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ORE EXTENSIONS OVER (α, δ)-WEAKLY RIGID RINGS

WANG Yao[1], ZHANG Jiu-lin[1], REN Yan-li[2]

(*1.School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*)

(*2.School of Mathematics and Information Technology, Nanjing Xiaozhuang University, Nanjing 211171, China*)

Abstract: In this paper, we investigate the weak symmetric, weak zip, nilpotent p.p. and nilpotent Baer property of the Ore extension $R[x;\alpha,\delta]$ of a ring R, respectively. By using the itemized analysis method on polynomials, we prove that if R is (α,δ)-weakly rigid and semicommutative, then $R[x;\alpha,\delta]$ is weak symmetric (resp., weak zip, nilpotent p.p., nilpotent Baer) if and only if R is weak symmetric (resp., weak zip, nilpotent p.p., nilpotent Baer). These results unify and extend nontrivially the previously known results.

Keywords: (α,δ)-weakly rigid ring; Ore extension; weak symmetric ring; weak zip ring; nilpotent p.p.-ring; nilpotent Baer ring

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1 Introduction

Throughout this paper, R denotes an associative ring with identity, $\alpha : R \to R$ is an endomorphism, and δ is an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for all $a, b \in R$. If $\alpha = id$ is identity endomorphism, then an *id*-derivation of R is directly referred to as a derivation of R. We denote $S = R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x+\delta(a)$ for any $a \in R$. In the special case that $\delta = 0$, we call $S = R[x;\alpha]$ the Ore extension of endomorphism type, or skew polynomial ring; and when $\alpha = 0$, we call $S = R[x;\delta]$ the Ore extension of derivation type, or differential polynomial ring. According to Krempa [1], an endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring R is called α-rigid if there exists a rigid endomorphism α of R. Any rigid endomorphism of a ring R is a monomorphism and α-rigid rings are reduced. Following [2], a ring R is called α-compatible if for each $a, b \in R$, $ab = 0$ if

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Biography: Wang Yao (1962–), male, born at Hailun, Heilongjiang, professor, major in associative rings and associative algebras.

Corresponding author: Ren Yanli.

and only if $a\alpha(b) = 0$. Moreover, R is called δ-compatible if for each $a, b \in R$, $ab = 0$ implies $a\delta(b) = 0$. If R is both α-compatible and δ-compatible, then R is called (α, δ)-compatible. α-rigid rings and (α, δ)-compatible rings were investigate by many authors (see [1, 3–6]). From Hashemi and Moussavi [4], a ring R is α-rigid if and only if R is (α, δ)-compatible and reduced. According to Nasr-isfahani and Moussavi [7], a ring R with a monomorphism α is called α-weakly rigid if for each $a, b \in R$, $aRb = 0$ holds iff $a\alpha(Rb) = 0$. Moreover, R is called δ-weakly rigid if for each $a, b \in R$, $aRb = 0$ implies $a\delta(b) = 0$. A ring R with a monomorphism α and an α-derivation δ is called (α, δ)-weakly rigid if it is both α-weakly rigid and δ-weakly rigid. Every (α, δ)-compatible ring is (α, δ)-weakly rigid. Notice that the class (α, δ)-compatible rings is too narrow, and there are many rich classes of (α, δ)-weakly rigid rings (see [7]). We will provide an example which is an (α, δ)-weakly rigid ring but is not an (α, δ)-compatible ring.

