mathbb{R}^n)}+\|\partial_t u\|_{\dot H_q^{-n(\frac1p-\frac1q)}(\mathbb{R}^n)})\le M\}\end{aligned}\tag{4.1}$$

for a constant $M>0$ and the distant of elements in S_M, $d(u,v)=\sup\limits_{t\in[0,T]}\|u-v\|_M$.

Consider the map F defined on S_M,

$$F(u)=u_0+E(u|u|^\beta),\tag{4.2}$$

where u_0 is defined by (3.8) in Lemma 3.3 and $E(u|u|^\beta)$ is defined by (3.12) in Lemma 3.4. Set $q=(\beta+1)p$, then in terms of the embedding theorem in homogeneous Sobolev space under the condition $n-(\beta+1)p(\frac{2}{2+m}-\frac{n\beta}{2+\beta})=1$, we obtain

$$\begin{aligned}\|F(u)\|_M\le&\|u_0\|_M+\|E(u|u|^\beta)\|_M\le\|u_0\|_M+Ct\int_0^1\|u(t\tau,\cdot)\|^{\beta+1}_{L^q(\mathbb{R}^n)}d\tau\\ \le&\|u_0\|_M+Ct\sup_{0\le\tau\le t}(\|u(\tau,\cdot)\|_{L^q(\mathbb{R}^n)})^{\beta+1}\le\|u_0\|_M+Ct\sup_{0\le\tau\le t}(\|u(\tau,\cdot)\|_M)^{\beta+1}\\ \le&C(\|\varphi_1\|_{\dot H_p^{\frac{2}{2+m}}(\mathbb{R}^n)}+\|\varphi_2\|_{L^p(\mathbb{R}^n)})+CtM^{\beta+1}.\end{aligned}\tag{4.3}$$

This implies that $F(u)\in S_M$ for smallness of T.

Meanwhile, $q=\frac{\beta pq}{q-p}$, $\beta<q$, then for any $u\in S_M$ and $v\in S_M$, we obtain

$$\begin{aligned}
&\|F(u)-F(v)\|_M=\|E(u|u|^\beta-v|v|^\beta)\|_M\\
\le& Ct\int_0^1\|(u|u|^\beta-v|v|^\beta)(t\tau,\cdot)\|_{L^p(\mathbb{R}^n)}d\tau\\
\le& Ct\int_0^1\|((\beta+1)(u-v)\int_0^1|\theta(u-v)+v|^\beta d\theta)(t\tau,\cdot)\|_{L^p(\mathbb{R}^n)}d\tau\\
\le& Ct\int_0^1\|(u-v)(t\tau,\cdot)\|_{L^q(\mathbb{R}^n)}\times(\int_{\mathbb{R}^n}(\int_0^1|(\theta(u-v)+v)(t\tau,\vartheta)|^\beta d\theta)^{\frac{pq}{q-p}}d\vartheta)^{\frac{q-p}{pq}}d\tau\\
\le& Ct\int_0^1\|(u-v)(t\tau,\cdot)\|_{L^q(\mathbb{R}^n)}(\|u\|^\beta_{L^{\frac{\beta pq}{q-p}}(\mathbb{R}^n)}+\|v\|^\beta_{L^{\frac{\beta pq}{q-p}}(\mathbb{R}^n)})d\tau\\
\le& Ct\int_0^1\|(u-v)(t\tau,\cdot)\|_M(\|u\|^\beta_M+\|v\|^\beta_M)d\tau\\
\le& Ctd(u,v)M^\beta. \qquad (4.4)
\end{aligned}$$

This yields the contraction property of map F on S_M for smallness of T.

Note that $p=1+\frac{1}{\beta+1}$, then in terms of (4.3) and (4.4), we establish Theorem 1.1.

