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GLOBAL SMOOTH SOLUTIONS TO THE 1-D COMPRESSIBLE NAVIER-STOKES-KORTEWEG SYSTEM WITH LARGE INITIAL DATA

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Abstract: This paper is concerned with the Cauchy problem of the one-dimensional isothermal compressible Navier-Stokes-Korteweg system when the viscosity coefficient and capillarity coefficient are general smooth functions of the density. By using the elementary energy method and Kanel's technique [25], we obtain the global existence and time-asymptotic behavior of smooth non-vacuum solutions with large initial data, which improves the previous ones in the literature.

Keywords: compressible Navier-Stokes-Korteweg system; global existence; time-asymptotic behavior; large initial data

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1 Introduction

This paper is concerned with the Cauchy problem of the one-dimensional isothermal compressible Navier-Stokes-Korteweg system with density-dependent viscosity coefficient and capillarity coefficient in the Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = (\mu(\rho) u_x)_x + K_x, \end{cases} \quad t > 0,\, x \in \mathbb{R} \tag{1.1}$$

with the initial data

$$(\rho(0,x), u(0,x)) = (\rho_0(x), u_0(x)), \quad \lim_{x\to\pm\infty} (\rho_0(x), u_0(x)) = (\bar{\rho}, 0), \tag{1.2}$$

here t and x represent the time variable and the spatial variable, respectively, K is the Korteweg tensor given by

$$K = \rho\kappa(\rho)\rho_{xx} + \frac{1}{2}\left(\rho\kappa'(\rho) - \kappa(\rho)\right)\rho_x^2.$$

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The unknown functions $\rho > 0, u, P = P(\rho)$ denote the density, the velocity, and the pressure of the fluids respectively. $\mu = \mu(\rho) > 0$ and $\kappa = \kappa(\rho) > 0$ are the viscosity coefficient and the capillarity coefficient, respectively, and $\bar{\rho} > 0$ is a given constant. Throughout this paper, we assume that

$$P(\rho) = \rho^{\gamma}, \quad \gamma > 1 \text{ is a constant.} \tag{1.3}$$

System (1.1) can be used to model the motions of compressible isothermal viscous fluids with internal capillarity, see [1–3] for its derivations. Notice that when $\kappa = 0$, system (1.1) is reduced to the compressible Navier-Stokes system.

There were extensive studies on the mathematical aspects on the compressible Navier-Stokes-Korteweg system. For small initial data, we refer to [8, 9, 13–15, 19–23] for the global existence and large time behavior of smooth solutions in Sobolev space, [5, 7, 11] for the global existence and uniqueness of strong solutions in Besov space, and [5, 6] for the global existence of weak solutions near constant states in the whole space $\mathbb{R}^2$.

For large initial data, Kotschote [12], Hattori and Li [10] proved the local existence of strong solutions. Bresch et al. [4] investigated the global existence of weak solutions for an isothermal fluid with the viscosity coefficients $\mu(\rho) = \tilde{\nu}\rho, \lambda(\rho) = 0$ and the capillarity coefficient $\kappa(\rho) \equiv \tilde{\kappa}$ in a periodic domain $\mathbb{T}^d (d = 2, 3)$, where $\tilde{\nu}, \tilde{\kappa} > 0$ are positive constants. Later, such a result was improved by Haspot [6] to some more general density-dependent viscosity coefficients. Tsyganov [16] studied the global existence and time-asymptotic convergence of weak solutions for an isothermal compressible Navier-Stokes-Korteweg system with the viscosity coefficient $\mu(\rho) \equiv 1$ and the capillarity coefficient $\kappa(\rho) = \rho^{-5}$ on the interval $[0, 1]$. Charve and Haspot [17] showed the global existence of strong solutions to system (1.1) with $\mu(\rho) = \varepsilon\rho$ and $\kappa(\rho) = \varepsilon^2\rho^{-1}$. Recently, Germain and LeFloch [18] studied the global existence of weak solutions to the Cauchy problem (1.1)–(1.2) with general density-dependent viscosity and capillarity coefficients. Both the vacuum and non-vacuum weak solutions were obtained in [18]. Moreover, Chen et al. [23, 24] discussed the global existence and large time behavior of smooth and non-vacuum solutions to the Cauchy problem of system (1.1) with the viscosity and capillarity coefficients being some power functions of the density.

However, few results were obtained for the global smooth, large solutions of the isothermal compressible Navier-Stokes-Korteweg system with general density-dependent viscosity coefficient and capillarity coefficient up to now. This paper is devoted to this problem, and we are concerned with the global existence and large time behavior of smooth, non-vacuum solutions to the Cauchy problems (1.1)–(1.2) when the the viscosity coefficient $\mu(\rho)$ and the capillarity coefficient $\kappa(\rho)$ are general smooth functions of the density ρ.

The main result of this paper is stated as follows.

Theorem 1.1 Suppose the following conditions hold:

(i) The initial data $(\rho_0(x) - \bar{\rho}, u_0(x)) \in H^4(\mathbb{R}) \times H^3(\mathbb{R})$, and there exist two positive constants m_0, m_1 such that $m_0 \leq \rho_0(x) \leq m_1$ for all $x \in \mathbb{R}$.

