

LOCAL TIME OF MIXED BROWNIAN MOTION AND SUBFRACTIONAL BROWNIAN MOTION

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Abstract: In this article, local time of mixed Brownian motion and subfractional Brownian motion is studied. By using an alternative expression of subfractional Brownian motion, it is proved that the local time is a Hida distribution through white noise approach. Moreover, the chaos expansion of the local time is given by S -transform. Lastly, regularized condition of the local time is also obtained. Some results of local time of Brownian motion are popularized.

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1 Introduction

Let S_t^k be a subfractional Brownian motion (sfBm) with parameter $k \in (-\frac{1}{2}, \frac{1}{2})$ on \mathbb{R}^d . S_t^k is a centered Gaussian process with representation

$$S_t^k = \frac{1}{c(k)} \int_{\mathbb{R}} [(t-s)_+^k + (t+s)_-^k - 2(-s)_+^k] dB_s, \quad (1.1)$$

where $c(k) = [2(\int_0^\infty ((1+s)^k - s^k)^2 + \frac{1}{2k+1})]^\frac{1}{2}$, $a_+ = \max\{a, 0\}$, $a_- = \max\{-a, 0\}$ and B_s is a Brownian motion (Bm). It is well known that subfractional Brownian motion is an extension of Bm.

The object of study in this article is the local time of mixed Bm B_t and sfBm S_t^k (mBs), which is formally defined as

$$L_k(T, b) = \int_0^T \delta(\alpha B_t + \beta S_t^k - b) dt, \quad (1.2)$$

where b is some fixed point, $T > 0$, $\delta(x)$ is a Dirac delta function, α and β are two real constants such that $(\alpha, \beta) \neq (0, 0)$. The local time $L_k(T, b)$ characterizes time of the mixed process spending on the interval $[0, T]$.

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In recent years, some authors focused on the research on fractional Brownian motion (fBm), due to its interesting properties and its applications. Moreover, some authors introduce another Gaussian process so called sfBm instead of fBm. In spite of sfBm has many properties analogous of fBm such as self-similar and long-rang dependent, sfBm has non-stationary increments and is weakly correlated in comparing with fBm. Indeed, sfBm is intermediate between Bm and fBm. At the same time, local times of sfBm were studied by many authors as well, see e.g. Liu et al. [1], Yan et al. [2] and Guo et al. [3].

In this article, motivated by [3–4], in order to deal with problem of complexity structure of sfBm, we use an alternative expression of sfBm in Guo et al. (2015), and study the existence of local time of mBs in white noise analysis framework. The outline of the paper is as follows. In Section 2, we give some background materials in white noise analysis. In Section 3, we present the main results and their demonstrations.

2 White Noise Analysis

We briefly recall some notions and facts, and for detail see Refs. [4–6]. The first real Gelfand triple is $\mathcal{S}_{2d}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathbb{R}^{2d}) \subset \mathcal{S}_{2d}^*(\mathbb{R})$, where $\mathcal{S}_{2d}(\mathbb{R})$ (resp. $\mathcal{S}_{2d}^*(\mathbb{R})$) is the space of the vector valued Schwartz test functions (resp. tempered distributions). We consider $2d$ -tuple of Gaussian white noise $\vec{\omega} = (\vec{\omega}_1, \vec{\omega}_2) = (\omega_{1,1}, \dots, \omega_{1,d}, \omega_{2,1}, \dots, \omega_{2,d})$.

