

## STUDY ON HYPERBOLIC DIFFERENTIAL EQUATION WITH MIXED BOUNDARIES

WEI Li<sup>1</sup>, Ravi P. Agarwal<sup>2,3</sup>, Patricia J. Y. Wong<sup>4</sup>

(1. School of Mathematics and Statistics, Hebei University of Economics and Business,  
Shijiazhuang 050061, China)

(2. Department of Mathematics, Texas A&M University - Kingsville,  
Kingsville TX 78363, USA)

(3. Department of Mathematics, Faculty of Science, King Abdulaziz University,  
Jeddah 21589, Saudi Arabia)

(4. School of Electrical and Electronic Engineering, Nanyang Technological University,  
Singapore 639798, Singapore)

**Abstract:** In this paper, one kind of general form of hyperbolic differential equation with mixed boundaries is studied. By using the method of splitting the equation and the Neumann boundary condition respectively, and by defining some nonlinear operators and checking some properties of theirs in view of the result of almost equality of the ranges for maximal monotone operators presented by Reich, we prove that the hyperbolic boundary value problem has a unique solution in  $L^p(0, T; W^{1,p}(\Omega))$ . Some new techniques can be found since the main parts in the hyperbolic equation are nonlinear, which can be regarded as the complement and extension of the previous work.

**Keywords:** strictly monotone operator; subdifferential; trace operator; almost equality of the ranges; hyperbolic equation

**2010 MR Subject Classification:** 47H05; 47H09

**Document code:** A                    **Article ID:** 0255-7797(2017)03-0598-15

### 1 Introduction and Preliminaries

Elliptic differential equations, parabolic differential equations and hyperbolic differential equations are three kinds of important differential equations. Inspired by Calvert and Gupta's perturbation result on the ranges of nonlinear  $m$ -accretive mappings presented in [1], the elliptic  $p$ -Laplacian boundary value problems and their general forms were extensively studied in work of [2–6]. Actually, [6] can be regarded as the summary of the work

---

\* **Received date:** 2015-07-07

**Accepted date:** 2015-10-12

**Foundation item:** Supported by National Natural Science Foundation of China (11071053); Natural Science Foundation of Hebei Province (A2014207010); Key Project of Science and Research of Hebei Educational Department (ZH2012080) and Key Project of Science and Research of Hebei University of Economics and Business (2015KYZ03).

**Biography:** Wei Li (1967–), female, born at Leting, Hebei, professor, major in nonlinear analysis.

done in [2–5]. Namely, the following elliptic equation involving the generalized  $p$ -Laplacian operator with Neumann boundaries is studied

$$\begin{cases} -\operatorname{div}[(C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon|u|^{q-2}u + g(x, u(x)) = f(x) & \text{a.e. in } \Omega, \\ -\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \rangle \in \beta_x(u(x)) & \text{a.e. on } \Gamma, \end{cases} \tag{1.1}$$

where  $\beta_x$  is the subdifferential of a proper, convex and lower-semi continuous function. It was shown in [6] that (1.1) has solutions in  $L^s(\Omega)$  under some conditions, where  $\frac{2N}{N+1} < p \leq s < +\infty$ ,  $1 \leq q < +\infty$  if  $p \geq N$ , and  $1 \leq q \leq \frac{Np}{N-p}$  if  $p < N$  for  $N \geq 1$ .

Recently, the work done in [2–6] is extended to the following one

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{q-2}u + \lambda_2|u|^{r-2}u + g(x, u(x)), \varepsilon\nabla u(x) = f(x), \\ -\langle \vartheta, \alpha(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \rangle \in \beta_x(u(x)) & \text{a.e. on } \Gamma. \end{cases} \tag{1.2}$$

By using the properties of  $H$ -accretive mappings, it is shown in [7] that (1.2) has solutions in  $L^2(\Omega)$  under some conditions, where  $\frac{2N}{N+1} < p < +\infty$ ,  $1 \leq q, r < +\infty$  if  $p \geq N$ , and  $1 \leq q, r \leq \frac{Np}{N-p}$  if  $p < N$  for  $N \geq 1$ .

As for parabolic differential equation, Wei and Agarwal [8] studied the following one

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(\alpha(\nabla u)) + \varepsilon|u|^{q-2}u = f(x, t), & (x, t) \in \Omega \times (0, T), \\ -\langle \vartheta, \alpha(\nabla u) \rangle \in \beta(u(x, t)) - h(x, t), & (x, t) \in \Gamma \times (0, T), \\ u(x, 0) = u(x, T), & x \in \Omega. \end{cases} \tag{1.3}$$

By using some results on the ranges for bounded pseudo-monotone operator and maximal monotone operator presented in [9, 10], they obtained that (1.3) has solutions in  $L^p(0, T; W^{1,p}(\Omega))$  for  $1 < q \leq p < +\infty$ .

How about hyperbolic differential equations? Can we use the perturbation theories for nonlinear operators to do the analysis?

In this paper, we shall study the following hyperbolic problem with mixed boundaries

$$\begin{cases} -\frac{\partial}{\partial t}(\alpha_1(\frac{\partial u}{\partial t})) - \operatorname{div}(\alpha_2(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{r_1-2}u + \lambda_2|u|^{r_2-2}u = f(x, t), \\ (x, t) \in \Omega \times (0, T), \\ -\langle \vartheta, \alpha_2(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \rangle \in \beta(u(x, t)) - h(x, t), & (x, t) \in \Gamma \times (0, T), \\ \gamma u(x, t) = w(x, t) & \text{a.e. } (x, t) \in \Gamma \times (0, T), \\ u(x, 0) = u(x, T), & x \in \Omega, \\ \alpha_1(\frac{\partial u}{\partial t}(x, T)) = \alpha_1(\frac{\partial u}{\partial t}(x, 0)), & x \in \Omega, \end{cases} \tag{1.4}$$

where  $\alpha_1$  is the subdifferential of  $j$ , i.e.,  $\alpha_1 = \partial j$ , and  $j : R \rightarrow R$  is a proper, convex and lower-semi continuous function;  $\alpha_2 : R^+ \cup \{0\} \rightarrow R^+$  is a continuous nonlinear mapping such that  $pt\alpha_2'(t) + (p - 1)\alpha_2(t) > 0$ ,  $0 \leq \alpha_2(t) \leq k_1$  for  $t \geq 0$ ,  $\lim_{t \rightarrow +\infty} \alpha_2(t) = k_2 > 0$ ,

where  $k_1$  and  $k_2$  are positive constants;  $\beta : R \rightarrow R$  is maximal monotone. More details of (1.4) will be presented in Section 2. We shall discuss the existence of solution of (1.4) in  $L^p(0, T; W^{1,p}(\Omega))$ .

