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THE FRACTIONAL NOLINEAR BI-INTEGRABLE COUPLINGS OF KAUP-NEWELL HIERARCHY AND ITS HAMILTONIAN STRUCTURES

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Abstract: In this paper, we study the fractional nolinear bi-integrable couplings of Kaup-Newell hierarchy. By using fractional isospectral problems and non-semisimple matrix Lie algebras on which there exist non-degenerate, symmetric and ad-invariant bilinear forms, the fractional nonlinear bi-integrable couplings of Kaup-Newell hierarchy are presented. Furthermore, we also obtained the fractional Hamiltonian structures of the fractional integrable couplings of Kaup-Newell hierarchy. The methods derived by us can be generalized to other fractional integrable couplings of soliton hierarchy.

Keywords: matrix Lie algebras; Kaup-Newell hierarchy; bi-integrable couplings; fractional Hamiltonian structures

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1 Introduction

The theory of integrals and derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov. The fractional analysis attracted the interest of many researchers, because fractional analysis has numerous applications: kinetic theories [1, 2], statistical mechanics [3, 4], dynamics in complex media [5, 6], and many others [7–9]. The main advantage of fractional derivative in comparison with classical integer-order models is that it provides an effective instrument for the description of memory and hereditary properties of various materials and progress. Also, the advantages of the fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields.

Tu proposed a Lie algebras and trace identity to constructing the integrable system and Hamiltonian structure of integrable systems [10]. Afterwards, Ma called the method as Tu scheme [11]. From then on, many integrable system and theirs Hamiltonian structure

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Biography: Wei Hanyu (1982–), male, born at Zhoukou, Henan, lecturer, major in solitons and inegrable systems.

were obtained [12, 13]. Then how to obtain new fractional integrable hierarchies of the fractional soliton equations is an important and interesting work in soliton theory. In [14], Wu firstly proposed the generalized Tu formula and research for the Hamiltonian structures of fractional AKNS hierarchy, Yu presented the generalized fractional KN equation hierarchy and its fractional Hamiltonian structure [15]. Integrable couplings [16] were coupled systems of integrable equations, which has been introduced when we study of Virasoro symmetric algebras. It is an important and interesting topic to search for integrable couplings in soliton theory.

Let us consider an integrable evolution equation

$$D_t^{\beta} u = K(u). \tag{1.1}$$

An integrable coupling of eq.(1.1)

$$D_t^{\beta} \bar{u} = \bar{K}_1(\bar{u}) = \begin{pmatrix} K(u) \\ S_1(u, u_1) \end{pmatrix}, \bar{u} = \begin{pmatrix} u \\ u_1 \end{pmatrix} \tag{1.2}$$

is called nonlinear, if $S_1(u, u_1)$ is nonlinear with respect to the sub-vector u_1 of dependent variables [17]. An integrable system of the form

$$D_t^{\beta} \bar{u} = \bar{K}_2(\bar{u}) = \begin{pmatrix} K(u) \\ S_1(u, u_1) \\ S_2(u, u_1, u_2) \end{pmatrix}, \bar{u} = \begin{pmatrix} u \\ u_1 \\ u_2 \end{pmatrix} \tag{1.3}$$

is called a bi-integrable coupling of eq.(1.1). Note that in (1.3), S_1 does not depend on u_2, and the whole systemis of triangular form. In this paper, we would like to explore some mathematical structures of Lie algebras and zero curvature equations, to construct bi-integrable couplings and their Hamiltonian structures by using the generalized fractional trace variational identity associated with the enlarged Lax pairs. In this paper, we shall introduce a kind of explicit Lie algebra for which fractional couplings of Kaup-Newell hierarchy can be generated. Then we construct the fractional Hamiltonian structures of the fractional integrable couplings of Kaup-Newell hierarchy by using generalized fractional trace variational identity.