Introduce the following notations

$$\mathbf{n} = (n_1, \dots, n_d), \quad n = \sum_{i=1}^d n_i, \quad \mathbf{n}! = \prod_{i=1}^d n_i!$$

Let $(L^2) \equiv L^2(\mathcal{S}_{2d}^*(\mathbb{R}), d\mu)$ be the Hilbert space of square integrable functionals on $\mathcal{S}_{2d}^*(\mathbb{R})$ with respect to measure μ . By the Wiener-Itô-Segal isomorphism theorem, we have the chaos expansion $F(\vec{\omega}_1, \vec{\omega}_2) = \sum_{\mathbf{m} \in \mathbb{N}^d} \sum_{\mathbf{k} \in \mathbb{N}^d} \langle : \vec{\omega}^{\otimes \mathbf{m}} : \otimes : \vec{\omega}^{\otimes \mathbf{k}} : , \mathbf{f}_{\mathbf{m}, \mathbf{k}} \rangle$ for each $F \in (L^2)$.

Let $\Gamma(A)$ be the second quantization of A , where A is defined by $(Ag)_i(t) = (-\frac{d^2}{dt^2} + t^2 + 1)g_i(t)$. For each integer p , let (\mathcal{S}_p) be the completion of $\text{Dom}\Gamma(A)^p$ with respect to the Hilbert norm $\| \cdot \|_p = \| \Gamma(A)^p \cdot \|_0$, where we denote the norm of (L^2) by $\| \cdot \|_0$. Let $(\mathcal{S}) = \bigcap_{p \geq 0} (\mathcal{S}_p)$ be the projective limit of $\{(\mathcal{S}_p) \mid p \geq 0\}$, and let $(\mathcal{S})^* = \bigcup_{p \geq 0} (\mathcal{S}_{-p})$ be the inductive limit of $\{(\mathcal{S}_{-p}) \mid p \geq 0\}$. Thus there is the second Gelfand triple $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$. Elements of (\mathcal{S}) (resp. $(\mathcal{S})^*$) are called Hida testing (resp. generalized) functionals. For each test function $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) = (f_{1,1}, \dots, f_{1,d}, f_{2,1}, \dots, f_{2,d}) \in \mathcal{S}_{2d}(\mathbb{R})$, \mathcal{S} -transform of Hida generalized functionals Φ is defined by $S\Phi(\mathbf{f}) = \ll \Phi, : \exp \langle \cdot, \mathbf{f} \rangle : \gg$.

Lemma 2.1 [4] Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space, and let Φ be a mapping defined on Ω with values in $(\mathcal{S})^*$. We assume \mathcal{S} -transform of Φ :

- (1) Φ is an μ -measurable function of λ for $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R})$; and
- (2) Φ obeys a U-functional estimate

$$|S\Phi(z\mathbf{f})| \leq C_1(\lambda) \exp\{C_2(\lambda) |z|^2 |A^p \mathbf{f}|_0^2\} \quad (2.1)$$

for some fixed p and for $C_1 \in L^1(\mu)$, $C_2 \in L^\infty(\mu)$. Then

$$\int_{\Omega} \Phi d\mu(\lambda) \in (\mathcal{S})^* \text{ and } S\left(\int_{\Omega} \Phi d\mu(\lambda)\right)(\mathbf{f}) = \int_{\Omega} (S\Phi)(\mathbf{f}) d\mu(\lambda).$$

3 Local Time of Mixed Independent Bm and SfBm

Let Bm and sfBm be independent processes. For any $\varepsilon > 0$, we define

$$L_{k,\varepsilon}(T, b) = \int_0^T p_\varepsilon(\alpha B_t + \beta S_t^k - b) dt, \tag{3.1}$$

where $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\{-\frac{x^2}{2\varepsilon}\}$.

Lemma 3.1 [3] Let $k \in (-\frac{1}{2}, \frac{1}{2})$ be given. S_t^k has a continuous version of $\langle \cdot, \frac{1}{c(k)} I_{-}^k \mathbb{I}_{[0,t]}^o \rangle$, where $\mathbb{I}_{[0,t]}^o$ denotes the odd extension of $\mathbb{I}_{[0,t]}$ and $c(k) = [2(\int_0^\infty ((1+s)^k - s^k)^2 ds + \frac{1}{2k+1})]^\frac{1}{2}$.