Recall that a ring R is called reversible if for all $a, b \in R$, $ab = 0$ implies $ba = 0$, R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$, and R is called weak symmetric if $abc \in \mathrm{nil}(R)$ implies $acb \in \mathrm{nil}(R)$ for all $a, b, c \in R$, where $\mathrm{nil}(R)$ denotes the set of all nilpotent elements in R. For a nonempty subset $X \subseteq R$, $r_R(X) = \{r \in R | Xr = 0\}$ and $l_R(X) = \{r \in R \mid rX = 0\}$ are called the right annihilator of X in R and the left annihilator of X in R, respectively. Faith in [8] called a ring R right zip if the right annihilator of a subset X of R is zero, $r_R(X) = 0$, then $r_R(Y) = 0$ for a finite subset $Y \subseteq X$. Put $Nr_R(X) = \{a \in R \mid xa \in \mathrm{nil}(R), \text{for all } x \in X\}$ and $Nl_R(X) = \{b \in R \mid bx \in \mathrm{nil}(R), \text{for all } x \in X\}$. A ring R is called right weak zip provided that $Nr_R(X) \subseteq \mathrm{nil}(R)$, where X is a subset of R, then there exists a finite subset $Y \subseteq X$ such that $Nr_R(Y) \subseteq \mathrm{nil}(R)$. Left weak zip rings are defined similarly. If a ring R is both left and right weak zip, then R is called weak zip. Notice that by a simple computation we can see that $Nr_R(X) = Nl_R(X) = N_R(X)$ and hence the weak zip property is left-right symmetric. We call a ring R weak zip provided that $N_R(X) \subseteq \mathrm{nil}(R)$, where X is a subset of R, then there exists a finite subset $Y \subseteq X$ such that $N_R(Y) \subseteq \mathrm{nil}(R)$. Obviously, for any nonempty subset X of a ring R, we have $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$. We call $N_R(X)$ the nilpotent annihilator of X in R. A nilpotent annihilator is a nature and non-trivial generalization of an annihilator. Notice that if R is semicommutative, then $N_R(X)$ and $\mathrm{nil}(R)$ are ideals of R, and if R is reduced, then $r_R(X) = N_R(X) = l_R(X)$. In [9], Kaplansky introduced the Baer rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Closely related to Baer rings are p.p.-rings. A ring R is called a right p.p.-ring if the right annihilator for each element of R is generated by an idempotent. A ring R is called nilpotent p.p.-ring if for any element $p \in R$ with $N_R(p) \neq R$, $N_R(p)$ is generated as a right ideal by a nilpotent element. We call a ring R nilpotent Baer ring, if for any non-empty subset $X \subseteq R$ with $N_R(X) \neq R$, $N_R(X)$ is generated as a right ideal by a nilpotent element.

Due to the fact that many of the quantized algebras and their representations can be expressed in terms of iterated skew polynomial rings, it is interesting to know if the general Ore extension $S = R[x; \alpha, \delta]$ of a ring R share the same property with the ring R. We investigate in this note the weak symmetric, weak zip, nilpotent p.p. and nilpotent Baer property of the Ore extension $R[x;\alpha,\delta]$ of a ring R, respectively.

2 Ore Extensions over (α,δ)-Weakly Rigid Rings

Let α be an endomorphism and δ an α-derivation of a ring R. It is easy to see that for any subring S of the full matrix ring $M_n(R)$, $\overline{\alpha}: S\to S$, given by $\overline{\alpha}((a_{ij}))=(\alpha(a_{ij}))$, is a homomorphism, and $\overline{\delta}: S\to S$, given by $\overline{\delta}((a_{ij}))=(\delta(a_{ij}))$, for each $(a_{ij})\in S$, is an $\overline{\alpha}$-derivation.

The following example shows that there exists an (α,δ)-weakly rigid ring which is not (α,δ)-compatible.

Example 2.1 Let D be a domain and α be the automorphism of the polynomial ring $R=D[x_1,x_2,\cdots,x_m]$, with indeterminates $x_1,x_2\cdots,x_m$, given by $\alpha(x_i)=x_{i+1}$ for $1\leqslant i\leqslant m-1$ and $\alpha(x_m)=x_1$. Then R is α-rigid, and hence R is (α,δ)-weakly rigid. From Corollary 2.5 in [7], we get $M_n(R)$ is $(\overline{\alpha},\overline{\delta})$-weakly rigid. Taking

$$A=\begin{pmatrix} x_1 & x_2\\ 0 & 0\end{pmatrix},\quad B=\begin{pmatrix} 0 & x_2\\ 0 & -x_1\end{pmatrix},$$

we have $AB=\begin{pmatrix} 0 & x_1x_2-x_2x_1\\ 0 & 0\end{pmatrix}=0$, but

$$A\overline{\alpha}(B)=\begin{pmatrix} x_1 & x_2\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & x_3\\ 0 & -x_2\end{pmatrix}=\begin{pmatrix} 0 & x_1x_3-x_2x_2\\ 0 & 0\end{pmatrix}\neq 0.$$

Hence $M_n(R)$ is not $(\overline{\alpha},\overline{\delta})$-compatible.

Proposition 2.2 Let R be a semicommutative ring. If R is (α,δ)-weakly rigid, then R is (α,δ)-compatible.

Proof Suppose $ab=0$ for $a,b\in R$. Then we have $aRb=0$ since R is semicommutative. This implies that $a\delta(b)=0$ and $a\alpha(Rb)=0$ since R is (α,δ)-weakly rigid. It follows that $a\alpha(b)=0$. On the other hand, if $a\alpha(b)=0$ for $a,b\in R$, then $aR\alpha(b)=0$ since R is semicommutative. This implies that $a\alpha(Rb)=a\alpha(R)\alpha(b)=0$ since $\alpha(R)\alpha(b)\subseteq R\alpha(b)$. It follows that $aRb=0$, and hence $ab=0$. Therefore, R is (α,δ)-compatible.