We may notice that the traditional part  $-\frac{\partial^2 u}{\partial t^2}$  in hyperbolic differential equations is replaced by  $-\frac{\partial}{\partial t}(\alpha_1(\frac{\partial u}{\partial t}))$ , which leads to the differences in the proofs of the main result. Furthermore, if we set  $j \equiv I$ ,  $\alpha_2(t) = 1 + \frac{t}{\sqrt{1+t^2}}$ , for  $t \geq 0$ , and if  $\lambda_1 \equiv \lambda_2 \equiv \lambda$ , then (1.4) becomes to the following hyperbolic capillarity equation with mixed boundaries

$$\left\{ \begin{array}{l} -\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left[\left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] + \lambda|u|^{r_1-2}u + \lambda|u|^{r_2-2}u = f(x, t), \\ (x, t) \in \Omega \times (0, T), \\ -\langle \vartheta, \left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u \rangle \in \beta(u(x, t)) - h(x, t), \quad (x, t) \in \Gamma \times (0, T), \\ \gamma u(x, t) = w(x, t), \quad \text{a.e. } (x, t) \in \Gamma \times (0, T), \\ u(x, 0) = u(x, T), \quad x \in \Omega, \\ \frac{\partial u}{\partial t}(x, T) = \frac{\partial u}{\partial t}(x, 0), \quad x \in \Omega. \end{array} \right. \quad (1.5)$$

For  $1 < p \leq 2$ , if we set  $j \equiv I$ ,  $\alpha_2(t) = (C + t^{\frac{2}{p}})^{\frac{p-2}{2}} t^{\frac{2-p}{p}}$ ,  $t > 0$ ,  $C \geq 0$ , and  $\lambda_2 \equiv 0$ , then (1.4) becomes to the following hyperbolic equation involving generalized  $p$ -Laplacian with mixed boundaries

$$\left\{ \begin{array}{l} -\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left[(C(x) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right] + \lambda_1|u|^{r_1-2}u = f(x, t), \quad (x, t) \in \Omega \times (0, T), \\ -\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u \rangle \in \beta(u(x, t)) - h(x, t), \quad (x, t) \in \Gamma \times (0, T), \\ \gamma u(x, t) = w(x, t) \quad \text{a.e. } (x, t) \in \Gamma \times (0, T), \\ u(x, 0) = u(x, T), \quad x \in \Omega, \\ \frac{\partial u}{\partial t}(x, T) = \frac{\partial u}{\partial t}(x, 0), \quad x \in \Omega. \end{array} \right. \quad (1.6)$$

If, in (1.6),  $C(x) \equiv 0$ , then (1.6) becomes to the hyperbolic  $p$ -Laplacian boundary value problem.

For  $s \leq 0$ , if we set  $j \equiv I$ ,  $\alpha_2(t) = (1 + t^{\frac{2}{p}})^{\frac{s}{2}} t^{\frac{m-p+1}{p}}$ ,  $t > 0$ ,  $m \geq 0$ ,  $m + s + 1 = p$ , and  $\lambda_2 \equiv 0$ , then (1.4) becomes to the following hyperbolic curvature equation with mixed boundaries

$$\left\{ \begin{array}{l} -\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left[(1 + |\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\right] + \lambda_1|u|^{r_1-2}u = f(x, t), \quad (x, t) \in \Omega \times (0, T), \\ -\langle \vartheta, (1 + |\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u \rangle \in \beta(u(x, t)) - h(x, t), \quad (x, t) \in \Gamma \times (0, T), \\ \gamma u(x, t) = w(x, t) \quad \text{a.e. } (x, t) \in \Gamma \times (0, T), \\ u(x, 0) = u(x, T), \quad x \in \Omega, \\ \frac{\partial u}{\partial t}(x, T) = \frac{\partial u}{\partial t}(x, 0), \quad x \in \Omega. \end{array} \right. \quad (1.7)$$

We need the following knowledge to begin our discussion.

Let  $X$  be a real Banach space with a strictly convex dual space  $X^*$ . We shall use  $(\cdot, \cdot)$  to denote the generalized duality pairing between  $X$  and  $X^*$ . We shall use “ $\rightarrow$ ” and “ $w$ -lim” to denote strong and weak convergence, respectively. Let “ $X \hookrightarrow Y$ ” and “ $X \hookrightarrow\hookrightarrow Y$ ” denote that space  $X$  embedded continuously or compactly in space  $Y$ , respectively. For any subset  $G$  of  $X$ , we denote by  $\text{int}G$  its interior and  $\overline{G}$  its closure, respectively. For two subsets  $G_1$  and  $G_2$  in  $X$ , if  $\overline{G_1} = \overline{G_2}$  and  $\text{int}G_1 = \text{int}G_2$ , then we say  $G_1$  is almost equal to  $G_2$ , which is denoted by  $G_1 \simeq G_2$ . A mapping  $T : X \rightarrow X^*$  is said to be hemi-continuous on  $X$  (see [11, 12]) if  $w - \lim_{t \rightarrow 0} T(x + ty) = Tx$  for any  $x, y \in X$ .

A function  $\Phi$  is called a proper convex function on  $X$  (see [11, 12]) if  $\Phi$  is defined from  $X$  to  $(-\infty, +\infty]$ , not identically  $+\infty$ , such that

$$\Phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\Phi(x) + \lambda\Phi(y),$$

whenever  $x, y \in X$  and  $0 \leq \lambda \leq 1$ . A function  $\Phi : X \rightarrow (-\infty, +\infty]$  is said to be lower-semi continuous on  $X$  (see [11, 12]) if  $\liminf_{y \rightarrow x} \Phi(y) \geq \Phi(x)$ , for any  $x \in X$ . Given a proper convex function  $\Phi$  on  $X$  and a point  $x \in X$ , we denote by  $\partial\Phi(x)$  the set of all  $x^* \in X^*$  such that

$$\Phi(x) \leq \Phi(y) + (x - y, x^*)$$

for every  $y \in X$ . Such element  $x^*$  is called the subgradient of  $\Phi$  at  $x$ , and  $\partial\Phi(x)$  is called the subdifferential of  $\Phi$  at  $x$  (see [11]).

Let  $J$  denote the normalized duality mapping from  $X$  into  $2^{X^*}$  defined by

$$Jx = \{f \in X^* : (x, f) = \|x\|\|f\|, \|f\| = \|x\|\}, \quad \forall x \in X.$$

Since  $X^*$  is strictly convex,  $J$  is single-valued.

A multi-valued operator  $B : X \rightarrow 2^{X^*}$  is said to be monotone (see [12]) if its graph  $G(B)$  is a monotone subset of  $X \times X^*$  in the sense that  $(u_1 - u_2, w_1 - w_2) \geq 0$ , for any  $[u_i, w_i] \in G(B)$ ,  $i = 1, 2$ . Further,  $B$  is called strictly monotone if  $(u_1 - u_2, w_1 - w_2) \geq 0$  and the equality holds if and only if  $u_1 = u_2$ . The monotone operator  $B$  is said to be maximal monotone if  $G(B)$  is maximal among all monotone subsets of  $X \times X^*$  in the sense of inclusion. Also,  $B$  is maximal monotone if and only if  $R(B + \lambda J) = X^*$ , for any  $\lambda > 0$ . The mapping  $B$  is said to be coercive (see [12]) if

$$\lim_{n \rightarrow +\infty} (x_n, x_n^*) / \|x_n\| = +\infty$$

for all  $[x_n, x_n^*] \in G(B)$  such that  $\lim_{n \rightarrow +\infty} \|x_n\| = +\infty$ .

Let  $B : X \rightarrow 2^{X^*}$  be a maximal monotone operator such that  $[0, 0] \in G(B)$ , then the equation  $J(u_t - u) + tBu_t \ni 0$  has a unique solution  $u_t \in D(B)$  for every  $u \in X$  and  $t > 0$ . The resolvent  $J_t^B$  and the Yosida approximation  $B_t$  of  $B$  are defined by the following (see [12]):  $J_t^B u = u_t$ ,  $B_t u = -\frac{1}{t}J(u_t - u)$ , for every  $u \in X$  and  $t > 0$ . Hence,  $[J_t^B u, B_t u] \in G(B)$ .

Let  $1 < p < +\infty$ , then  $L^p(0, T; X)$  denotes the space of all  $X$ -valued strongly measurable functions  $x(t)$  defined a.e. on  $(0, T)$  such that  $\|x(t)\|_X^p$  is Lebesgue integrable over  $(0, T)$ . It is well-known that  $L^p(0, T; X)$  is a Banach space with the norm defined by

$$\|x\|_{L^p(0, T; X)} = \left( \int_0^T \|x(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

If  $X$  is reflexive, then  $L^p(0, T; X)$  is reflexive, and its dual space coincides with  $L^{p'}(0, T; X^*)$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ . Moreover,  $L^p(0, T; X)$  is reflexive in the case where  $X$  is reflexive and  $L^p(0, T; X)$  is strictly (uniformly) convex in the case where  $X$  is strictly (uniformly) convex.