The following lemma is very useful to prove our main results. Bender [7] gave the similar estimate in discussion the local time of fBm.

Lemma 3.2 [3] Let $k \in (-\frac{1}{2}, \frac{1}{2})$ and $f \in \mathcal{S}_1(\mathbb{R})$ be given. Then there exists a non-negative constant C_k such that

$$|\int_{\mathbb{R}} f(x) \frac{1}{c(k)} (I_{-}^k \mathbb{I}_{[0,t]}^o)(x) dx| \leq C_k |t| \|f\|, \tag{3.2}$$

where $c(k) = [2(\int_0^\infty ((1+s)^k - s^k)^2 ds + \frac{1}{2k+1})]^\frac{1}{2}$, C_k is some constant independent of f and $\|f\| \equiv \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |f'(x)|$.

Theorem 3.1 For each $k \in (-\frac{1}{2}, \frac{1}{2})$ and given $b \in \mathbb{R}$, the local time of mBs given by

$$\begin{aligned} L_{k,\varepsilon}(T, b) &= \int_0^T p_\varepsilon(\alpha B_t + \beta S_t^k - b) dt \\ &= \frac{1}{(2\pi\varepsilon)^\frac{d}{2}} \int_0^T \exp\left\{-\frac{(\alpha B_t + \beta S_t^k - b)^2}{2\varepsilon}\right\} dt \end{aligned} \tag{3.3}$$

is a Hida distribution. Moreover, S -transform of $L_{k,\varepsilon}(T, b)$ is given by

$$\begin{aligned} S(L_{k,\varepsilon}(T, b))(\mathbf{f}) &= \int_0^T \frac{1}{(2\pi(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1}))^\frac{d}{2}} \\ &\quad \cdot \exp\left\{-\frac{(\alpha \int_0^t \mathbf{f}(s) ds + \beta \int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-}^k \mathbb{I}_{[0,t]}^o)(s) ds - b)^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} dt \end{aligned} \tag{3.4}$$

for all $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) = (f_{1,1}, \dots, f_{1,d}, f_{2,1}, \dots, f_{2,d}) \in \mathcal{S}_{2d}(\mathbb{R})$.

Proof Set

$$\Phi_{k,\varepsilon}(\vec{\omega}) \equiv \left(\frac{1}{2\pi\varepsilon}\right)^\frac{d}{2} \exp\left\{-\frac{(\alpha B_t + \beta S_t^k - b)^2}{2\varepsilon}\right\},$$

where $\vec{\omega} = (\vec{\omega}_1, \vec{\omega}_2) = (\omega_{1,1}, \dots, \omega_{1,d}, \omega_{2,1}, \dots, \omega_{2,d})$. For each $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R})$, calculate S -transform of $L_{k,\varepsilon}(T, b)$ as follows

$$S(\Phi_{k,\varepsilon}(\vec{\omega}))(\mathbf{f}) = \frac{1}{(2\pi(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1}))^{\frac{d}{2}}} \cdot \exp\left\{-\frac{(\alpha \int_0^t \mathbf{f}(s) ds + \beta \int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds - b)^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}.$$

For the bounded condition we observe that

$$\begin{aligned} |S(\Phi_{k,\varepsilon}(\vec{\omega}))(z\mathbf{f})| &= \left(\frac{1}{2\pi(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right)^{\frac{d}{2}} \\ &\cdot \left|\exp\left\{-\frac{(z\alpha \int_0^t \mathbf{f}(s) ds + z\beta \int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds - b)^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}\right| \\ &\leq \left(\frac{1}{2\pi(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right)^{\frac{d}{2}} \exp\left\{\frac{b^2 + 2|z| \|b\| \alpha \left|\int_0^t \mathbf{f}(s) ds\right|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} \\ &\cdot \exp\left\{\frac{2|z| \|b\| \beta \left|\int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds\right|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} \\ &\cdot \exp\left\{\frac{2|z|^2 \alpha \|\beta\| \left|\int_0^t \mathbf{f}(s) ds\right| \left|\int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds\right|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} \\ &\cdot \exp\left\{\frac{|z|^2 \alpha^2 \left|\int_0^t \mathbf{f}(s) ds\right|^2 + |z|^2 \beta^2 \left|\int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds\right|^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} \end{aligned}$$