2 Brief Overview of Fractional Derivatives and Integrals

Kolwankar and Gangal [18, 19] defined the local fractional derivative as

$$D_{x^+}^{\alpha} f(x) = \lim_{y \to x^+} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dy} \int_x^y \frac{(f(\xi) - f(x))}{(y-\xi)^{\alpha}} d\xi \quad (0 < \alpha < 1). \tag{2.1}$$

Chen et al. [20] gave the necessary conditions for the following relationship

$$D_{x^+}^{\alpha} f(x) = \lim_{y \to x^+} \frac{\Gamma(1+\alpha)(f(y) - f(x))}{(y-x)^{\alpha}} \quad (0 < \alpha < 1). \tag{2.2}$$

We adopt derivative (2.2) for simplicity. Some fractional derivative properties are proposed as follows.

(i) The generalized Leibniz product law. If $f(x), g(x)$ are α order differentiable functions, one can have

$$D_x^\alpha(f(x)g(x)) = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x). \tag{2.3}$$

Proof By the simplicity definition (2.2), we have

$$\begin{aligned} D_x^\alpha(f(x)g(x)) &= \lim_{y\to x^+} \frac{\Gamma(1+\alpha)(f(y)g(y)-f(x)g(x))}{(y-x)^\alpha} \\ &= \lim_{y\to x^+} \frac{\Gamma(1+\alpha)(f(y)g(y)-f(x)g(y)+f(x)g(y)-f(x)g(x))}{(y-x)^\alpha} \\ &= \lim_{y\to x^+} \frac{\Gamma(1+\alpha)(f(y)-f(x))g(y)}{(y-x)^\alpha} + \lim_{y\to x^+} \frac{\Gamma(1+\alpha)(g(y)-g(x))f(x)}{(y-x)^\alpha} \\ &= g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x), \end{aligned} \tag{2.4}$$

the proof is completed.

(ii) The Leibniz Formula for fractional differentiable functions reads.

$${}_0I_x^\alpha D_x^\alpha f(x) = f(x) - f(0) \quad (0 < a < 1), \tag{2.5}$$

where ${}_0I_x^\alpha$ denotes the Riemann-Liouville integration, which is defined as

$${}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi \quad (0 < a < 1). \tag{2.6}$$

Therefore from the defined fractional integration, properties (i) and (ii), the integration by parts can be used during the fractional calculus

$${}_0I_x^\alpha[g(x)D_x^\alpha f(x)] = f(x)g(x)|_0^x - {}_0I_x^\alpha[f(x)D_x^\alpha g(x)]. \tag{2.7}$$

(iii) Fractional variational derivative

$$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} + \sum_{k=1} (-1)^k (D_x^\alpha)^k (\frac{\partial L}{\partial (D_x^\alpha)^k y}), \tag{2.8}$$

where k is a positive integer. Properties (ii) and (iii) can be proved similarly, we omit these proofs in this paper.

3 Matrix Lie Algebras and Bi-Integrable Couplings

To generate bi-integrable couplings, we introduce a kind of block matrices

$$M(A_1, A_2, A_3) = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \xi A_2 & A_2 \\ 0 & 0 & A_1 \end{pmatrix}, \tag{3.1}$$

where ξ be an arbitrary fixed constant, which could be zero. $A_i = A_i(\lambda), i = 1, 2, 3$ are square matrices of the same order, depending on the free parameter λ. Then we have the commutator relation

$$[M(A_1, A_2, A_3), M(B_1, B_2, B_3)] = M(C_1, C_2, C_3) \tag{3.2}$$

with

$$\begin{cases} C_1 = [A_1, B_1], \\ C_2 = [A_1, B_2] + [A_2, B_1] + \xi[A_2, B_2], \\ C_3 = [A_1, B_3] + [A_3, B_1] + [A_2, B_2]. \end{cases} \tag{3.3}$$

This closure property implies that all block matrices defined by (3.1) form a matrix Lie algebra. Such matrix Lie algebras create a basis for us to generate nonlinear Hamiltonian bi-integrable couplings. The block A_1 corresponds to the original integrable equation, and the other two blocks A_2 and A_3 are used to generate the supplementary vector fields S_1 and S_2. The commutator $[A_2, B_2]$ yields nonlinear terms in the resulting bi-integrable couplings.