For  $1 \leq r < p < +\infty$ , if  $X \hookrightarrow Y$ , then  $L^p(0, T; X) \hookrightarrow L^r(0, T; Y)$ .

**Lemma 1.1** (see [12]) If  $A : X \rightarrow 2^{X^*}$  is a everywhere defined, monotone and hemi-continuous mapping, then  $A$  is maximal monotone.

**Lemma 1.2** (see [12]) If  $\Phi : X \rightarrow (-\infty, +\infty]$  is a proper convex and lower-semi continuous function, then  $\partial\Phi$  is maximal monotone from  $X$  to  $X^*$ .

**Lemma 1.3** (see [12]) If  $A_1$  and  $A_2$  are two maximal monotone operators in  $X$  such that  $(\text{int}D(A_1)) \cap D(A_2) \neq \emptyset$ , then  $A_1 + A_2$  is maximal monotone.

**Theorem 1.1** (see [10]) Let  $X$  be a real reflexive Banach space with both  $X$  and its dual  $X^*$  being strictly convex. Let  $J : X \rightarrow X^*$  be the normalized duality mapping on  $X$ . Let  $A$  and  $B$  be two maximal monotone operators in  $X$ . If there exist  $0 \leq k < 1$  and  $C_1, C_2 > 0$  such that

$$(a, J^{-1}(B_t v)) \geq -k\|B_t v\|^2 - C_1\|B_t v\| - C_2, \quad \forall v \in D(A), a \in Av$$

and  $t > 0$ , where  $B_t$  is the Yosida approximation of  $B$ . Then  $R(A) + R(B) \simeq R(A + B)$ .

**Lemma 1.4** (see [13]) Let  $\Omega$  be a bounded conical domain in  $R^N$ . If  $mp > N$ , then  $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$ ; if  $0 < mp < N$  and  $q = \frac{Np}{N-mp}$ , then  $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ ; if  $mp = N$  and  $p > 1$ , then for  $1 \leq q < +\infty$ , then  $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ .

**Lemma 1.5** (see [13]) Let  $\Omega$  be a domain of  $R^N$  with its boundary  $\Gamma \in C^1$ , then we have the following results

- (i) if  $u \in W^{1,p}(\Omega)$ , then the trace  $\gamma u \in W^{1-\frac{1}{p},p}(\Gamma)$  and  $\|\gamma u\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq K_1\|u\|_{W^{1,p}(\Omega)}$ ;
- (ii) if  $v \in W^{1-\frac{1}{p},p}(\Gamma)$ , then there exists  $u \in W^{1,p}(\Omega)$  such that  $v = \gamma u$  and  $\|u\|_{W^{1,p}(\Omega)} \leq K_2\|v\|_{W^{1-\frac{1}{p},p}(\Gamma)}$ , where  $\gamma : W^{1,p}(\Omega) \rightarrow W^{1-\frac{1}{p},p}(\Gamma)$  denotes the trace operator.

**Lemma 1.6** (see [12]) Let  $A : X \rightarrow 2^{X^*}$  be a maximal monotone operator and let  $B : X \rightarrow X^*$  be a hemi-continuous, bounded, coercive and monotone operator with  $D(B) = X$ , then  $R(A + B) = X^*$ .

## 2 Main Results

In this paper, unless otherwise stated, we shall assume that  $N \geq 1$ ,  $2 \leq p < +\infty$ ,  $1 \leq r_i \leq p$  for  $i = 1, 2$ . And  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\frac{1}{r_1} + \frac{1}{r_1'} = 1$  and  $\frac{1}{r_2} + \frac{1}{r_2'} = 1$ .

In (1.4),  $\Omega$  is a bounded conical domain of a Euclidean space  $R^N$  with its boundary  $\Gamma \in C^1$  (see [2]),  $T$  is a positive constant,  $\lambda_1$  and  $\lambda_2$  are non-negative constants, and  $\vartheta$  denotes the exterior normal derivative of  $\Gamma$ ,

$$\gamma : L^p(0, T; W^{1,p}(\Omega)) \rightarrow L^p(0, T; W^{1-\frac{1}{p},p}(\Gamma))$$

denotes the trace operator. We shall assume that Green's Formula is available.

Suppose that  $\alpha_1 = \partial j$  is continuous, where  $j : R \rightarrow R$  is a proper, convex and lower-semi continuous function. Suppose  $\alpha_2 : R^+ \cup \{0\} \rightarrow R^+$  is a continuous nonlinear mapping such that  $pt\alpha_2'(t) + (p - 1)\alpha_2(t) > 0$ ,  $0 \leq \alpha_2(t) \leq k_1$ , for  $t \geq 0$ ,  $\lim_{t \rightarrow +\infty} \alpha_2(t) = k_2 > 0$ , where  $k_1$  and  $k_2$  are positive constants.  $\beta : R \rightarrow R$  is maximal monotone such that, for each  $w(x, t) \in L^p(0, T; L^p(\Gamma))$ ,  $\beta(w) \in L^p(0, T; L^p(\Gamma))$ .

Now, we present our discussion in the sequel.

**Lemma 2.1** (see [14]) For  $u(x, t) \in L^p(0, T; W^{1,p}(\Omega))$ ,

$$\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \leq k_3 \left( \int_0^T \int_{\Omega} |\nabla u|^p dxdt \right)^{\frac{1}{p}} + k_4,$$

where  $k_3$  and  $k_4$  are positive constants,  $\frac{2N}{N+1} < p < +\infty$  and  $N \geq 1$ .

**Lemma 2.2** Define the mapping  $B : L^p(0, T; W^{1,p}(\Omega)) \rightarrow L^{p'}(0, T; (W^{1,p}(\Omega))^*)$  by

$$\begin{aligned} (w, Bu) &= \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u|^p) |\nabla u|^{p-2} \nabla u, \nabla w \rangle dxdt \\ &\quad + \lambda_1 \int_0^T \int_{\Omega} |u|^{r_1-2} u w dxdt + \lambda_2 \int_0^T \int_{\Omega} |u|^{r_2-2} u w dxdt \end{aligned}$$

for any  $u, w \in L^p(0, T; W^{1,p}(\Omega))$ . Then  $B$  is everywhere defined, bounded, hemi-continuous, monotone and coercive.

Here  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  denote the Euclidean inner-product and Euclidean norm in  $R^N$ .