for any complex number z and each $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R})$. By Lemma 3.1 and Lemma 3.2, we can obtain the following inequalities

$$\begin{aligned} \exp\left\{\frac{2|z| \|a\| \beta \left|\int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds\right|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} &\leq \exp\left\{\frac{2|z| \|a\| \beta C_{k,1} t \|\mathbf{f}\|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}, \\ \exp\left\{\frac{|z|^2 \alpha^2 \left|\int_0^t \mathbf{f}(s) ds\right|^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} &\leq \exp\left\{\frac{|z|^2 \alpha^2 \|\mathbf{f}\|^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}, \\ \exp\left\{\frac{|z|^2 \beta^2 \left|\int_{\mathbb{R}} \mathbf{f}(s) (\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^o}) (s) ds\right|^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} &\leq \exp\left\{\frac{C_{k,1}^2 t^2 |z|^2 \beta^2 \|\mathbf{f}\|^2}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}, \\ \exp\left\{\frac{2|z| \|a\| \alpha \left|\int_0^t \mathbf{f}(s) ds\right|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} &\leq \exp\left\{\frac{2|z| \|a\| \alpha \|\mathbf{f}\|}{2(\varepsilon + \alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}, \end{aligned}$$

$$\begin{aligned} & \exp\left\{\frac{2|z|^2|\alpha||\beta|\int_0^t \mathbf{f}(s)ds \|\int_{\mathbb{R}} \mathbf{f}(s)(\frac{1}{c(k)}I_{c(k)}^k \mathbb{I}_{[0,t]}^o)(s)ds\|}{2(\varepsilon + \alpha^2t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} \\ \leq & \exp\left\{\frac{2C_{k,1}t|z|^2|\alpha||\beta|\|\mathbf{f}\|\|\mathbf{f}\|}{2(\varepsilon + \alpha^2t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}, \end{aligned}$$

where $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathcal{S}_{2d}(\mathbb{R})$, $\|\mathbf{f}\| \equiv \sum_{i=1}^{2d} |f_i|$,

$$\|\mathbf{f}\| \equiv \left(\sum_{i=1}^{2d} (\sup_{x \in \mathbb{R}} |f_i(x)| + \sup_{x \in \mathbb{R}} |f'_i(x)|)^2\right)^{\frac{1}{2}},$$

and $C_{k,1}$ is some constant, which depends on parameter k . Hence we obtain

$$\begin{aligned} & |S(\Phi_\varepsilon(z\mathbf{f}))| \\ \leq & \left(\frac{1}{2\pi(\alpha^2t + \beta^2(2 - 2^{2k})t^{2k+1})}\right)^{\frac{d}{2}} \exp\left\{\frac{b^2 + |z|^2\alpha^2\|f\|^2 + 2|z|\|b\|\alpha\|f\|}{2\pi(\alpha^2t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\} \\ & \cdot \exp\left\{\frac{2(|z|\|b\|\beta\|\|f\| + C_{k,1}|z|^2|\alpha||\beta|\|f\|\|f\|)t + C_{k,1}^2|z|^2\beta^2\|f\|^2t^2}{2\pi(\alpha^2t + \beta^2(2 - 2^{2k})t^{2k+1})}\right\}. \end{aligned}$$

We see that the first part in above inequality is integrable as a function on $[0, T]$, and the third part is bound on $[0, T]$. Then by an application of Lemma 2.1, the proof is completed.