(ii) The smooth functions $\mu(\rho)$ and $\kappa(\rho)$ satisfy $\mu(\rho), \kappa(\rho) > 0$ for $\rho > 0$, and one of the following two conditions hold:

(a) $\displaystyle\int_0^1 \sqrt{\kappa(s)}\, ds = +\infty, \qquad \int_1^{+\infty} \sqrt{s\kappa(s)}\, ds = +\infty,$

(b) $\displaystyle\int_0^1 \frac{\mu(s)}{s^{3/2}}\, ds = +\infty, \qquad \int_1^{+\infty} \frac{\mu(s)}{s}\, ds = +\infty.$

(iii) $\displaystyle -\frac{2}{3}\left(\sqrt{\frac{\mu(\rho)\kappa(\rho)}{\rho}}\left(\sqrt{\frac{\mu(\rho)\kappa(\rho)}{\rho}}\right)'\right)' + \left(\left(\sqrt{\frac{\mu(\rho)\kappa(\rho)}{\rho}}\right)'\right)^2 + \frac{\kappa(\rho)}{3}\left(\frac{\mu(\rho)}{\rho}\right)'' + \frac{\mu(\rho)\kappa''(\rho)}{6\rho} \leq 0$. Then the Cauchy problems (1.1)–(1.2) admits a unique global smooth solution $(\rho, u)(t, x)$ satisfying

$$C_1^{-1} \leq \rho(t,x) \leq C_1, \quad \forall\, (t,x) \in [0,\infty) \times \mathbb{R}, \tag{1.4}$$

$$\begin{aligned} &\|(\rho-\bar{\rho})(t)\|^2_{H^4(\mathbb{R})} + \|u(t)\|^2_{H^3(\mathbb{R})} + \int_0^t \left(\|\rho_x(s)\|^2_{H^4(\mathbb{R})} + \|u_x(s)\|^2_{H^3(\mathbb{R})}\right) ds \\ \leq\ & C_2\left(\|\rho_0-\bar{\rho}\|^2_{H^4(\mathbb{R})} + \|u_0\|^2_{H^3(\mathbb{R})}\right), \quad \forall\, t > 0, \end{aligned} \tag{1.5}$$

and the time-asymptotic behavior

$$\lim_{t\to+\infty} \left\{\|(\rho-\bar{\rho}, u)(t)\|_{L^\infty(\mathbb{R})} + \|\rho_x(t)\|_{H^3(\mathbb{R})} + \|u_x(t)\|_{H^2(\mathbb{R})}\right\} = 0, \tag{1.6}$$

here C_1 is a positive constant depending only on $m_0, m_1, \|\rho_0-\bar{\rho}\|_{H^1(\mathbb{R})}$ and $\|u_0\|_{L^2(\mathbb{R})}$, and C_2 is a positive constant depending only on $m_0, m_1, \|\rho_0-\bar{\rho}\|_{H^4(\mathbb{R})}$ and $\|u_0\|_{H^3(\mathbb{R})}$.

When the viscosity coefficient $\mu(\rho)$ and the capillarity coefficient $\kappa(\rho)$ are given by

$$\mu(\rho) = \rho^\alpha, \quad \kappa(\rho) = \rho^\beta, \tag{1.7}$$

where $\alpha, \beta \in \mathbb{R}$ are some constants, condition (ii) of Theorem 1.1 corresponds to

$$(a) -3 \leq \beta \leq -2, \qquad (b)\ 0 \leq \alpha \leq \frac{1}{2};$$

while condition (iii) of Theorem 1.1 is equivalent to

$$\frac{\beta+3}{3} - \frac{1}{3}\sqrt{-2\beta^2-6\beta} \leq \alpha \leq \frac{\beta+3}{3} + \frac{1}{3}\sqrt{-2\beta^2-6\beta}, \quad -3 \leq \beta \leq 0$$

or

$$\alpha - 2 - \sqrt{-2\alpha^2+2\alpha+1} \leq \beta \leq \alpha - 2 + \sqrt{-2\alpha^2+2\alpha+1}, \quad \frac{1-\sqrt{3}}{2} \leq \alpha \leq \frac{1+\sqrt{3}}{2}.$$

Thus from Theorem 1.1, we have the following corollary.

Corollary 1.1 Let condition (i) of Theorem 1.1 holds. Suppose that the viscosity coefficient $\mu(\rho)$ and the capillarity coefficient $\kappa(\rho)$ are given by (1.7) and the constants α, β satisfy one of the following conditions:

(A) $\dfrac{\beta+3}{3}-\dfrac{1}{3}\sqrt{-2\beta^2-6\beta}\le\alpha\le\dfrac{\beta+3}{3}+\dfrac{1}{3}\sqrt{-2\beta^2-6\beta},\quad -3\le\beta\le-2;$

(B) $\alpha-2-\sqrt{-2\alpha^2+2\alpha+1}\le\beta\le\alpha-2+\sqrt{-2\alpha^2+2\alpha+1},\quad 0\le\alpha\le\dfrac{1}{2};$

then the same conclusions of Theorem 1.1 hold.