A ring R is called quasi-IFP, provided that $\sum\limits_{i=0}^{n} Ra_iR$ is nilpotent whenever $\sum\limits_{i=0}^{n} a_ix^i\in R[x]$ is nilpotent. Semicommutative rings are quasi-IFP. From Theorem 2.2 in [10] and Proposition 2.2 we have the following

Theorem 2.3 Let α be a monomorphism and δ an α-derivation of a ring R, and $f(x)=a_0+a_1x+\cdots+a_nx^n$. If R is (α,δ)-weakly rigid and semicommutative, then $f(x)\in \mathrm{nil}(R[x;\alpha,\delta])$ if and only if $a_i\in \mathrm{nil}(R)$ for all $0\leq i\leq n$. That is, we have

$$\mathrm{nil}(R[x;\alpha,\delta])=\mathrm{nil}(R)[x;\alpha,\delta].$$

Proposition 2.4 Let α be a monomorphism and δ an α-derivation of a ring R. Then R is α-rigid if and only if R is (α,δ)-weakly rigid and reduced.

Proof If R is α-rigid, then R is (α,δ)-compatible and reduced from Lemma 2.2 in [4]. Clearly, (α,δ)-compatible rings are (α,δ)-weakly rigid. So R is (α,δ)-weakly rigid. Conversely, if R is (α,δ)-weakly rigid and reduced, then R is semicommutative, and hence R is (α,δ)-compatible by Proposition 2.2. Thus we have that R is α-rigid since R is (α,δ)-compatible and reduced from Lamma 2.2 in [4].

Proposition 2.5 Let α be a monomorphism and δ an α-derivation of a ring R. If R is (α,δ)-weakly rigid and semicommutative, then we have the following

(1) if $ab=0$, then $a\alpha^n(b)=0$, $\alpha^m(a)b=0$ for all positive integers m,n;

(2) if $\alpha^k(a)b=0$ for some positive integer k, then $ab=0$;

(3) if $a\alpha^s(b)=0$ for some positive integer s, then $ab=0$;

(4) if $ab=0$, then $\alpha^n(a)\delta^m(b)=0$ and $\delta^s(a)\alpha^t(b)=0$ for all nonnegative integers m,n,s,t.

Proof (1) If $ab=0$, then $aRb=0$ since R is semicommutative. It follows that $a\alpha(Rb)=0$ by the definition of (α,δ)-weakly rigid, and hence $a\alpha(b)=0$. Since R is (α,δ)-compatible, by Proposition 2.2 we have $a\alpha^2(b)=0,\cdots,a\alpha^n(b)=0$ for any positive integer n. On the other hand, if $ab=0$, then $aRb=0$, whence $\alpha(a)\alpha(Rb)=\alpha(aRb)=0$. This implies $\alpha(a)Rb=0$ by the definition of (α,δ)-weakly rigid. It follows that $\alpha(a)b=0$. Continuing this procedure yields that $\alpha^m(a)b=0$ for any positive integer m.

(2) If $\alpha^k(a)b=0$ for some positive integer k, then $\alpha^k(ab)=\alpha^k(a)\alpha^k(b)=0$ by (1). So $ab=0$ since α is a monomorphism.

(3) If $a\alpha^s(b)=0$ for some positive integer s, then $\alpha^s(ab)=\alpha^s(a)\alpha^s(b)=0$ by (1). It follows that $ab=0$ since α is a monomorphism.

(4) If $ab=0$, then $aRb=0$, and hence $a\delta(b)=0$ since R is (α,δ)-weakly rigid. Continuing this procedure, we have $a\delta^m(b)=0$ for any positive integer m. So $\alpha^n(a)\delta^m(b)=0$ by (1) for any non-negative integers m,n. On the other hand, if $ab=0$, then $\delta(ab)=0$ and $\alpha(a)\delta(b)=0$ by (1) and the definition of (α,δ)-compatible. Since $\delta(a)b=\delta(ab)-\alpha(a)\delta(b)$, $\delta(a)b=0$. Continuing this procedure, we have $\delta^s(a)b=0$ for any positive integer s. This implies $\delta^s(a)\alpha^t(b)=0$ by (1) for all nonnegative integers s,t.

In the following, let α be an endomorphism of R and δ be an α-derivation of R. For integers i,j with $0\le i\le j$, $f_i^j\in \mathrm{End}(R,+)$ will denote the map which is the sum of all possible words in α, δ built with i letters α and $j-i$ letters δ. By Lemma 4.1 in [11], for any positive integer n and $r\in R$, we have $x^nr=\sum\limits_{i=0}^{n} f_i^n(r)x^i$ in the ring $R[x;\alpha,\delta]$. For a subset I of R, we denote by $I[x;\alpha,\delta]$ the subset of $R[x;\alpha,\delta]$, where the coefficients of elements in $I[x;\alpha,\delta]$ are in subset I.