Similar to [4, 5], we can prove that $L_{k,\varepsilon}(T, b)$ convergence to $L_k(T, b)$ in (L^2) . The following theorem provides the chaos expansion of the local time of mixed independent Bm and sfBm. For simplicity, we only consider the expansion of $L_k(T, 0)$ and case $a \neq 0$ is similar.

Theorem 3.2 For each $k \in (-\frac{1}{2}, \frac{1}{2})$, the local time of mBs

$$L_k(T, b) \equiv \frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} \exp\{i\lambda(\alpha B_t + \beta S_t^k - b)\} d\lambda dt$$

is a Hida distribution for $b \in \mathbb{R}$. Moreover, kernel functions of chaos expansion of $L_k(T, 0)$ are given by

$$\begin{aligned} & \mathbf{G}_{\mathbf{m},\mathbf{k}}(u_1, \dots, u_m, v_1, \dots, v_k) \\ = & \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left(\frac{1}{2}\right)^{\frac{m+k}{2}} \frac{1}{(\frac{\mathbf{m}+\mathbf{k}}{2})!} \frac{(\mathbf{m}+\mathbf{k})!}{\mathbf{m}!\mathbf{k}!} \\ & \cdot \int_0^T \left(\frac{1}{\alpha^2t + \beta^2(2 - 2^{2k})t^{2k+1}}\right)^{\frac{d+m+k}{2}} \alpha^m \prod_{i=1}^m \mathbb{I}_{[0,t]}(u_i) \beta^k \prod_{j=1}^k \left(\frac{1}{c(k)}I_{c(k)}^k \mathbb{I}_{[0,t]}^o\right)(v_j) dt \end{aligned}$$

for each $\mathbf{n} \in \mathbb{N}^d$ such that $n \geq N$. All other kernel functions $\mathbf{G}_{\mathbf{m},\mathbf{k}}$ are identically equal to zero.

Proof To show this result, we need apply Lemma 2.1 to the S -transform of the integral with respect to Lebesgue measure dt on $[0, T]$. For $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R})$ and any complex number z ,

by the definition of S -transform, these is

$$\begin{aligned}
 & S(\delta(\alpha B_t + \beta S_t^k - b))(\mathbf{f}) \\
 &= \left(\frac{1}{2\pi(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right)^{\frac{d}{2}} \\
 & \cdot \left| \exp \left\{ - \frac{(z\alpha \int_0^t \mathbf{f}(s)ds + z\beta \int_{\mathbb{R}} \mathbf{f}(s) \left(\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^k} \right)(s)ds - b)^2}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\} \right| \\
 & \leq \left(\frac{1}{2\pi(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right)^{\frac{d}{2}} \exp \left\{ \frac{b^2 + 2|z||b||\alpha| \left| \int_0^t \mathbf{f}(s)ds \right|}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\} \\
 & \cdot \exp \left\{ \frac{2|z||b||\beta| \left| \int_{\mathbb{R}} \mathbf{f}(s) \left(\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^k} \right)(s)ds \right|}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\} \\
 & \cdot \exp \left\{ \frac{2|z|^2|\alpha||\beta| \left| \int_0^t \mathbf{f}(s)ds \right| \left| \int_{\mathbb{R}} \mathbf{f}(s) \left(\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^k} \right)(s)ds \right|}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\} \\
 & \cdot \exp \left\{ \frac{|z|^2\alpha^2 \left| \int_0^t \mathbf{f}(s)ds \right|^2 + |z|^2\beta^2 \left| \int_{\mathbb{R}} \mathbf{f}(s)(\beta^2(2 - 2^{2k})t^{2k+1})(s)ds \right|^2}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\} \\
 & \leq \left(\frac{1}{2\pi(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right)^{\frac{d}{2}} \exp \left\{ \frac{a^2 + |z|^2\alpha^2 \|\mathbf{f}\|^2 + 2|z||b||\beta| C_{k,1} \|\mathbf{f}\|}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\} \\
 & \cdot \exp \left\{ \frac{C_{k,1} |z|^2\beta^2 \|\mathbf{f}\|^2 + 2|z||b||\alpha| \|\mathbf{f}\| + 2C_{k,1} |z|^2|\alpha||\beta| \|\mathbf{f}\| \|\mathbf{f}\|}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right\},
 \end{aligned}$$

where the first part is integrable as a function on $[0, T]$, and the third part is bound.