**Proof Step 1**  $B$  is everywhere defined.  $\forall u, w \in L^p(0, T; W^{1,p}(\Omega))$ ,

$$\begin{aligned} |(w, Bu)| &\leq \int_0^T \int_{\Omega} k_1 |\nabla u|^{p-1} |\nabla w| dxdt + \lambda_1 \int_0^T \int_{\Omega} |u|^{r_1-1} |w| dxdt \\ &\quad + \lambda_2 \int_0^T \int_{\Omega} |u|^{r_2-1} |w| dxdt \\ &\leq k_1 \|u\|_{L^p(0,T;W^{1,p}(\Omega))}^{\frac{p}{p-1}} \|w\|_{L^p(0,T;W^{1,p}(\Omega))} + \lambda_1 \|w\|_{L^{r_1}(0,T;L^{r_1}(\Omega))} \|u\|_{L^{r_1}(0,T;L^{r_1}(\Omega))}^{\frac{r_1}{r_1-1}} \\ &\quad + \lambda_2 \|w\|_{L^{r_2}(0,T;L^{r_2}(\Omega))} \|u\|_{L^{r_2}(0,T;L^{r_2}(\Omega))}^{\frac{r_2}{r_2-1}}. \end{aligned}$$

Since  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^{r_1}(\Omega)$ ,  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^{r_2}(\Omega)$ , then  $\forall v \in W^{1,p}(\Omega)$ ,

$$\|v\|_{L^{r_1}(\Omega)} \leq k_5 \|v\|_{W^{1,p}(\Omega)}, \quad \|v\|_{L^{r_2}(\Omega)} \leq k_6 \|v\|_{W^{1,p}(\Omega)},$$

where  $k_5$  and  $k_6$  are positive constants. Hence,

$$\begin{aligned} |(w, Bu)| &\leq k_1 \|u\|_{L^p(0,T;W^{1,p}(\Omega))}^{\frac{p}{p-1}} \|w\|_{L^p(0,T;W^{1,p}(\Omega))} + \lambda_1 k_5 \|u\|_{L^p(0,T;W^{1,p}(\Omega))}^{\frac{r_1}{r_1-1}} \|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\ &\quad + \lambda_2 k_6 \|u\|_{L^p(0,T;W^{1,p}(\Omega))}^{\frac{r_2}{r_2-1}} \|w\|_{L^p(0,T;W^{1,p}(\Omega))}, \end{aligned}$$

which implies that  $B$  is everywhere defined. Actually, from the above proof, we know that  $B$  is bounded.

**Step 2**  $B$  is strictly monotone.  $\forall u, v \in L^p(0, T; W^{1,p}(\Omega))$ ,

$$(u - v, Bu - Bv) \geq \int_0^T \int_{\Omega} (\alpha_2(|\nabla u|^p)|\nabla u|^{p-1} - \alpha_2(|\nabla v|^p)|\nabla v|^{p-1})(|\nabla u| - |\nabla v|) dx dt \\ + \lambda_1 \int_0^T \int_{\Omega} (|u|^{r_1-1} - |v|^{r_1-1})(|u| - |v|) dx dt + \lambda_2 \int_0^T \int_{\Omega} (|u|^{r_2-1} - |v|^{r_2-1})(|u| - |v|) dx dt.$$

If we set  $f(s) = s^{1-\frac{1}{p}}\alpha_2(s)$ ,  $s > 0$ , then

$$f'(s) = [(1 - \frac{1}{p})\alpha_2(s) + s\alpha_2'(s)]s^{-\frac{1}{p}} > 0$$

in view of the assumption of  $\alpha_2$ , which implies that  $f$  is strictly monotone. And then  $B$  is strictly monotone.

**Step 3**  $B$  is hemi-continuous. In fact, it suffices to show that, for any  $u, v, w \in L^p(0, T; W^{1,p}(\Omega))$  and  $t \in [0, 1]$ ,  $(w, B(u + tv) - Bu) \rightarrow 0$ , as  $t \rightarrow 0$ . Since

$$|(w, B(u + tv) - Bu)| \\ \leq \int_0^T \int_{\Omega} |\alpha_2(|\nabla u + t\nabla v|^p)|\nabla u + t\nabla v|^{p-2}(\nabla u + t\nabla v) - \alpha_2(|\nabla u|^p)|\nabla u|^{p-2}\nabla u||\nabla w| dx dt \\ + \lambda_1 \int_0^T \int_{\Omega} ||u + tv|^{r_1-2}(u + tv) - |u|^{r_1-2}u||w| dx dt \\ + \lambda_2 \int_0^T \int_{\Omega} ||u + tv|^{r_2-2}(u + tv) - |u|^{r_2-2}u||w| dx dt$$

by Lebesgue's dominated convergence theorem and noticing that  $\alpha_2$  is continuous, we know that  $\lim_{t \rightarrow 0} (w, B(u + tv) - Bu) = 0$ , and hence  $B$  is hemi-continuous.

**Step 4**  $B$  is coercive. For  $u \in L^p(0, T; W^{1,p}(\Omega))$ , let  $\|u\|_{L^p(0, T; W^{1,p}(\Omega))} \rightarrow +\infty$ . Using Lemma 2.1, we find

$$\frac{(u, Bu)}{\|u\|_{L^p(0, T; W^{1,p}(\Omega))}} \\ = \frac{\int_0^T \int_{\Omega} \alpha_2(|\nabla u|^p)|\nabla u|^p dx dt}{\|u\|_{L^p(0, T; W^{1,p}(\Omega))}} + \lambda_1 \frac{\int_0^T \int_{\Omega} |u|^{r_1} dx dt}{\|u\|_{L^p(0, T; W^{1,p}(\Omega))}} + \lambda_2 \frac{\int_0^T \int_{\Omega} |u|^{r_2} dx dt}{\|u\|_{L^p(0, T; W^{1,p}(\Omega))}} \\ > \frac{1}{\|u\|_{L^p(0, T; W^{1,p}(\Omega))}} [k_2 \int_0^T \int_{\Omega} |\nabla u|^p dx dt + \lambda_1 \int_0^T \int_{\Omega} |u|^{r_1} dx dt + \lambda_2 \int_0^T \int_{\Omega} |u|^{r_2} dx dt] \\ > \frac{1}{\|u\|_{L^p(0, T; W^{1,p}(\Omega))}} k_2 \int_0^T \int_{\Omega} |\nabla u|^p dx dt \rightarrow +\infty.$$

This completes the proof.

**Lemma 2.3** Define

$$S : D(S) = \{u(x, t) \in L^p(0, T; W^{1,p}(\Omega)) : u(x, 0) = u(x, T), \alpha_1(\frac{\partial u}{\partial t}(x, 0)) = \alpha_1(\frac{\partial u}{\partial t}(x, T)), \\ \gamma u = w, \frac{\partial}{\partial t}(\alpha_1(\frac{\partial u}{\partial t}(x, t))) \in L^p(0, T; (W^{1,p}(\Omega))^*)\} \rightarrow (-\infty, +\infty]$$

by

$$Su(x, t) = \begin{cases} \int_0^T \int_{\Omega} j(\frac{\partial u}{\partial t}) dxdt, & j(\frac{\partial u}{\partial t}) \in L^1(0, T; \Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the mapping  $S$  is proper, convex and lower-semi continuous.

**Proof** It is only need to show that  $S$  is lower-semi continuous on  $L^p(0, T; W^{1,p}(\Omega))$ .

For this, let  $\{u_n\}$  be such that  $u_n \rightarrow u$  in  $L^p(0, T; W^{1,p}(\Omega))$  as  $n \rightarrow \infty$ . Then there exists a subsequence of  $\{u_n\}$ , which is still denoted by  $\{u_n\}$  such that

$$\frac{\partial u_n(x, t)}{\partial t} \rightarrow \frac{\partial u(x, t)}{\partial t} \quad \text{a.e. } (x, t) \in \Omega \times (0, T).$$

Since  $j$  is lower-semi continuous, then  $j(\frac{\partial u(x, t)}{\partial t}) \leq \liminf_{n \rightarrow \infty} j(\frac{\partial u_n(x, t)}{\partial t})$  a.e. on  $\Omega \times (0, T)$ .

Using Fatou's lemma, we have

$$\int_0^T \int_{\Omega} j(\frac{\partial u(x, t)}{\partial t}) dxdt \leq \liminf_{n \rightarrow \infty} \int_0^T \int_{\Omega} j(\frac{\partial u_n(x, t)}{\partial t}) dxdt.$$

Therefore,  $Su \leq \liminf_{n \rightarrow \infty} S(u_n)$ , whenever  $u_n \rightarrow u$  in  $L^p(0, T; W^{1,p}(\Omega))$ . The result follows.