Proposition 2.6 Let α be a monomorphism and δ an α-derivation of a ring R. If R is (α,δ)-weakly rigid and semicommutative, then we have the following:

(1) $ab=0$ implies $af_i^j(b)=0$ for all $0\le i\le j$ and $a,b\in R$;

(2) $abc=0$ implies $a\delta(b)c=0$ for all $a,b,c\in R$;

(3) $abc = 0$ implies $af_i^j(b)c = 0$ for all $0 \le i \le j$ and $a, b, c \in R$;

(4) $ab \in \mathrm{nil}(R)$ implies $af_i^j(b) \in \mathrm{nil}(R)$ for all $0 \le i \le j$ and $a, b \in R$;

(5) $a\alpha^m(b) \in \mathrm{nil}(R)$ implies $ab \in \mathrm{nil}(R)$ for all $m \ge 0$ and $a, b \in R$.

Proof (1) R is (α, δ)-compatible by Proposition 2.2. If $ab = 0$, then $a\alpha^i(b) = 0$ and $a\delta^j(b) = 0$ for all $i \ge 0$ and $j \ge 0$. So $af_i^j(b) = 0$ for all $0 \le i \le j$.

(2) If $abc = 0$, then $0 = a\delta(bc) = a(\delta(b)c + \alpha(b)\delta(c)) = a\delta(b)c + a\alpha(b)\delta(c)$ by Proposition 2.5 (4). On the other hand, we have $\alpha(ab)\delta(c) = 0$ by Proposition 2.5 (4), and hence $a\alpha(b)\delta(c) = 0$ by Proposition 2.5 (2). These implies $a\delta(b)c = 0$.

(3) If $abc = 0$, then $a\delta(b)c = 0$ by (2). Continuing this procedure, we have $a\delta^i(b)c = 0$ for all $i \ge 0$. On the other hand, $abc = 0$ implies $\alpha(abc) = \alpha(a)\alpha(b)\alpha(c) = 0$. It follows that $a\alpha(b)c = 0$ by Proposition 2.5. Continuing this procedure yields that $a\alpha^j(b)c = 0$ for all $j \ge 0$. So $af_i^j(b)c = 0$.

(4) If $ab \in \mathrm{nil}(R)$, then there exists some positive integer k such that $(ab)^k = abab\cdots ab = 0$, and hence $af_i^j(b)af_i^j(b)\cdots af_i^j(b) = 0$ by (3). Thus we have $(af_i^j(b))^k = 0$, and hence $af_i^j(b) \in \mathrm{nil}(R)$.

(5) If $a\alpha^m(b) \in \mathrm{nil}(R)$, then there exists some positive integer k such that $(a\alpha^m(b))^k = 0$, that is $a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b) = 0$. Using freely Proposition 2.5(1) and (3), we get $\alpha^m(a)\alpha^m(b)\alpha^m(a)\alpha^m(b)\cdots\alpha^m(a)\alpha^m(b) = 0$. This implies $abab\cdots ab = (ab)^k = 0$ since α is a monomorphism. So $ab \in \mathrm{nil}(R)$.

Lemma 2.7 Let α be a monomorphism and δ an α-derivation of a ring R. If R is an (α, δ)-weakly rigid and semicommutative ring, then for

$$f(x) = \sum_{i=0}^{m} a_i x^i, \quad g(x) = \sum_{j=0}^{n} b_j x^j, \quad h(x) = \sum_{k=0}^{p} c_k x^k \in R[x; \alpha, \delta]$$

and $c \in R$, we have the following

(1) $fg \in \mathrm{nil}(R[x; \alpha, \delta])$ if and only if $a_i b_j \in \mathrm{nil}(R)$ for all $0 \le i \le m$, $0 \le j \le n$;

(2) $fgc \in \mathrm{nil}(R[x; \alpha, \delta])$ if and only if $a_i b_j c \in \mathrm{nil}(R)$ for all $0 \le i \le m$, $0 \le j \le n$;

(3) $fgh \in \mathrm{nil}(R[x; \alpha, \delta])$ if and only if $a_i b_j c_k \in \mathrm{nil}(R)$ for all $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le p$.

Proof Since R is (α, δ)-weakly rigid and semicommutative, then R is (α, δ)-compatible by Proposition 2.2. By analogy with the proof of Proposition 4.4 in [12], the proofs carry over mutatis mutandis for these conclusions by using of Lemma 2.6(4), Lemma 2.6(5) and Theorem 2.3.

Theorem 2.8 Let α be a monomorphism and δ an α-derivation of a ring R. If R is (α, δ)-weakly rigid and semicommutative, then $R[x; \alpha, \delta]$ is weak symmetric if and only if R is weak symmetric.