Proof of Theorem 1.2 Define a closed set

$$\begin{aligned}
S_\epsilon=&\{u\in C((0,\infty);\dot{H}_q^{\frac{m+4}{2(2+m)}-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n))\cap C^1((0,\infty);\dot{H}_q^{-\frac{m}{2(2+m)}-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)):\\
&\|u\|_\epsilon=\sup_{t\in(0,\infty)}t^{\frac{(m+2)(n-1)}{4}(\frac{1}{p}-\frac{1}{q})}(\|t^{\frac{m}{4}}u\|_{\dot{H}_q^{\frac{m+4}{2(2+m)}-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)}\\
&+\|t^{-\frac{m}{4}}\partial_tu\|_{\dot{H}_q^{-\frac{m}{2(2+m)}-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n)})\le 2\epsilon\} \qquad (4.5)
\end{aligned}$$

with the distant of elements in S_ϵ, $d(u,v)=\sup\limits_{t\in(0,\infty)}\|u-v\|_\epsilon$.

Consider the map F defined on S_ϵ, $F(u)=u_0+E(t^\alpha u|u|^\beta)$, where u_0 is defined by (3.8) in Lemma 3.3 and $E(u|u|^\beta)$ is defined by (3.12) in Lemma 3.4. Set $q=(\beta+1)p$, $\gamma=\frac{m}{4}+\frac{(m+2)(n-1)}{4}(\frac{1}{p}-\frac{1}{q})$, in terms of the embedding theorem in homogeneous Sobolev space under the condition $n-(\beta+1)p(\frac{2}{2+m}-\frac{n\beta}{2+\beta})=1$, we obtain

$$\begin{aligned}
\|F(u)\|_\epsilon\le&\|u_0\|_\epsilon+\|E(\tau^\alpha u|u|^\beta)\|_\epsilon\le\|u_0\|_\epsilon+Ct\int_0^1((t\tau)^\alpha\|u(t\tau,\cdot)\|^{\beta+1}_{L^q(\mathbb{R}^n)}d\tau\\
\le&\|u_0\|_\epsilon+Ct\int_0^1(t\tau)^{\alpha-\gamma(\beta+1)}\|(t\tau)^\gamma u(t\tau,\cdot)\|^{\beta+1}_{L^q(\mathbb{R}^n)}d\tau\\
\le&\|u_0\|_\epsilon+Ct^{1+\alpha-\gamma(\beta+1)}\int_0^1\|(t\tau)^\gamma u(t\tau,\cdot)\|^{\beta+1}_{L^q(\mathbb{R}^n)}d\tau\\
\le&C(\|\varphi_1\|_{\dot{H}_p^{\frac{2}{2+m}}(\mathbb{R}^n)}+\|\varphi_2\|_{L^p(\mathbb{R}^n)})+Ct^{1+\alpha-\gamma(\beta+1)}\epsilon^{\beta+1}. \qquad (4.6)
\end{aligned}$$

Then, in terms of the condition

$$1+\alpha-(\beta+1)(\frac{m}{4}+\frac{(m+2)(n-1)}{4}(\frac{1}{p}-\frac{1}{q}))=0,$$

this implies that $F(u) \in S_\epsilon$ for the smallness of ϵ.

Meanwhile, $q = \frac{\beta pq}{q-p}$, $\beta < q$, then for any $u \in S_\epsilon$ and $v \in S_\epsilon$, we obtain

$$\begin{aligned}
&\|F(u) - F(v)\|_\epsilon = \|E(\tau^\alpha(u|u|^\beta - v|v|^\beta)\|_\epsilon \\
\le& Ct \int_0^1 \|(t\tau)^\alpha(u|u|^\beta - v|v|^\beta)(t\tau,\cdot)\|_{L^p(\mathbb{R}^n)} d\tau \\
\le& Ct^{1+\alpha} \int_0^1 \|((\beta+1)(u-v)\int_0^1 |\theta(u-v)+v|^\beta d\theta)(t\tau,\cdot)\|_{L^p(\mathbb{R}^n)} d\tau \\
\le& Ct^{1+\alpha-\gamma(\beta+1)} \int_0^1 \|(t\tau)^\gamma(u-v)(t\tau,\cdot)\|_{L^q(\mathbb{R}^n)} \times (\|(t\tau)^\gamma u\|^\beta_{L^{\frac{\beta pq}{q-p}}} + \|(t\tau)^\gamma v\|^\beta_{L^{\frac{\beta pq}{q-p}}}) d\tau \\
\le& Ct^{1+\alpha-\gamma(\beta+1)} d(u,v)\epsilon^\beta \le Cd(u,v)\epsilon^\beta. \qquad (4.7)
\end{aligned}$$

This yields the contraction property of map F on S_ϵ.