This completes the proof.

**Lemma 2.4** Let  $S$  be the same as that in Lemma 2.3. If  $w(x, t) \in \partial S(u(x, t))$  then

$$w(x, t) = -\frac{\partial}{\partial t}(\alpha_1(\frac{\partial u}{\partial t})) \quad \text{a.e. in } \Omega \times (0, T).$$

**Proof** Let  $w(x, t) = \frac{\partial \bar{w}(x, t)}{\partial t}$ . In view of the definition of subdifferential, we have if  $w(x, t) \in \partial S(u(x, t))$ , then

$$\begin{aligned} & \int_0^T \int_{\Omega} [j(\frac{\partial u}{\partial t}) - j(\frac{\partial v}{\partial t})] dxdt \leq \int_0^T \int_{\Omega} w(x, t)[u(x, t) - v(x, t)] dxdt \\ & = \int_0^T \int_{\Omega} \frac{\partial \bar{w}(x, t)}{\partial t} [u(x, t) - v(x, t)] dxdt = - \int_0^T \int_{\Omega} \bar{w}(x, t) (\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) dxdt. \end{aligned} \quad (2.1)$$

Let  $E$  be any measurable subset of  $\Omega$  such that for  $t \in (0, T)$ ,

$$\tilde{w}(x, t) = \begin{cases} v(x, t), & x \in E, \\ u(x, t) & x \in E^C, \end{cases}$$

where  $E^C$  is the complement of  $E$  in  $\Omega$ . Taking  $v(x, t) = \tilde{w}(x, t)$  in (2.1), we have

$$\int_0^T \int_E [j(\frac{\partial u}{\partial t}) - j(\frac{\partial v}{\partial t}) + \bar{w}(x, t)] (\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) dxdt \leq 0.$$

In as much as  $E$  was any measurable subset of  $\Omega$ , we have

$$j(\frac{\partial u}{\partial t}) - j(\frac{\partial v}{\partial t}) \leq -\bar{w}(x, t) (\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) \quad \text{a.e. } (x, t) \in \Omega \times (0, T).$$

Thus

$$\bar{w}(x, t) = -\partial j\left(\frac{\partial u}{\partial t}\right) = -\alpha_1\left(\frac{\partial u}{\partial t}\right) \quad \text{a.e. } (x, t) \in \Omega \times (0, T).$$

Then

$$w(x, t) = -\frac{\partial}{\partial t}\left(\alpha_1\left(\frac{\partial u}{\partial t}\right)\right) \quad \text{a.e. } (x, t) \in \Omega \times (0, T).$$

This completes the proof.

**Theorem 2.1** For each  $w(x, t) \in L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))$  and  $f(x, t) \in L^{p'}(0, T; (W^{1,p}(\Omega))^*)$ , there exists  $u(x, t) \in L^p(0, T; W^{1,p}(\Omega))$  which is the unique solution of the following boundary value problem

$$\begin{cases} -\frac{\partial}{\partial t}\left(\alpha_1\left(\frac{\partial u}{\partial t}\right)\right) - \operatorname{div}(\alpha_2(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{r_1-2}u + \lambda_2|u|^{r_2-2}u = f(x, t), \\ (x, t) \in \Omega \times (0, T), \\ \gamma u(x, t) = w(x, t) \quad \text{a.e. } (x, t) \in \Gamma \times (0, T), \\ u(x, 0) = u(x, T), \quad x \in \Omega, \\ \alpha_1\left(\frac{\partial u}{\partial t}(x, T)\right) = \alpha_1\left(\frac{\partial u}{\partial t}(x, 0)\right), \quad x \in \Omega. \end{cases} \quad (2.2)$$

In the following of the paper, we denote  $u_{w,f}$  the unique solution of (2.2).

**Proof** From Lemmas 1.2, 2.3, 1.6 and 2.2, we know that there exists

$$u(x, t) \in L^p(0, T; W^{1,p}(\Omega)),$$

which satisfies

$$\partial S(u(x, t)) + Bu(x, t) = f(x, t). \quad (2.3)$$

Then  $\forall \varphi \in C_0^\infty(0, T; \Omega)$ ,

$$\begin{aligned} & \int_0^T \int_\Omega -\frac{\partial}{\partial t}\left(\alpha_1\left(\frac{\partial u}{\partial t}\right)\right)\varphi dxdt + \int_0^T \int_\Omega \langle \alpha_2(|\nabla u|^p)|\nabla u|^{p-2}\nabla u, \nabla \varphi \rangle dxdt \\ & + \int_0^T \int_\Omega \lambda_1|u|^{r_1-2}u\varphi dxdt + \int_0^T \int_\Omega \lambda_2|u|^{r_2-2}u\varphi dxdt = \int_0^T \int_\Omega f\varphi dxdt. \end{aligned}$$

From the properties of generalized function, we have

$$-\frac{\partial}{\partial t}\left(\alpha_1\left(\frac{\partial u}{\partial t}\right)\right) - \operatorname{div}(\alpha_2(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{r_1-2}u + \lambda_2|u|^{r_2-2}u = f(x, t) \quad \text{a.e. in } \Omega \times (0, T).$$

Combining with the definition of  $S$ , we know that (2.2) has a solution in  $L^p(0, T; W^{1,p}(\Omega))$ .

Uniqueness: let both  $u(x, t)$  and  $v(x, t)$  be solutions of (2.2), then they satisfy (2.3). Thus  $(u - v, Bu - Bv) = -(u - v, \partial S(u) - \partial S(v)) \leq 0$ , since  $\partial S$  is monotone. But  $B$  is monotone too, so  $(u - v, Bu - Bv) = 0$ , which implies that  $u(x, t) = v(x, t)$  since  $B$  is strictly monotone.

This completes the proof.

**Lemma 2.5** Define the operator

$$A : L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma)) \rightarrow (L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma)))^*$$

by

$$Aw = \langle \vartheta, \alpha_2(|\nabla u_{w,f}|^p)|\nabla u_{w,f}|^{p-2}\nabla u_{w,f} \rangle, \quad \forall w \in L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma)),$$

then  $A$  is maximal monotone on  $L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))$ .

**Proof Step 1**  $A$  is everywhere defined.  $\forall w_1, w_2 \in L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))$ , noticing Lemma 1.5, there exists  $\bar{w}_2 \in L^p(0, T; W^{1,p}(\Omega))$  such that for  $t \in (0, T)$ ,  $\gamma\bar{w}_2 = w_2$  and

$$\|\bar{w}_2\|_{L^p(0, T; W^{1,p}(\Omega))} \leq K_2 \|w_2\|_{L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))}.$$