Proof ($\Leftarrow$) Let

$$f(x) = \sum_{i=0}^{m} a_i x^i, \quad g(x) = \sum_{j=0}^{n} b_j x^j, \quad h(x) = \sum_{k=0}^{p} c_k x^k \in R[x; \alpha, \delta]$$

such that $fgh \in \mathrm{nil}(R[x;\alpha,\delta])$. Then we get $a_ib_jc_k \in \mathrm{nil}(R)$ for all i, j, k by Lemma 2.7(3). Since R is weak symmetric, $a_ic_kb_j \in \mathrm{nil}(R)$. By Lemma 2.7(3), we have $fhg \in \mathrm{nil}(R[x;\alpha,\delta])$. Therefore, $R[x;\alpha,\delta]$ is weak symmetric.

($\Rightarrow$) Since any subring of a weak symmetric ring is weak symmetric, the result follows.

Hirano [13] observed the relations between annihilators in a ring R and annihilators in $R[x]$. In the following we investigate the relations between weak annihilators in a ring R and weak annihilators in Ore extension ring $S = R[x;\alpha,\delta]$. Given a polynomial $h(x) = a_0+a_1x+\cdots+a_nx^n \in R[x;\alpha,\delta]$, we denote by C_h the subset of R consisting of the coefficients of $h(x)$, and for a subset $V \subseteq R[x;\alpha,\delta]$, $C_V = \cup_{h\in V} C_h$. Given a ring R, we define

$$N\mathrm{Ann}_R(2^R) = \{N_R(U) \mid U \subseteq R\},\ \ N\mathrm{Ann}_S(2^S) = \{N_S(V) \mid V \subseteq S\}.$$

Clearly, if $U_1 \subseteq U_2$, we have $N_R(U_1) \supseteq N_R(U_2)$.

Lemma 2.9 Let α be a monomorphism and δ an α-derivation of a ring R. If R is an (α,δ)-weakly rigid and semicommutative ring, then for any subet $U \subseteq R$, we have $N_S(U) = N_R(U)[x;\alpha,\delta]$.

Proof For any $f(x) = a_0 + a_1x + \cdots + a_nx^n \in N_S(U)$, we have $uf(x) \in \mathrm{nil}(S)$ for all $u \in U$, and hence $ua_i \in \mathrm{nil}(R)$ for $0 \le i \le n$ by Theorem 2.3. This implies that $a_i \in N_R(U)$ and hence $f(x) \in N_R(U)[x;\alpha,\delta]$. So $N_S(U) \subseteq N_R(U)[x;\alpha,\delta]$.

Conversely, for any $f(x) = a_0 + a_1x + \cdots + a_nx^n \in N_R(U)[x;\alpha,\delta]$, we have $ua_i \in \mathrm{nil}(R)$ for $0 \le i \le n$ and any $u \in U$. Thus $uf(x) \in \mathrm{nil}(R)[x;\alpha,\delta]$. By Theorem 2.3, we obtain $uf(x) \in \mathrm{nil}(S)$, and hence $f(x) \in N_S(U)$. Therefore, $N_S(U) \supseteq N_R(U)[x;\alpha,\delta]$.

Lemma 2.10 Let α be a monomorphism and δ an α-derivation of a ring R. If R is an (α,δ)-weakly rigid and semicommutative ring, then $\phi : N\mathrm{Ann}_R(2^R) \to N\mathrm{Ann}_S(2^S)$ defined by $\phi(I) = I[x;\alpha,\delta]$ for every $I \in N\mathrm{Ann}_R(2^R)$ is bijective.

Proof We know that ϕ is well defined by Lemma 2.9. Obviously, ϕ is injective. In the following, we show that ϕ is surjective. Let $N_S(V) \in N\mathrm{Ann}_S(2^S)$ and $g(x) = \sum\limits_{j=0}^{n} b_jx^j \in N_S(V)$, where $V \subseteq S$. Then for any $f(x) = \sum\limits_{i=0}^{m} a_ix^i \in V$, we have $f(x)g(x) \in \mathrm{nil}(S)$. By Lemma 2.7(1), we get $a_ib_j \in \mathrm{nil}(R)$ for each i, j. This implies $b_j \in N_R(C_v)$ for all $0 \le j \le n$, and hence $g(x) \in N_R(C_v)[x;\alpha,\delta]$. It shows that $N_S(V) \subseteq N_R(C_v)[x;\alpha,\delta]$.