Let us consider linear spectral problem

$$D_x^\alpha \varphi = U(u, \lambda)\varphi, \quad D_t^\beta \varphi = V(u, \lambda)\varphi \tag{3.4}$$

from the fractional zero curvature equation

$$D_t^\beta U - D_x^\alpha V + [U, V] = 0, \tag{3.5}$$

we get an integrable system

$$D_{t_n}^\beta u = K_n(u). \tag{3.6}$$

Now we introduce an enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{pmatrix} U(u, \lambda) & U_1(u_1, \lambda) & U_2(u_2, \lambda) \\ 0 & U(u, \lambda) + \xi U_1(u_1, \lambda) & U_1(u_1, \lambda) \\ 0 & 0 & U(u, \lambda) \end{pmatrix}, \tag{3.7}$$

where $\bar{u}$ consists of u, u_1, u_2. From an enlarged generalized zero curvature equation

$$D_t^\beta \bar{U} - D_x^\alpha \bar{V} + [\bar{U}, \bar{V}] = 0 \tag{3.8}$$

with

$$\bar{V} = \bar{V}(\bar{u}, \lambda) = \begin{pmatrix} V(u, \lambda) & V_1(u, u_1, \lambda) & V_2(u, u_1, u_2, \lambda) \\ 0 & V(u, \lambda) + \xi V_1(u, u_1, \lambda) & V_1(u, u_1, \lambda) \\ 0 & 0 & V(u, \lambda) \end{pmatrix} \tag{3.9}$$

gives rise to

$$\begin{cases} D_t^\beta U - D_x^\alpha V + [U, V] = 0, \\ D_t^\beta U_t - D_x^\alpha V_1 + [U, V_1] + [U_1, V] + \xi[U_1, V_1] = 0, \\ D_t^\beta U_t - D_x^\alpha V_2 + [U, V_2] + [U_2, V] + [U_1, V_1] = 0, \end{cases} \tag{3.10}$$

i.e.,

$$D_{t_n}^{\beta}\bar{u}=\begin{pmatrix} D_{t_n}^{\beta}u \\ D_{t_n}^{\beta}u_1 \\ D_{t_n}^{\beta}u_2 \end{pmatrix}=\begin{pmatrix} K(u) \\ S_1(u,u_1) \\ S_2(u,u_1,u_2) \end{pmatrix}. \tag{3.11}$$

This is a bi-integrable coupling of the evolution eq.(3.6), noting the zero curvature representation (3.5) of (3.4). Normally, it is nonlinear with respect to u_1 and u_2, thereby providing a nonlinear bi-integrable coupling.

We take a solution $\bar{W}$ to the enlarged stationary zero curvature equation

$$D_x^{\alpha}\bar{W}=[\bar{U},\bar{W}] \tag{3.12}$$

apply the associated generalized fractional trace variational identity [18]

$$\frac{\delta}{\delta\bar{u}}\langle\bar{W},\bar{U}_{\lambda}\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle\bar{W},\bar{U}_{\bar{u}}\rangle\ \ (\ \gamma \text{ is a constant}) \tag{3.13}$$

to furnish Hamiltonian structures for the bi-integrable couplings described above.

In the next section, we will apply the above computational paradigm to the Kaup-Newell hierarchy, thus generating ahierarchy of nonlinear Hamiltonian bi-integrable couplings for the Kaup-Newell equations. We remark that our general ideaworks for both positive and negative soliton hierarchies.

4 Application to the Kaup-Newell Hierarchy

The Kaup-Newell spectral problem [21]

$$D_x^{\alpha}\varphi=U(u,\lambda)\varphi, U=\begin{pmatrix} -\lambda & \lambda q \\ r & \lambda \end{pmatrix}, u=\begin{pmatrix} q \\ r \end{pmatrix}. \tag{4.1}$$