Given a $\mathbf{f} = (f_{1,1}, \dots, f_{1,d}, f_{2,1}, \dots, f_{2,d}) \in \mathcal{S}_{2d}(\mathbb{R})$, there is

$$\begin{aligned}
 & S(L(T, 0))(\mathbf{f}) \\
 &= \left(\frac{1}{\pi} \right)^{\frac{d}{2}} \int_0^T \sum_{n \geq N} \left\{ \left(\frac{1}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right)^{n + \frac{d}{2}} \right\} \sum_{n_1, \dots, n_d} \frac{1}{n_1! \dots n_d!} \prod_{i=1}^d \sum_{m_i + k_i = 2n_i} \\
 & \cdot \frac{(m_i + k_i)!}{m_i! k_i!} \alpha^{m_i} \left(\int_0^t f_i(s)ds \right)^{m_i} \beta^{k_i} \left(\int_{\mathbb{R}} f_i(s) \left(\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^k} \right)(s)ds \right)^{k_i} dt \\
 &= \left(\frac{1}{\pi} \right)^{\frac{d}{2}} \int_0^T \sum_{n \geq N} \sum_{n_1, \dots, n_d} \sum_{m_i + k_i = 2n_i} \left\{ \prod_{i=1}^d \left(\frac{1}{2(\alpha^2 t + \beta^2(2 - 2^{2k})t^{2k+1})} \right)^{\frac{1}{2} + \frac{m_i + k_i}{2}} \frac{1}{\frac{m_i + k_i!}{2}} \right\} \\
 & \cdot \left\{ \prod_{i=1}^d \frac{(m_i + k_i)!}{m_i! k_i!} \alpha^{m_i} \left(\int_0^t f_i(s)ds \right)^{m_i} \beta^{k_i} \left(\int_{\mathbb{R}} f_i(s) \left(\frac{1}{c(k)} I_{-\mathbb{I}_{[0,t]}^k} \right)(s)ds \right)^{k_i} \right\}.
 \end{aligned}$$

Comparing with the general form of the chaos expansion,

$$L_k(T, 0) = \sum_{\mathbf{m}} \sum_{\mathbf{k}} \langle : \omega_1^{\otimes \mathbf{m}} : \otimes : \omega_2^{\otimes \mathbf{k}} : , \mathbf{G}_{\mathbf{m}, \mathbf{k}} \rangle,$$

the kernel functions of chaos expansion are obtained.

4 Regularized Condition of Local Time

Next we discuss the regularized condition of local time of mBs.

Theorem 4.1 For $k \in (-\frac{1}{2}, \frac{1}{2})$, local time of mBs is a Hölder continuous function with fractional order

$$1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \},$$

i.e., there exists some constant $C_{k,2}$ such that

$$E[| L_t - L_s |] \leq C_{k,2} | t - s |^{1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}}.$$

Proof Set

$$L_t = \int_0^t \delta(\alpha B_u + \beta S_u^k) du, \quad L_{t,\varepsilon} = \int_0^t p_\varepsilon(\alpha B_u + \beta S_u^k) du.$$