Using Green's formula and (2.2), we have

$$\begin{aligned} |(w_2, Aw_1)| &= \left| \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_1,f}|^p)|\nabla u_{w_1,f}|^{p-2}\nabla u_{w_1,f} \rangle w_2 d\Gamma(x) dt \right| \\ &= \left| \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1,f}|^p)|\nabla u_{w_1,f}|^{p-2}\nabla u_{w_1,f}, \nabla \bar{w}_2 \rangle dx dt \right. \\ &\quad \left. + \int_0^T \int_{\Omega} \operatorname{div}[\alpha_2(|\nabla u_{w_1,f}|^p)|\nabla u_{w_1,f}|^{p-2}\nabla u_{w_1,f}] \bar{w}_2 dx dt \right| \\ &\leq k_1 \int_0^T \int_{\Omega} |\nabla u_{w_1,f}|^{p-1} |\nabla \bar{w}_2| dx dt + \int_0^T \int_{\Omega} \left| \frac{\partial}{\partial t} \left( \alpha_1 \left( \frac{\partial u_{w_1,f}}{\partial t} \right) \right) \bar{w}_2 \right| dx dt \\ &\quad + \int_0^T \int_{\Omega} |\lambda_1 |u_{w_1,f}|^{r_1-2} u_{w_1,f} + \lambda_2 |u_{w_1,f}|^{r_2-2} u_{w_1,f} - f(x, t)| |\bar{w}_2| dx dt \\ &\leq (k_1 \|u_{w_1,f}\|_{L^p(0, T; W^{1,p}(\Omega))}^{\frac{p}{p'}} + \lambda_1 \|u_{w_1,f}\|_{L^p(0, T; W^{1,p}(\Omega))}^{\frac{r_1}{r_1-1}} \\ &\quad + \lambda_2 \|u_{w_1,f}\|_{L^p(0, T; W^{1,p}(\Omega))}^{\frac{r_2}{r_2-1}} + \|f\|_{L^{p'}(0, T; (W^{1,p}(\Omega))^*)} \\ &\quad + \left\| \frac{\partial}{\partial t} \alpha_1 \left( \frac{\partial u_{w_1,f}}{\partial t} \right) \right\|_{L^{p'}(0, T; (W^{1,p}(\Omega))^*)} \| \bar{w}_2 \|_{L^p(0, T; W^{1,p}(\Omega))}), \end{aligned}$$

which implies that  $A$  is everywhere defined.

**Step 2**  $A$  is monotone.  $\forall w_1, w_2 \in L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))$ , using Theorem 2.1, there exists

$$u_{w_1,f}, u_{w_2,f} \in L^p(0, T; W^{1,p}(\Omega))$$

such that for  $t \in (0, T)$ ,  $\gamma u_{w_1,f} = w_1$  and  $\gamma u_{w_2,f} = w_2$ . Using Green's formula, we have the following

$$\begin{aligned} &(w_1 - w_2, Aw_1 - Aw_2) \\ &= \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_1,f}|^p)|\nabla u_{w_1,f}|^{p-2}\nabla u_{w_1,f} \rangle w_1 d\Gamma(x) dt \\ &\quad - \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_1,f}|^p)|\nabla u_{w_1,f}|^{p-2}\nabla u_{w_1,f} \rangle w_2 d\Gamma(x) dt \\ &\quad - \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_2,f}|^p)|\nabla u_{w_2,f}|^{p-2}\nabla u_{w_2,f} \rangle w_1 d\Gamma(x) dt \\ &\quad + \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_2,f}|^p)|\nabla u_{w_2,f}|^{p-2}\nabla u_{w_2,f} \rangle w_2 d\Gamma(x) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1,f}|^p) |\nabla u_{w_1,f}|^{p-2} \nabla u_{w_1,f}, \nabla u_{w_1,f} \rangle dx dt \\
&\quad + \int_0^T \int_{\Omega} [-f + \lambda_1 |u_{w_1,f}|^{r_1-2} u_{w_1,f} + \lambda_2 |u_{w_1,f}|^{r_2-2} u_{w_1,f} - \frac{\partial}{\partial t} (\alpha_1(\frac{\partial u_{w_1,f}}{\partial t}))] u_{w_1,f} dx dt \\
&\quad - \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1,f}|^p) |\nabla u_{w_1,f}|^{p-2} \nabla u_{w_1,f}, \nabla u_{w_2,f} \rangle dx dt \\
&\quad + \int_0^T \int_{\Omega} [f - \lambda_1 |u_{w_1,f}|^{r_1-2} u_{w_1,f} - \lambda_2 |u_{w_1,f}|^{r_2-2} u_{w_1,f} + \frac{\partial}{\partial t} (\alpha_1(\frac{\partial u_{w_1,f}}{\partial t}))] u_{w_2,f} dx dt \\
&\quad - \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_2,f}|^p) |\nabla u_{w_2,f}|^{p-2} \nabla u_{w_2,f}, \nabla u_{w_1,f} \rangle dx dt \\
&\quad + \int_0^T \int_{\Omega} [f - \lambda_1 |u_{w_2,f}|^{r_1-2} u_{w_2,f} - \lambda_2 |u_{w_2,f}|^{r_2-2} u_{w_2,f} + \frac{\partial}{\partial t} (\alpha_1(\frac{\partial u_{w_2,f}}{\partial t}))] u_{w_1,f} dx dt \\
&\quad + \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_2,f}|^p) |\nabla u_{w_2,f}|^{p-2} \nabla u_{w_2,f}, \nabla u_{w_2,f} \rangle dx dt \\
&\quad + \int_0^T \int_{\Omega} [-f + \lambda_1 |u_{w_2,f}|^{r_1-2} u_{w_2,f} + \lambda_2 |u_{w_2,f}|^{r_2-2} u_{w_2,f} - \frac{\partial}{\partial t} (\alpha_1(\frac{\partial u_{w_2,f}}{\partial t}))] u_{w_2,f} dx dt \\
&= (u_{w_1,f} - u_{w_2,f}, (B + \partial S)u_{w_1,f} - (B + \partial S)u_{w_2,f}) \geq 0.
\end{aligned}$$

**Step 3**  $A$  is hemi-continuous. It suffices to show that for  $w_1, w_2, w_3 \in L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))$  and  $k \in [0, 1]$ ,

$$(w_3, A(w_1 + kw_2) - Aw_1) \rightarrow 0$$

as  $k \rightarrow 0$ .

In fact, notice again that for  $w_3 \in L^p(0, T; W^{1-\frac{1}{p}, p}(\Gamma))$ , there exists  $\overline{w_3} \in L^p(0, T; W^{1,p}(\Omega))$  such that  $\gamma \overline{w_3} = w_3$  for  $t \in (0, T)$ .

Now we shall compute the following

$$\begin{aligned}
&|(w_3, A(w_1 + kw_2) - Aw_1)| \\
&= \left| \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_1+kw_2,f}|^p) |\nabla u_{w_1+kw_2,f}|^{p-2} \nabla u_{w_1+kw_2,f} \rangle w_3 d\Gamma(x) dt \right. \\
&\quad \left. - \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w_1,f}|^p) |\nabla u_{w_1,f}|^{p-2} \nabla u_{w_1,f} \rangle w_3 d\Gamma(x) dt \right| \\
&= \left| \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1+kw_2,f}|^p) |\nabla u_{w_1+kw_2,f}|^{p-2} \nabla u_{w_1+kw_2,f}, \nabla \overline{w_3} \rangle dx dt \right. \\
&\quad + \int_0^T \int_{\Omega} \operatorname{div} [\alpha_2(|\nabla u_{w_1+kw_2,f}|^p) |\nabla u_{w_1+kw_2,f}|^{p-2} \nabla u_{w_1+kw_2,f}] \overline{w_3} dx dt \\
&\quad - \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1,f}|^p) |\nabla u_{w_1,f}|^{p-2} \nabla u_{w_1,f}, \nabla \overline{w_3} \rangle dx dt \\
&\quad \left. - \int_0^T \int_{\Omega} \operatorname{div} [\alpha_2(|\nabla u_{w_1,f}|^p) |\nabla u_{w_1,f}|^{p-2} \nabla u_{w_1,f}] \overline{w_3} dx dt \right| \\
&= \left| \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1+kw_2,f}|^p) |\nabla u_{w_1+kw_2,f}|^{p-2} \nabla u_{w_1+kw_2,f}, \nabla \overline{w_3} \rangle dx dt \right.
\end{aligned}$$