Setting

$$W=\begin{pmatrix} a & \lambda b \\ c & -a \end{pmatrix}=\sum_{m\geq 0}W_m\lambda^{-m}=\sum_{m\geq 0}\begin{pmatrix} a_m & \lambda b_m \\ c_m & -a_m \end{pmatrix}\lambda^{-m}, \tag{4.2}$$

the stationary zero curvature equation $D_x^{\alpha}W=[U,W]$ gives

$$\begin{cases} D_x^{\alpha}a_m=-rb_{m+1}+qc_{m+1}, \\ D_x^{\alpha}b_m=-2b_{m+1}-2qa_m, \\ D_x^{\alpha}c_m=2ra_m+2c_{m+1}, \\ b_0=c_0=0, a_0=1, b_1=-q, c_1=-r, a_1=-\frac{1}{2}qr, \\ b_2=\frac{1}{2}D_x^{\alpha}q+\frac{1}{2}q^2r, c_2=-\frac{1}{2}D_x^{\alpha}r+\frac{1}{2}qr^2, a_2=\frac{1}{4}rD_x^{\alpha}q-\frac{1}{4}qD_x^{\alpha}r+\frac{3}{8}q^2r^2,\cdots. \end{cases} \tag{4.3}$$

Let

$$V^{(n)}=(\lambda^nW)_++\triangle_n, \tag{4.4}$$

where $\triangle_n=-a_ne_1$. The fractional zero curvature equation

$$D_{t_n}^{\beta}U-D_x^{\alpha}V^{(n)}+[U,V^{(n)}]=0 \tag{4.5}$$

generate the fractional Kaup-Newell hierarchy of soliton equations

$$D_{t_n}^{\beta} u = K_n(u) = (D_x^{\alpha} b_n, D_x^{\alpha} c_n)^T = J\frac{\delta H_n}{\delta u}, \quad n \geq 1 \tag{4.6}$$

with the Hamiltonian operator J, the Hamiltonian functions and the hereditary recursion operator L:

$$J = \begin{pmatrix} 0 & D_x^{\alpha} \\ D_x^{\alpha} & 0 \end{pmatrix}, \quad H_n = \frac{2a_n - qc_n}{n}, \quad L = \begin{pmatrix} -\frac{1}{2}D_x^{\alpha} - \frac{1}{2}D_x^{\alpha} q D_x^{-\alpha} r & -\frac{1}{2}D_x^{\alpha} q D_x^{-\alpha} q \\ -\frac{1}{2}D_x^{\alpha} r D_x^{-\alpha} r & \frac{1}{2}D_x^{\alpha} - \frac{1}{2}D_x^{\alpha} r D_x^{-\alpha} q \end{pmatrix}. \tag{4.7}$$

When we take $n = 2$, hierarchy (4.6) can be reduced to 2-order fractional Kaup-Newell equations

$$\begin{cases} D_{t_2}^{\beta} q = D_x^{\alpha}(\frac{1}{2}D_x^{\alpha} q + \frac{1}{2}q^2 r), \\ D_{t_2}^{\beta} r = D_x^{\alpha}(-\frac{1}{2}D_x^{\alpha} r + \frac{1}{2}q r^2). \end{cases} \tag{4.8}$$

We begin with an enlarged spectral matrix

$$\bar{U}(\bar{u}) = \begin{pmatrix} U & U_1 & U_2 \\ 0 & U + \xi U_1 & U_1 \\ 0 & 0 & U \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} q \\ r \\ p_1 \\ p_2 \\ w_1 \\ w_2 \end{pmatrix}, \tag{4.9}$$

where U is defined as in (4.1) and the supplementary spectral matrices U_1 and U_2 read

$$\begin{aligned} U_1 = U_1(u_1) &= \begin{pmatrix} 0 & \lambda p_1 \\ p_2 & 0 \end{pmatrix}, u_1 = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \\ U_2 = U_2(u_2) &= \begin{pmatrix} 0 & \lambda w_1 \\ w_2 & 0 \end{pmatrix}, u_2 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \end{aligned} \tag{4.10}$$

To solve the enlarged stationary zero curvature eq.(3.12), we take a solution of the following form

$$\bar{W}(\bar{u}) = \begin{pmatrix} W & W_1 & W_2 \\ 0 & W + \xi W_1 & W_1 \\ 0 & 0 & W \end{pmatrix}, \tag{4.11}$$

where W, defined by (4.2), and

$$W_1 = \begin{pmatrix} e & \lambda f \\ g & -e \end{pmatrix}, \quad W_2 = \begin{pmatrix} e' & \lambda f' \\ g' & -e' \end{pmatrix}. \tag{4.12}$$