Finally, in terms of (4.6)–(4.7) and $p = 1 + \frac{1}{\beta+1}$, we complete the proof of Theorem 1.2.

References

[1] Tricomi F G. Sulle equazioni lineari alle derivate parziali di 2^o ordine ditipo misto[J]. Atti Accad. Naz. dei Lincei, 1923, 15(5): 133–247.

[2] Morawetz C S. Mixed equations and transonic flow[J]. J. Hyper. Diff. Equ., 2004, 1(1): 1–26.

[3] Rassias J M. Mixed type partial differential equations with initial and boundary values in fluid mechanics[J]. Int. J. Appl. Math. Stat., 2008, 13(J08): 77–107.

[4] Beals M. Singularities due to cusp interaction in nonlinear waves[J]. Nonl. Hyper. Equ. Field The. (Lake Como, 1990), 36–51, Pitman Res. Notes Math. Ser., 253, Harlow: Longman Sci. Tech., 1992.

[5] Zhang Kangqun. Existence and regularity of solution to the generalized Tricomi equation with a singular initial datum at a point[J]. Acta Math. Sin., 2012, 28(6): 1135–1154.

[6] Ruan Zhuoping, Witt I, Yin Huicheng. On the existence and cusp singularity of solutions to semi-linear generalized Tricomi equations with discontinuous initial data[J]. Commun. Contemp. Math., arXiv: 1211.0334v1[math.AP], 2012.

[7] Yagdjian K. Global existence for the n-dimensional semilinear Tricomi-type equations[J]. Commun. Part. Diff. Equ., 2006, 31(6): 907–944.

[8] Yagdjian K. The self-similar solutions of the one-dimentional semilinear Tricomi-type equations[J]. J. Diff. Equ., 2007, 236(1): 82–115.

[9] Galstian A. Global existence for the one-dimensional second order semilinear hyperbolic equations[J]. J. Math. Anal. Appl., 2008, 344(1): 76–98.

[10] Zhang Kangqun. A note on initial value problem for the generalized Tricomi equation in a mixed-type domain[J]. Acta Math. Sin., 2013, 29(8): 1581–1596.

[11] Stein E M. Singular integrals and differentiability properties of functions[M]. Princeton: Princeton Univ. Press, 1970.

[12] Erdelyi A, Magnus W, Oberhettinger F, et al. Higher transcendental functions, vol. 1[M]. New York: McGraw-Hill, 1953.

[13] Hormander L. Estimates for translation invariant operators in L^p spaces[J]. Acta Math., 1960, 104(1): 93–140.

[14] Brenner P. On $L^p - L^q$ estimates for the wave equation[J]. Math. Zeitschrift, 1975, 145(3): 251–254.

[15] Taniguchi K, Tozaki Y. A hyperbolic equation with double characteristics which has a solution with branching singularities[J]. Math. Japon., 1980, 25(3): 279–300.

[16] Li Mei, Xie Chunhong. Global existence and blow-up of solutions for degenerate parabolic systems[J]. J. Math., 2004, 24(2), 197–203.

半线性广义Tricomi方程初值问题的解的存在性

张康群, 李宇尘

(南京工程学院数理部, 江苏 南京 211167)

摘 要: 本文研究了半线性广义Tricomi方程初值问题解的存在性. 基于对两个傅立叶积分算子的$\dot{H}_q^{s_1}$-$\dot{H}_p^{s_0}$加权估计不等式, 建立了半线性广义Tricomi方程在双曲半平面解的局部和全局存在性. 同时, 给出了解在退化区域附近的正则性损失和在无穷远处衰减律.

关 键 词: 广义Tricomi方程; 半线性; 加权估计; 存在性

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