$$\begin{aligned}
 & - \int_0^T \int_{\Omega} \frac{\partial}{\partial t} (\alpha_1(\frac{\partial u_{w_1+kw_2,f}}{\partial t})) \bar{w}_3 dxdt \\
 & - \int_0^T \int_{\Omega} f \bar{w}_3 dxdt + \int_0^T \int_{\Omega} \lambda_1 |u_{w_1+kw_2,f}|^{r_1-2} u_{w_1+kw_2,f} \bar{w}_3 dxdt \\
 & + \int_0^T \int_{\Omega} \lambda_2 |u_{w_1+kw_2,f}|^{r_2-2} u_{w_1+kw_2,f} \bar{w}_3 dxdt \\
 & - \int_0^T \int_{\Omega} \langle \alpha_2(|\nabla u_{w_1,f}|^p) |\nabla u_{w_1,f}|^{p-2} \nabla u_{w_1,f}, \nabla \bar{w}_3 \rangle dxdt + \int_0^T \int_{\Omega} \frac{\partial}{\partial t} (\alpha_1(\frac{\partial u_{w_1,f}}{\partial t})) \bar{w}_3 dxdt \\
 & + \int_0^T \int_{\Omega} f \bar{w}_3 dxdt - \int_0^T \int_{\Omega} \lambda_1 |u_{w_1,f}|^{r_1-2} u_{w_1,f} \bar{w}_3 dxdt - \int_0^T \int_{\Omega} \lambda_2 |u_{w_1,f}|^{r_2-2} u_{w_1,f} \bar{w}_3 dxdt \\
 \leq & |(\bar{w}_3, Bu_{w_1+kw_2,f} - Bu_{w_1,f})| + \int_0^T \int_{\Omega} |\alpha_1(\frac{\partial u_{w_1+kw_2,f}}{\partial t}) - \alpha_1(\frac{\partial u_{w_1,f}}{\partial t})| |\frac{\partial \bar{w}_3}{\partial t}| dxdt. \tag{2.4}
 \end{aligned}$$

Notice that

$$\gamma u_{w_1+kw_2,f} = w_1 + kw_2 = \gamma u_{w_1,f} + k\gamma u_{w_2,f} \text{ a.e. on } \Gamma \times (0, T),$$

then by using Lemma 1.5, we have

$$u_{w_1+kw_2,f} = u_{w_1,f} + ku_{w_2,f} \text{ a.e. in } \Omega \times (0, T).$$

Thus Lemma 2.2, (2.4) and the assumption on  $\alpha_1$  ensure that

$$(w_3, A(w_1 + kw_2) - Aw_1) \rightarrow 0,$$

as  $k \rightarrow 0$ , which implies that  $A$  is hemi-continuous.

Therefore, Lemma 1.1 implies that  $A$  is maximal monotone. This completes the proof.

**Definition 2.1** Define a mapping  $P : L^p(0, T; W^{1-\frac{1}{p},p}(\Gamma)) \rightarrow R$  by

$$\begin{aligned}
 Pw(x, t) &= \lambda_1 \int_0^T \int_{\Omega} |u_{w,f}|^{r_1-2} u_{w,f} dxdt + \lambda_2 \int_0^T \int_{\Omega} |u_{w,f}|^{r_2-2} u_{w,f} dxdt, \\
 \forall w &\in L^p(0, T; W^{1-\frac{1}{p},p}(\Gamma)),
 \end{aligned}$$

where  $u_{w,f}$  is the unique solution of (2.2).

**Theorem 2.2** If

$$\frac{1}{T \cdot \text{meas}(\Gamma)} (\int_0^T \int_{\Gamma} h(x, t) d\Gamma(x) dt + \int_0^T \int_{\Omega} f(x, t) dxdt) \in R(\beta + \frac{1}{T \cdot \text{meas}(\Gamma)} P),$$

then nonlinear problem (1.4) has a solution in  $L^p(0, T; W^{1,p}(\Omega))$ .

**Proof** Let  $A$  be defined in Lemma 2.5. Then  $A$  is maximal monotone operator on  $L^p(0, T; W^{1-\frac{1}{p},p}(\Gamma))$  or in  $L^{p'}(0, T; L^{p'}(\Gamma))$ . Define  $Lw = \beta(w)$ , where

$$D(L) := \{w \in L^{p'}(0, T; L^{p'}(\Gamma)) : \beta(w) \in L^p(0, T; L^p(\Gamma))\},$$

then  $L$  is maximal monotone in  $L^{p'}(0, T; L^{p'}(\Gamma))$  since  $\beta$  is maximal monotone.

Next, we shall verify that the conditions of Theorem 1.1 are satisfied in  $L^{p'}(0, T; L^p(\Gamma))$ .

$$\begin{aligned}
& (Aw, J^{-1}(L_\mu(w))) \\
&= \int_0^T \int_\Gamma \langle \vartheta, \alpha_2(|\nabla u_{w,f}|^p) |\nabla u_{w,f}|^{p-2} \nabla u_{w,f} \rangle J^{-1}(\beta_\mu(w)) d\Gamma(x) dt \\
&= \int_0^T \int_\Omega \operatorname{div}[\alpha_2(|\nabla u_{w,f}|^p) |\nabla u_{w,f}|^{p-2} \nabla u_{w,f}] \|\beta_\mu(u_{w,f})\|_p^{2-p} \beta_\mu(u_{w,f}) |\beta_\mu(u_{w,f})|^{p-2} dx dt \\
&\quad + \int_0^T \int_\Omega \langle \alpha_2(|\nabla u_{w,f}|^p) |\nabla u_{w,f}|^{p-2} \nabla u_{w,f}, \nabla J^{-1}(\beta_\mu(u_{w,f})) \rangle dx dt \\
&= \int_0^T \int_\Omega \left( -\frac{\partial}{\partial t} \left( \alpha_1 \left( \frac{\partial u_{w,f}}{\partial t} \right) \right) + \lambda_1 |u_{w,f}|^{r_1-2} u_{w,f} \right. \\
&\quad \left. + \lambda_2 |u_{w,f}|^{r_2-2} u_{w,f} - f \right) \|\beta_\mu(u_{w,f})\|_p^{2-p} \beta_\mu(u_{w,f}) |\beta_\mu(u_{w,f})|^{p-2} dx dt \\
&\quad + (p-1) \int_0^T \int_\Omega \alpha_2(|\nabla u_{w,f}|^p) |\nabla u_{w,f}|^p \|\beta_\mu(u_{w,f})\|_p^{2-p} \beta'_\mu(u_{w,f}) |\beta_\mu(u_{w,f})|^{p-2} dx dt \\
&\geq -\left( \left\| \frac{\partial}{\partial t} \left( \alpha_1 \left( \frac{\partial u_{w,f}}{\partial t} \right) \right) \right\|_{L^{p'}(0,T;(W^{1,p}(\Omega))^*)} + \|f\|_{L^{p'}(0,T;(W^{1,p}(\Omega))^*)} + \lambda_1 \|u_{w,f}\|_{L^p(0,T;W^{1,p}(\Omega))}^{r_1} \right. \\
&\quad \left. + \lambda_2 \|u_{w,f}\|_{L^p(0,T;W^{1,p}(\Omega))}^{r_2} \right) \|L_\mu(w)\|_{L^p(0,T;L^p(\Gamma))}.
\end{aligned}$$

Thus from Theorem 1.1, we have  $R(A) + R(L) \simeq R(A + L)$ .