On the other hand, if $g(x) = \sum\limits_{j=0}^{n} b_jx^j \in N_R(C_v)[x;\alpha,\delta]$, then $b_j \in N_R(C_v)$ for all $0 \le j \le n$. Thus for all $f(x) = \sum\limits_{i=0}^{m} a_ix^i \in V$, we have $a_ib_j \in \mathrm{nil}(R)$ for each i, j. By Lemma 2.7(1), we get $f(x)g(x) \in \mathrm{nil}(S)$. It follows that $g(x) \in N_S(V)$ and $N_R(C_v)[x;\alpha,\delta] \subseteq N_S(V)$. Therefore $N_S(V) = N_R(C_v)[x;\alpha,\delta] = \phi(N_R(C_v))$. This proves that ϕ is surjective, and hence ϕ is bijective.

Theorem 2.11 Let α be a monomorphism and δ an α-derivation of a ring R. If R is (α,δ)-weakly rigid and semicommutative, then $R[x;\alpha,\delta]$ is weak zip if and only if R is weak zip.

Proof ($\Leftarrow$) Suppose that R is weak zip. Let $X \subseteq S = R[x;\alpha,\delta]$ such that $N_S(X) \subseteq \mathrm{nil}(S)$. For all $r \in N_R(C_X)$ and $a \in C_X$, we have $ar \in \mathrm{nil}(R)$. Then by Proposition 2.6(4) we obtain $af_i^j(r) \in \mathrm{nil}(R)$ for any $0 \le i \le j$. For any $f(x) = \sum\limits_{i=0}^{m} a_i x^i \in X$, we have

$$\begin{aligned} f(x)r &= (\sum_{i=0}^{m} a_i x^i)r \\ &= \sum_{i=0}^{m} a_i f_0^i(r) + (\sum_{i=1}^{m} a_i f_1^i(r))x + \cdots + (\sum_{i=s}^{m} a_i f_s^i(r))x^s + \cdots + a_m\alpha^m(r)x^m \\ &= \Delta_0 + \Delta_1 x + \cdots + \Delta_s x^s + \cdots + \Delta_m x^m. \end{aligned}$$

Since R is semicommutative, then $\mathrm{nil}(R)$ is an ideal. It implies that $\Delta_j \in \mathrm{nil}(R)$ for all $0 \le j \le m$, and hence $f(x)r \in \mathrm{nil}(S)$ by Theorem 2.3. So $r \in N_S(X) \subseteq \mathrm{nil}(S)$ and $N_R(C_X) \subseteq \mathrm{nil}(R)$. Since R is weak zip, there exists a finite subset $Y' \subseteq C_X$ such that $N_R(Y') \subseteq \mathrm{nil}(R)$. For each $b \in Y'$, we can find $g_b(x) \in X$ such that some of the coefficients of $g_b(x)$ are b. Let X' be a minimal subset of X such that $g_b(x) \in X'$ for each $b \in Y'$. Clearly $Y' \subseteq C_{X'}$, so $N_R(C_{X'}) \subseteq N_R(Y') \subseteq \mathrm{nil}(R)$. For any $f(x) = \sum\limits_{i=0}^{m} a_i x^i \in N_S(X')$ and $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in X'$, we have $g(x)f(x) \in \mathrm{nil}(S)$. By Lemma 2.7(1), we obtain $b_j a_i \in \mathrm{nil}(R)$ for each i,j. Thus $a_i \in N_R(Y') \subseteq \mathrm{nil}(R)$ for $0 \le i \le m$. By Theorem 2.3, we have $f(x) \in \mathrm{nil}(S)$ and $N_S(X') \subseteq \mathrm{nil}(S)$. Therefore, S is a weak zip ring.

($\Rightarrow$) Suppose that $S = R[x;\alpha,\delta]$ is weak zip. Let X be a subset of R such that $N_R(X) \subseteq \mathrm{nil}(R)$ and $f(x) = \sum\limits_{i=0}^{m} a_i x^i \in N_S(X)$. By Lemma 2.9, we have $a_i \in N_R(X) \subseteq \mathrm{nil}(R)$ for all $0 \le i \le m$. This implies $f(x) \in \mathrm{nil}(S)$ and $N_S(X) \subseteq \mathrm{nil}(S)$ by Theorem 2.3. Since S is weak zip, there exists a finite set $X' \subseteq X$ such that $N_S(X') \subseteq \mathrm{nil}(S)$. Thus $N_R(X') = N_S(X') \cap R \subseteq \mathrm{nil}(R)$, and hence R is a weak zip ring.

Corollary 2.12 (1) Let α be a monomorphism of a ring R. If R is an α-weakly rigid and semicommutative ring, then $R[x;\alpha]$ is weak zip if and only if R is weak zip;

(2) Let δ be a derivation of a ring R. If R is δ-weakly rigid and semicommutative, then $R[x;\delta]$ is weak zip if and only if R is weak zip;

(3) Let R is a semicommutative ring. Then $R[x]$ is weak zip if and only if R is weak zip.