Setting

$$\begin{aligned} e &= \sum_{m\geq 0} e_m\lambda^{-m}, f = \sum_{m\geq 0} f_m\lambda^{-m}, g = \sum_{m\geq 0} g_m\lambda^{-m}, \\ e' &= \sum_{m\geq 0} e'_m\lambda^{-m}, f' = \sum_{m\geq 0} f'_m\lambda^{-m}, g' = \sum_{m\geq 0} g'_m\lambda^{-m}, \end{aligned} \tag{4.13}$$

which solves $D_x^\alpha W = [U, W]$, we have

$$\begin{cases} D_x^\alpha e_m = qg_{m+1} - rf_{m+1} + p_1c_{m+1} - p_2b_{m+1} + \beta p_1 g_{m+1} - \beta p_2 f_{m+1}, \\ D_x^\alpha f_m = -2f_{m+1} - 2qe_m - 2p_1a_m - 2\beta p_1 e_m, \\ D_x^\alpha g_m = 2g_{m+1} + 2re_m + 2u_2a_m + 2p_2e_m, \\ D_x^\alpha e'_m = qg'_{m+1} - rf'_{m+1} + w_1c_{m+1} - w_2b_{m+1} + p_1g_{m+1} - p_2f_{m+1}, \\ D_x^\alpha f'_m = -2f'_{m+1} - 2qe'_m - 2w_1a_m - 2p_1e_m, \\ D_x^\alpha g'_m = 2g'_{m+1} + 2re'_m + 2w_2a_m + 2p_2e_m. \end{cases} \tag{4.14}$$

We choose the initial data to be

$$f_0 = g_0 = f'_0 = g'_0 = 0, e_0 = e'_0 = 1 \tag{4.15}$$

from the recursion relation (4.14), we can get

$$\begin{aligned} f_1 &= -q - (1+\beta)p_1, g_1 = -r - (1+\beta)p_2, \\ e_1 &= -\frac{1}{2}(qr + qp_2 + \beta qp_2 + p_1r + \beta p_1 r - \beta p_1p_2 - \beta^2 p_1p_2), \\ f'_1 &= -q - w_1 - p_1, g'_1 = -r - w_2 - p_2, \\ e'_1 &= -\frac{1}{2}(qr + qw_2 + qp_2 + w_1r + p_1r + p_1p_2 + \beta p_1p_2), \cdots . \end{aligned} \tag{4.16}$$

For each integer $n \geq 0$, take

$$\bar{V}^{(n)} = \begin{pmatrix} V^{(n)} & V_1^{(n)} & V_2^{(n)} \\ 0 & V^{(n)} + \xi V_1^{(n)} & V_1^{(n)} \\ 0 & 0 & V^{(n)} \end{pmatrix}, \tag{4.17}$$

where $V^{(n)}$ define in (4.4), and $V_1^{(n)}$, $V_2^{(n)}$,

$$V_1^{(n)} = (\lambda^n W_1)_+ + \triangle_1, V_2^{(n)} = (\lambda^n W_2)_+ + \triangle_2 \tag{4.18}$$

with $\triangle_1 = -e_ne_1, \triangle_2 = -e'_ne_1$, and then from the enlarged generalized zero curvature equation (3.8), we have

$$D_{t_n}^\beta \bar{v} = S_n(\bar{v}) = \begin{pmatrix} S_{1n}(u, u_1) \\ S_{2n}(u, u_1, u_2) \end{pmatrix}, \tag{4.19}$$

where

$$S_{1n}(u, u_1) = \begin{pmatrix} D_x^\alpha f_n \\ D_x^\alpha g_n \end{pmatrix}, \quad S_{2n}(u, u_1, u_2) = \begin{pmatrix} D_x^\alpha f'_n \\ D_x^\alpha g'_n \end{pmatrix}. \tag{4.20}$$