For  $h(x, t) \in L^p(0, T; L^p(\Gamma))$ , we rewrite it as  $h(x, t) = g(x, t) - [g(x, t) - h(x, t)]$ , where  $g \in L^p(0, T; L^p(\Gamma))$ . For  $g(x, t)$ , we write it as  $g(x, t) = g_1(x, t) + g_2(x, t)$ , where

$$\begin{aligned}
g_1(x, t) &= g(x, t) - \frac{1}{T \cdot \operatorname{meas}(\Gamma)} \int_0^T \int_\Gamma g(x, t) d\Gamma(x) dt - \frac{1}{T \cdot \operatorname{meas}(\Gamma)} \int_0^T \int_\Omega f dx dt \\
&\quad + \frac{1}{T \cdot \operatorname{meas}(\Gamma)} \left( \lambda_1 \int_0^T \int_\Omega |u_{w,f}|^{r_1-2} u_{w,f} dx dt + \lambda_2 \int_0^T \int_\Omega |u_{w,f}|^{r_2-2} u_{w,f} dx dt \right)
\end{aligned}$$

and

$$\begin{aligned}
g_2(x, t) &= \frac{1}{T \cdot \operatorname{meas}(\Gamma)} \left( \int_0^T \int_\Omega f dx dt + \int_0^T \int_\Gamma h d\Gamma(x) dt \right. \\
&\quad \left. - \lambda_1 \int_0^T \int_\Omega |u_{w,f}|^{r_1-2} u_{w,f} dx dt - \lambda_2 \int_0^T \int_\Omega |u_{w,f}|^{r_2-2} u_{w,f} dx dt \right) \\
&\quad - \frac{1}{T \cdot \operatorname{meas}(\Gamma)} \int_0^T \int_\Gamma (h - g) d\Gamma(x) dt.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_0^T \int_\Gamma g_1(x, t) d\Gamma(x) dt \\
&= -\int_0^T \int_\Omega f dx dt + \lambda_1 \int_0^T \int_\Omega |u_{w,f}|^{r_1-2} u_{w,f} dx dt + \lambda_2 \int_0^T \int_\Omega |u_{w,f}|^{r_2-2} u_{w,f} dx dt \\
&= \int_0^T \int_\Omega \operatorname{div}[\alpha_2(|\nabla u_{w,f}|^p) |\nabla u_{w,f}|^{p-2} \nabla u_{w,f}] dx dt + \int_0^T \int_\Omega \frac{\partial}{\partial t} \alpha_1 \left( \frac{\partial u_{w,f}}{\partial t} \right) dx dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^T \int_{\Omega} \operatorname{div}[\alpha_2(|\nabla u_{w,f}|^p)|\nabla u_{w,f}|^{p-2}\nabla u_{w,f}]dxdt \\
&= \int_0^T \int_{\Gamma} \langle \vartheta, \alpha_2(|\nabla u_{w,f}|^p)|\nabla u_{w,f}|^{p-2}\nabla u_{w,f} \rangle d\Gamma(x)dt \\
&= \int_0^T \int_{\Gamma} Awd\Gamma(x)dt,
\end{aligned}$$

which implies that  $g_1 \in R(A)$ .

On the other hand, for the small perturbation  $h - g \in L^p(0, T; L^p(\Gamma))$ , from the given condition that

$$\frac{1}{T \cdot \operatorname{meas}(\Gamma)} \left( \int_0^T \int_{\Omega} f(x, t) dxdt + \int_0^T \int_{\Gamma} h d\Gamma(x)dt \right) \in R\left(\beta + \frac{1}{T \cdot \operatorname{meas}(\Gamma)}P\right),$$

we can easily know that  $g_2 \in R(L)$ . Therefore,  $g \in R(A+L)$ , which implies that  $h \in R(A+L)$ .

Thus

$$-\langle \vartheta, \alpha_2(|\nabla u_{w,f}|^p)|\nabla u_{w,f}|^{p-2}\nabla u_{w,f} \rangle \in \beta(u_{w,f}) - h(x, t).$$

That is, (1.4) has a unique solution in  $L^p(0, T; W^{1,p}(\Omega))$ . This completes the proof.

## References

- [1] Calvert B D, Gupta C P. Nonlinear elliptic boundary value problems in  $L^p$ -spaces and sums of ranges of accretive operators[J]. Nonl. Anal.,1978, 2: 1–26.
- [2] Wei Li, He Zhen. The applications of theories of accretive operators to nonlinear elliptic boundary value problems in  $L^p$ -spaces[J]. Nonl. Anal., 2001, 46: 199–211.
- [3] Wei Li, Zhou Haiyun. The existence of solutions of nonlinear boundary value problem involving the  $p$ -Laplacian operator in  $L^s$ -spaces[J]. J. Syst. Sci. Compl., 2005, 18: 511–521.
- [4] Wei Li, Zhou Haiyun. Research on the existence of solution of equation involving the  $p$ -Laplacian operator[J]. Appl. Math. J. Chinese Univ., Ser B, 2006, 21(2): 191–202.
- [5] Wei Li, Agarwal R P, Wong P J Y. Applications of perturbations on accretive mappings to nonlinear elliptic systems involving  $(p, q)$ -Laplacian[J]. Nonl. Oscil., 2009, 12: 199–212.
- [6] Wei Li, Agarwal R P. Existence of solutions to nonlinear Neumann boundary value problems with generalized  $p$ -Laplacian operator[J]. Comput. Math. Appl., 2008, 56: 530–541.
- [7] Wei Li, Liu Yuanxing. Study on one kind nonlinear Neumann boundary value problems[J]. Math. Appl., 2015, 18: 99–109.
- [8] Wei Li, Agarwal R P. Existence of solution to parabolic boundary value problem with generalized  $p$ -Laplacian operator (in Chinese)[J]. Acta. Math. Appl. Sinica, 2014, 37(1): 1–12.
- [9] Zeidler E. Nonlinear functional analysis and its applications, II, a linear monotone operators[M]. Berlin: Springer-Verlag, 1990.
- [10] Reich S. The range of sums of accretive and monotone operators[J]. J. Math. Anal. Appl., 1979, 68: 310–317.
- [11] Barbu V. Nonlinear semigroups and differential equations in Banach spaces[M]. Romania: Noordhoff, Leyden, 1976.

- [12] Pascali D, Sburlan S. Nonlinear mappings of monotone type[M]. Netherlands: Sijthoff and Noordhoff, 1978.
- [13] Adamas R A. The sobolev space (Version of Chinese translation)[M]. Beijing: People's Education Press, 1981.
- [14] Wei Li, Duan Liling, Agarwal R P. Existence and uniqueness of the solution to integro-differential equation involving the generalized  $p$ -Laplacian operator with mixed boundary conditions[J]. J. Math., 2013, 33(6): 1009–1018.

## 具混合边界的双曲微分方程在 $L^p(0, T; W^{1,p}(\Omega))$ 空间中的研究

魏利<sup>1</sup>, RAVI P. AGARWAL<sup>2,3</sup>, PATRICIA J. Y. WONG<sup>4</sup>

(1.河北经贸大学数学与统计学学院, 河北 石家庄 050061)

(2.Department of Mathematics, Texas A&M University - Kingsville, Kingsville, TX 78363, USA)

(3.Department of Mathematics, Faculty of Science, King Abdulaziz University,  
Jeddah 21589, Saudi Arabia)

(4. School of Electrical and Electronic Engineering, Nanyang Technological University,  
Singapore 639798, Singapore)

**摘要:** 本文研究了一类具混合边界的一般形式的双曲微分方程. 利用分裂方程和Neumann边界条件的方法,借助于定义非线性算子, 并利用Reich关于极大单调算子值域几乎相等的结论检验所定义算子具备某些性质的技巧, 获得了双曲边值问题在  $L^p(0, T; W^{1,p}(\Omega))$  空间中存在唯一解的结果. 基于双曲方程中的主项是非线性的, 所以本文应用了新的证明技巧, 推广和补充了以往的相关工作.

**关键词:** 严格单调算子; 次微分; 迹算子; 值域几乎相等; 双曲方程

MR(2010)主题分类号: 47H05; 47H09      中图分类号: O177.91