For $s < t$, consider

$$\begin{aligned} & E[| L_{t,\varepsilon} - L_{s,\varepsilon} |] \\ &= \frac{1}{2\pi} \int_s^t \int_{\mathbb{R}} E[\exp\{i\xi(\alpha B_u + \beta S_u^k)\}] \exp\{-\varepsilon\xi^2\} d\xi du \\ &= \frac{1}{2\pi} \int_s^t \int_{\mathbb{R}} E[\exp\{-\frac{1}{2}\xi^2(\alpha^2 \text{Var}(B_u) + \beta^2 \text{Var}(S_u^k))\}] \exp\{-\varepsilon\xi^2\} d\xi du \\ &\leq \frac{1}{2\pi} \int_s^t \int_{\mathbb{R}} \exp\{-\frac{1}{2}\xi^2(\alpha^2 u + \beta^2(2 - 2^{2k})u^{2k+1})\} d\xi du \\ &= \frac{1}{2\pi} \int_s^t \int_{\mathbb{R}} \sqrt{\frac{2}{\alpha^2 u + \beta^2(2 - 2^{2k})u^{2k+1}}} \exp\{-x^2\} dx du \\ &= \frac{1}{\sqrt{2\pi}} \int_s^t \sqrt{\frac{1}{\alpha^2 u + \beta^2(2 - 2^{2k})u^{2k+1}}} du \\ &\leq \frac{1}{\sqrt{2\pi}} \int_s^t u^{-\frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}} du \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{t^{1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}} - s^{1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}}}{1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}} \\ &\leq C_{k,2} | t - s |^{1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}}, \end{aligned}$$

where $C_{k,2} = \frac{1}{\sqrt{2\pi(1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) t^{2k+1} \})}}$.

By Fatou's lemma, we get

$$\begin{aligned} E[| L_t - L_s |] &= E[\lim_{\varepsilon \rightarrow 0} | L_{t,\varepsilon} - L_{s,\varepsilon} |] \\ &\leq \liminf_{\varepsilon \rightarrow 0} E[| L_{t,\varepsilon} - L_{s,\varepsilon} |] \\ &\leq C_{k,2} | t - s |^{1 - \frac{1}{2} \max_{u \in (s,t)} \{ \alpha^2 u, \beta^2 (2 - 2^{2k}) u^{2k+1} \}}. \end{aligned}$$

Therefore L_t is a Hölder continuous function.

References

- [1] Liu Junfeng, Peng Zhihang, Tang Donglei. On the self-intersection local time of subfractional Brownian motion[J]. *Abstr. Appl. Anal.*, 2012, 2012: 1–27.
- [2] Yan Litan, Shen Guangjun. On the collision local time of sub-fractional Brownian motions[J]. *Stat. Prob. Lett.*, 2010, 80(5-6): 296–308.
- [3] Guo Jingjun, Xiao Yanping. On collision local time of two independent subfractional Brownian motions[J]. *Compu. Model. Engin. Scien.*, 2015, 109(6): 519–536.
- [4] Oliveira M, Silva J, Streit L. Intersection local times of independent fractional Brownian motions as generalized white noise functionals[J]. *Acta Appl. Math.*, 2011, 113(1): 17–39.
- [5] Wang Xiangjun, Guo Jingjun, Jiang Guo. Collision local times of two independent fractional Brownian motions[J]. *Front. Math. China*, 2011, 6(2): 325–338.
- [6] Guo Jingjun, Jiang Guo, Xiao Yanping. Multiple intersection local time of fractional Brownian motion[J]. *J. Math.*, 2011, 31(3): 388–394.
- [7] Bender C. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter[J]. *Stoc. Proc. Appl.*, 2003, 104(1): 81–106.

布朗运动和次分数布朗运动混合的局部时

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摘要: 本文研究了布朗运动和次分数布朗运动混合的局部时问题. 利用白噪声分析方法和次分数布朗运动的另一种表示形式, 证明了该局部时是一个Hida广义泛函. 进一步, 借助于 S -变换给出了该局部时的混沌表示. 最后获得了该局部时的正则性条件. 推广了布朗运动局部时的一些结果.

关键词: 局部时; 次分数布朗运动; Hida广义泛函

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