Together with system (4.6), we present bi-integrable couplings of Kaup-Newell hierarchy

$$D^{\beta}_{t_n}\bar{u} = \begin{pmatrix} D^{\beta}_{t_n}q \\ D^{\beta}_{t_n}r \\ D^{\beta}_{t_n}p_1 \\ D^{\beta}_{t_n}p_2 \\ D^{\beta}_{t_n}w_1 \\ D^{\beta}_{t_n}w_2 \end{pmatrix} = \bar{K}_n(\bar{u}) = \begin{pmatrix} K_n(u) \\ S_{1n}(u,u_1) \\ S_{2n}(u,u_1,u_2) \end{pmatrix} = \begin{pmatrix} D^{\alpha}_x b_n \\ D^{\alpha}_x c_n \\ D^{\alpha}_x f_n \\ D^{\alpha}_x g_n \\ D^{\alpha}_x f'_n \\ D^{\alpha}_x g'_n \end{pmatrix}, n \geq 1. \tag{4.21}$$

When $n = 2$, we can obtain the first nonlinear bi-integrable coupling of (4.8)

$$\begin{cases} D^{\beta}_{t_2}q = & D^{\alpha}_x(\frac{1}{2}D^{\alpha}_x q + \frac{1}{2}q^2 r), \\ D^{\beta}_{t_2}r = & D^{\alpha}_x(-\frac{1}{2}D^{\alpha}_x r + \frac{1}{2}qr^2), \\ D^{\beta}_{t_2}p_1 = & \frac{1}{2}D^{\alpha}_x[D^{\alpha}_x q + qrp_1 + (1+\beta)D^{\alpha}_x p_1 + (q+\beta P_1) \\ & (qr + qp_2 + \beta qp_2 + p_1 r + \beta p_1 r + \beta p_1 p_2 + \beta^2 p_1 p_2)], \\ D^{\beta}_{t_2}p_2 = & -\frac{1}{2}D^{\alpha}_x[D^{\alpha}_x r - qrp_2 + (1+\beta)D^{\alpha}_x p_2 + (r+\beta p_2) \\ & (qr + qp_2 + \beta qp_2 + p_1 r + \beta p_1 r + \beta p_1 p_2 + \beta^2 p_1 p_2)], \\ D^{\beta}_{t_2}w_1 = & D^{\alpha}_x[\frac{1}{2}D^{\alpha}_x(q + w_1 + p_1) + \frac{1}{2}q^2(r + w_2 + p_2) + \frac{1}{2}p_1^2(r + \beta r + \beta p_2 + \beta^2 p_2) \\ & +qr(p_1 + w_1) + (1+\beta)qp_1p_2], \\ D^{\beta}_{t_2}w_2 = & D^{\alpha}_x[-\frac{1}{2}(r + w_2 + p_2) + \frac{1}{2}r^2(q + w_1 + p_1) + \frac{1}{2}p_2^2(q + \beta q + \beta p_1 + \beta^2 p_1) \\ & +qr(w_2 + p_2) + (1+\beta)p_1p_2r]. \end{cases} \tag{4.22}$$

5 Fractional Hamiltonian Structure

In order to generate Hamiltonian structures of the obtained fractional nonlinear bi-integrable couplings, we have to compute non-degenerate, symmetric and ad-invariant bilinear forms on the adopted Lie algebra

$$\bar{g} = \left(\begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \xi A_2 & A_2 \\ 0 & 0 & A_1 \end{pmatrix} | A_1, A_2, A_3 \in \tilde{sl}(2) \right). \tag{5.1}$$

As usual, we transform the Lie algebra $\bar{g}$ into a vector form through the mapping

$$\sigma : \bar{g} \longrightarrow R^9, \quad A \longmapsto (a_1, \cdots, a_9)^T \in R^9, \tag{5.2}$$

where

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \xi A_2 & A_2 \\ 0 & 0 & A_1 \end{pmatrix} \in \bar{g}, A_i = \begin{pmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{pmatrix}, i = 1,2,3. \tag{5.3}$$