Theorem 2.13 Let R be an (α,δ)-weakly rigid and semicommutative ring. Then $R[x;\alpha,\delta]$ is nilpotent p.p. if and only if R is nilpotent p.p..

Proof ($\Leftarrow$) Let $f(x) = \sum\limits_{i=0}^{m} a_i x^i \in S = R[x;\alpha,\delta]$ and $N_S(f(x)) \neq S$. Suppose $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in N_S(f(x))$. Then $f(x)g(x) \in \mathrm{nil}(S)$. By Lemma 2.7(1), we have $a_i b_j \in \mathrm{nil}(R)$ for each i,j, and hence $b_j \in N_R(a_i)$. If $N_R(a_i) = R$ for each i, then $a_i c_k \in \mathrm{nil}(R)$ for any $h(x) = \sum\limits_{k=0}^{l} c_k x^k \in S$ and any i,k. It implies that $f(x)h(x) \in \mathrm{nil}(S)$ by Lemma 2.7(1), and hence $h(x)\in N_S(f(x))$. Thus, we get $N_S(f(x))=S$, a contradiction. Therefore, there exists an $a_i\in C_f$ such that $N_R(a_i)\neq R$. By the definition of nilpotent p.p. rings, we have $N_R(a_i)=uR$ with $u\in \text{nil}(R)$, $b_j=ur_j$ with $r_j\in R$ and $g(x)=\sum\limits_{j=0}^{n}ur_jx^j=u\sum\limits_{j=0}^{n}r_jx^j\in uS$. This gives $N_S(f(x))\subseteq uS$. On the other hand, for any $p(x)=\sum\limits_{k=0}^{s}c_kx^k\in S=R[x;\alpha,\delta]$, we have $a_iuc_k\in \text{nil}(R)$ for each i,k since $\text{nil}(R)$ is an ideal. This implies $f(x)\cdot up(x)\in \text{nil}(S)$ by Lemma 2.7(1). It follows that $uS\subseteq N_S(f(x))$ and $N_S(f(x))=uS$, where $u\in\text{nil}(S)$. Therefore, S is a nilpotent p.p.-ring.

($\Rightarrow$) Suppose that $S=R[x;\alpha,\delta]$ is a nilpotent p.p.-ring and p is a element of R with $N_R(p)\neq R$. If $N_S(p)=S$, then $N_R(p)=N_S(p)\cap R=R$, which is a contradiction. Hence $N_S(p)\neq S$, and $N_S(p)=f(x)\cdot S$ since S is a nilpotent p.p.-ring, where $f(x)=a_0+a_1x+\cdots+a_nx^n\in\text{nil}(S)$. By Theorem 2.3, we have $a_i\in\text{nil}(R)$ for $0\leq i\leq n$. Since $\text{nil}(R)$ is an ideal of R, $pa_0r\in\text{nil}(R)$ for each $r\in R$. Thus $a_0r\in N_R(p)$, and hence $a_0\cdot R\subseteq N_R(p)$. On the other hand, assume $b\in N_R(p)$. Then $b\in N_S(p)$. It follows that there exists $h(x)=c_0+c_1x+\cdots+c_nx^n\in S$ such that $b=f(x)\cdot h(x)$. Thus we have $b=a_0c_0$, $a_0\cdot R\supseteq N_R(p)$ and $N_R(p)=a_0\cdot R$. So R is a nilpotent p.p.-ring.

Similar to the definition of nilpotent p.p.-rings, we call a ring R nilpotent Baer ring, if for any nonempty subset $X\subseteq R$ with $N_R(X)\neq R$, $N_R(X)$ is generated as a right ideal by a nilpotent element.

Example 2.14 For a domain R and positive integer n, consider the following set of triangular matrices

$$T_n(R,n)=\left\{\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ 0 & a_0 & a_1 & \dots & a_{n-2}\\ 0 & 0 & a_0 & \dots & a_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a_0 \end{pmatrix} |a_i\in R,\ i=0,1,\cdots,n-1\right\}$$