A required bilinear form [22] on the underlying Lie algebra $\bar{g}$ is given by

$$\begin{aligned} \langle A, B \rangle_{\bar{g}} = & \eta_1(a_1b_1 + \frac{1}{2}a_2b_3 + \frac{1}{2}a_3b_2) + \eta_2[a_1b_4 + \frac{1}{2}a_2b_6 + \frac{1}{2}a_3b_5 + a_4(b_1 + \xi b_4) \\ & + \frac{1}{2}a_5(b_3 + \xi b_6) + \frac{1}{2}a_6(b_2 + \xi b_5)] + \eta_3[2a_1b_7 + a_2b_9 + a_3b_8 + 2a_4b_4 \\ & + a_5b_6 + a_6b_5 + 2a_7b_1 + a_8b_3 + a_9b_2], \end{aligned} \tag{5.4}$$

where A and B are two block matrices of the form defined by (5.3), η_1, η_2, η_3 are constants.

Let us direct compute that

$$\langle \bar{W}\frac{\partial \bar{U}}{\partial \lambda}\rangle = \frac{1}{2}\eta_1(-2a+cq)+\frac{1}{2}\eta_2(cp_1-2e+gq+\beta gp_1)+\eta_3(cw_1+gp_1-2e'+qg'),$$
$$\langle \bar{W}\frac{\partial \bar{U}}{\partial q}\rangle = \frac{1}{2}\eta_1 c\lambda+\frac{1}{2}\eta_2 g\lambda+\eta_3 g'\lambda, \langle \bar{W}\frac{\partial \bar{U}}{\partial r}\rangle = \frac{1}{2}\eta_1 b\lambda+\frac{1}{2}\eta_2 f\lambda+\eta_3 f'\lambda, \langle \bar{W}\frac{\partial \bar{U}}{\partial w_1}\rangle = \eta_3 c\lambda,$$
$$\langle \bar{W}\frac{\partial \bar{U}}{\partial p_1}\rangle = \frac{1}{2}\eta_2 c\lambda+\frac{1}{2}\beta\eta_2 g\lambda+\eta_3 g\lambda, \langle \bar{W}\frac{\partial \bar{U}}{\partial p_2}\rangle = \frac{1}{2}\eta_2 b\lambda+\frac{1}{2}\beta\eta_2 f\lambda+\eta_3 f\lambda, \langle \bar{W}\frac{\partial \bar{U}}{\partial w_2}\rangle = \eta_3 b\lambda. \tag{5.5}$$

The corresponding fractional variational identity (3.13) leads to

$$\begin{aligned}&\frac{\delta}{\delta \bar{u}}\frac{\frac{1}{2}\eta_1(2a_n-qc_n)+\frac{1}{2}\eta_2(2e_n-p_1c_n-qg_n-\beta p_1 g_n)+\eta_3(2e'_n-w_1c_n-p_1g_n-qg'_n)}{n}\\ =\ &(\frac{1}{2}\eta_1 c_n+\frac{1}{2}\eta_2 g_n+\eta_3 g'_n, \frac{1}{2}\eta_1 b_n+\frac{1}{2}\eta_2 f_n+\eta_3 f'_n, \frac{1}{2}\eta_2 c_n+\frac{1}{2}\beta\eta_2 g_n+\eta_3 g_n,\\ &\frac{1}{2}\eta_2 b_n+\frac{1}{2}\beta\eta_2 f_n+\eta_3 f_n, \eta_3 c_n, \eta_3 b_n)^T, n\geq 1.\end{aligned} \tag{5.6}$$

It is easy to see that $\gamma=0$. Threrfor, the Kaup-Newell bi-integrable couplings in (4.21) possess the following Hamiltonian structures

$$D^{\beta}_{t_n}\bar{u} = \bar{K}_n(\bar{u}) = \bar{J}\frac{\delta \bar{H}_n}{\delta \bar{u}}, \quad n\geq 0, \tag{5.7}$$