with $n\geq 2$. It is easy to see that $T_n(R,n)$ is a subring of the triangular matrix ring, with matrix addition and multiplication. We denote elements of $T_n(R,n)$ by $(a_0,a_1,\cdots,a_{n-1})$, then $T_n(R,n)$ is a ring with addition pointwise and multiplication given by $(a_0,a_1,\cdots,a_{n-1})\cdot(b_0,b_1,\cdots,b_{n-1})=(a_0b_0,a_1b_2+a_2b_1,\cdots,a_1b_n+a_2b_{n-1}+\cdots+a_nb_1)$ for each $a_i,b_j\in R$. Let $X\subseteq T_n(R,n)$ with $N_R(X)\neq T_n(R,n)$. If $X_0=\{a_0\mid(a_0,a_1,\cdots,a_{n-1})\in X\}=\{0\}$, then $N_{T_n(R,n)}(X)=T_n(R,n)$. This is contrary to the fact that $N_R(X)\neq T_n(R,n)$. Thus there exists an $(a_0,a_1,\cdots,a_{n-1})\in X$ with $a_0\neq 0$. Since R is a domain, $N_{T_n(R,n)}(X)=\{(0,b_1,\cdots,b_{n-1})\mid b_i\in R\}=(0,1,0,\cdots,0)\cdot T_n(R,n)$, where $(0,1,0,\cdots,0)$ is a nilpotent element of $T_n(R,n)$. This shows that $T_n(R,n)$ is a nilpotent Baer ring.

Theorem 2.15 Let R be an (α,δ)-weakly rigid and semicommutative ring. Then $R[x;\alpha,\delta]$ is nilpotent Baer if and only if R is nilpotent Baer.

Proof ($\Leftarrow$) Let $\varnothing\neq X\subseteq S=R[x;\alpha,\delta]$ and $N_S(X)\neq S$. Suppose $g(x)=\sum\limits_{j=0}^{n}b_jx^j\in$

$N_S(X)$. Then $f(x)g(x)\in \mathrm{nil}(S)$ for any $f(x)=\sum\limits_{i=0}^{m}a_ix^i\in X$. By Lemma 2.7(1), we have $a_ib_j\in \mathrm{nil}(R)$ for each i,j and $b_j\in N_R(C_X)$ for all j. If $Nr_R(C_X)=R$, then $a_ic_k\in \mathrm{nil}(R)$ for any $h(x)=\sum\limits_{k=0}^{l}c_kx^k\in S$ and any i,k. It follows that $f(x)h(x)\in \mathrm{nil}(S)$ by Lemma 2.7(1), and hence $h(x)\in N_S(X)$. Thus we get $N_S(X)=S$, a contradiction. So $N_R(C_X)\neq R$. By the definition of nilpotent Baer rings, we have $N_R(C_X)=wR$ with $w\in \mathrm{nil}(R)$, $b_j=wr_j$ with $r_j\in R$ and $g(x)=\sum\limits_{j=0}^{n}wr_jx^j=w\sum\limits_{j=0}^{n}r_jx^j\in wS$. This implies $N_S(X)\subseteq wS$.

On the other hand, for any $p(x)=\sum\limits_{k=0}^{s}c_kx^k\in S$ and $f(x)=\sum\limits_{i=0}^{m}a_ix^i\in X$, we have $a_iwc_k\in \mathrm{nil}(R)$ for each i,k since $\mathrm{nil}(R)$ is an ideal. This implies that $f(x)\cdot wp(x)\in \mathrm{nil}(S)$ by Lemma 2.7(1). Thus we have $wS\subseteq N_S(X)$ and $N_S(X)=wS$ with $w\in \mathrm{nil}(S)$. So S is a nilpotent Baer ring.

($\Rightarrow$) It is similar to the proof of Theorem 2.13.

Corollary 2.16 (1) Let α be a monomorphism of a ring R. If R is α-weakly rigid and semicommutative, then $R[x;\alpha]$ is nilpotent Baer if and only if R is nilpotent Baer;

(2) Let δ be a derivation of a ring R. If R is δ-weakly rigid and semicommutative, then $R[x;\delta]$ is nilpotent Baer if and only if R is nilpotent Baer;

(3) Let R be a semicommutative ring. Then $R[x]$ is nilpotent Baer if and only if R is nilpotent Baer.

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(α,δ) -弱刚性环上的 Ore 扩张

王 尧[1], 张玖琳[1], 任艳丽[2]

(1.南京信息工程大学数学与统计学院, 江苏 南京 210044)

(2.南京晓庄学院数学与信息技术学院, 江苏 南京 211171)

摘要: 本文研究 (α,δ) -弱刚性环上的 Ore 扩张环 $R[x;\alpha,\delta]$ 的弱对称性、弱 zip 性、幂零 p.p. 性和幂零 Baer 性. 利用对多项式的逐项分析的方法, 证明了如果 R 是 (α,δ) -弱刚性环和半交换环, 则 Ore 扩张环 $R[x;\alpha,\delta]$ 是弱对称的 (弱 zip 的, 幂零 p.p. 的, 幂零 Baer 的) 当且仅当 R 是弱对称的 (弱 zip 的, 幂零 p.p. 的, 幂零 Baer 的). 这些结果统一和扩展了前面已有的相关结论.

关键词: (α,δ) -弱刚性环; Ore 扩张; 弱对称环; 弱 zip 环; 幂零 p.p. 环; 幂零 Baer 环

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