where the Hamiltonian operator is

$$\bar{J} = \begin{pmatrix} 0 & 0 & \frac{1}{\eta_3} \\ 0 & \frac{2D^{\alpha}_x}{\beta\eta_2+2\eta_3} & \frac{-\eta_2}{\eta_3(\beta\eta_2+2\eta_3)} \\ \frac{1}{\eta_3} & \frac{-\eta_2}{\eta_3(\beta\eta_2+2\eta_3)} & \frac{\eta_2^2-\beta\eta_1\eta_2-2\eta_1\eta_3}{2\eta_3(\beta\eta_2+2\eta_3)} \end{pmatrix} \otimes \begin{pmatrix} 0 & D^{\alpha}_x \\ D^{\alpha}_x & 0 \end{pmatrix} \tag{5.8}$$

with $\otimes$ denotes the Kronecker product of matrices, and the fractional Hamiltonian functionals read

$$\bar{H}_n = \frac{\frac{1}{2}\eta_1(2a_n-qc_n)+\frac{1}{2}\eta_2(2e_n-p_1c_n-qg_n-\beta p_1g_n)+\eta_3(2e'_n-w_1c_n-p_1g_n-qg'_n)}{n}. \tag{5.9}$$

A direct computation shows a recursion relation

$$\bar{K}_{n+1} = \bar{L}\bar{K}_n, \tag{5.10}$$

where the recursion operator $\bar{L}$ is given by

$$\bar{L} = \begin{pmatrix} L & 0 & 0 \\ L_1 & L+\xi L_1 & 0 \\ L_2 & L_1 & L \end{pmatrix} \tag{5.11}$$

with L being given by (4.7) and

$$L_1 = -\frac{1}{2}\begin{pmatrix} D_x^\alpha q D_x^{-\alpha} p_2 + D_x^\alpha p_1 D_x^{-\alpha}(r+\beta p_2) & D_x^\alpha q D_x^{-\alpha} p_1 + D_x^\alpha p_1 D_x^{-\alpha}(q+\beta p_1) \\ D_x^\alpha r D_x^{-\alpha} p_2 + D_x^\alpha p_2 D_x^{-\alpha}(r+\beta p_2) & D_x^\alpha r D_x^{-\alpha} p_1 + D_x^\alpha p_2 D_x^{-\alpha}(q+\beta p_1) \end{pmatrix},$$

$$L_2 = -\frac{1}{2}\begin{pmatrix} D_x^\alpha q D_x^{-\alpha} w_2 + D_x^\alpha w_1 D_x^{-\alpha} r + D_x^\alpha p_1 D_x^{-\alpha} p_2 & D_x^\alpha q D_x^{-\alpha} w_1 + D_x^\alpha w_1 D_x^{-\alpha} q + D_x^\alpha p_1 D_x^{-\alpha} p_1 \\ D_x^\alpha r D_x^{-\alpha} w_2 + D_x^\alpha w_2 D_x^{-\alpha} r + D_x^\alpha p_2 D_x^{-\alpha} p_2 & D_x^\alpha r D_x^{-\alpha} w_1 + D_x^\alpha w_2 D_x^{-\alpha} q + D_x^\alpha p_2 D_x^{-\alpha} p_1 \end{pmatrix}.$$

6 Conclusions

A way to construct fractional nonlinear bi-integrable couplings of fractional soliton hierarchy is presented. The fractional variational identity has been generalized to the fractional zero-curvature equation. As an application, the fractional Kaup-Newell hierarchy gave a hierarchy of the fractional nonlinear bi-integrable couplings and fractional Hamiltonian structures. As its reduction, we gain the fractional nonlinear integrable couplings of the Kaup-Newell equations. The solution of reduced equations is a very important and difficult work, we will take great efforts in our next work. The obtained results supplement the existing theories on the perturbation equations and classical integrable couplings. The method can be generalized to other fractional integrable couplings.

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Kaup-Newell族的分数阶非线性双可积耦合及其Hamilton结构

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摘要: 本文研究了Kaup-Newell族的分数阶非线性双可积耦合. 利用分数阶等谱问题和非半单矩阵Lie代数上的非退化、对称双线性形式, 得到了Kaup-Newell族的分数阶非线性双可积耦合, 并求出了Kaup-Newell族双可积耦合的分数阶Hamilton结构. 本文的方法还可以应用于其它孤子族分数阶可积耦合.

关键词: 矩阵Lie代数; Kaup-Newell族; 双可积耦合; 分数阶Hamilton结构

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