Remark 1.1 Some remarks on Theorem 1.1 and Corollary 1.1 are given as follows:

(1) Conditions (ii) and (iii) of Theorem 1.1 are used to deduce the positive lower and upper bounds of the density $\rho(t,x)$, see Lemmas 2.3–2.5 for details.

(2) In Theorem 1.1, the viscosity coefficient $\mu(\rho)$ and the capillarity coefficient $\kappa(\rho)$ are general smooth functions of ρ satisfying conditions (ii) and (iii) of Theorem 1.1, which are more general than those in [23, 24], where only some power like density-dependent viscosity and capillarity coefficients are studied.

On the other hand, Germain and LeFloch [18] also discussed the global existence of weak solutions away from vacuum for problems (1.1)–(1.2) with $\mu(\rho)=\rho^\alpha$ and $\kappa(\rho)=\rho^\beta$ under the condition that

$$0\le\alpha<\frac{1}{2},\quad 2\alpha-4\le\beta\le2\alpha-1$$

or

$$\alpha\ge0,\quad \beta<-2,\quad 2\alpha-4\le\beta\le2\alpha-1,$$

which means that $0\le\alpha<1$. From condition (A) of Corollary 1.1, we see that $\alpha\in[\frac{1-\sqrt{3}}{2},1]$, thus Corollary 1.1 also improves the results of [18] to the case $\alpha\in[\frac{1-\sqrt{3}}{2},0)$. Moreover, case (B) of Corollary 1.1 is completely new compared to the results in [18, 23, 24]. Thus in these sense, our main result Theorem 1.1 can be viewed as an extension of the works [18, 23, 24].

Now we make some comments on the analysis of this paper. The proof of Theorem 1.1 is motivated by the previous works [18, 23, 24]. When the viscosity coefficient $\mu(\rho)$ and the capillarity coefficient $\kappa(\rho)$ are some power functions of the density, the authors in [23, 24] studied the global existence and large time behavior of smooth solutions away from vacuum to the Cauchy problem of system (1.1) with large initial data in the Lagrangian coordinates. However, for the viscosity coefficient $\mu(\rho)$ and the capillarity coefficient $\kappa(\rho)$ being some general smooth functions of the density, it is much more easier for us to study such a problem in the Eulerian coordinates rather than the Lagrangian coordinates. To prove Theorem 1.1, we mainly use the method of Kanel [25] and the energy estimates. The key step is to derive the positive lower and upper bounds for the density $\rho(t,x)$. First, due to effect of the Korteweg tensor, an estimate of $\int_{\mathbb{R}}\kappa(\rho)\rho_x^2dx$ appears in the basic energy estimate (see Lemma 2.1). Based on this and a new inequality for the renormalized internal energy $G(\rho,\bar\rho)$ (see Lemma 2.2), the lower and upper bounds of $\rho(t,x)$ for cases (ii) (a) of Theorem 1.1 can be derived easily by applying Kanel's method [25] (see Lemma 2.3). Second, we perform an uniform-in-time estimate on $\int_{\mathbb{R}}\dfrac{\mu^2(\rho)}{2\rho^3}\rho_x^2dx$ under condition (iii) of Theorem 1.1 (see Lemma 2.4). We remark that Lemma 2.4 is proved by using the approach of Kanel [25], rather than introducing the effective velocity as [4, 17, 18]. Then by employing Kanel's method [25] again and Lemmas 2.1, 2.2 and 2.4, the lower and upper bounds of $\rho(t,x)$ for the cases (ii) (b) of Theorem 1.1 follows immediately (see Lemma 2.5). Having obtained the lower and upper bounds on $\rho(t,x)$, the higher order energy estimates of solutions to the Cauchy problem (1.1)–(1.2) can be deduced by using the lower order estimates and Gronwall's inequality, and then Theorem 1.1 follows by the standard continuation argument. In the next section, we will give the proof of Theorem 1.1.

Notations Throughout this paper, C denotes some generic constant which may vary in different estimates. If the dependence needs to be explicitly pointed out, the notation $C(\cdot,\cdots,\cdot)$ or $C_i(\cdot,\cdots,\cdot)(i\in\mathbb{N})$ is used. $f'(\rho)$ denotes the derivative of the function $f(\rho)$ with respect to ρ. For function spaces, $L^p(\mathbb{R})$ $(1\le p\le+\infty)$ is the standard Lebesgue space with the norm $\|\cdot\|_{L^p}$, and $H^l(\mathbb{R})$ stands for the usual l-th order Sobolev space with its norm

$$\|f\|_l=\left(\sum_{i=0}^{l}\|\partial_x^i f\|^2\right)^{\frac{1}{2}}\quad\text{with}\quad\|\cdot\|\triangleq\|\cdot\|_{L^2(\mathbb{R})}.$$

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. To do this, we seek the solutions of the Cauchy problems (1.1)–(1.2) in the following set of functions

$$X(0,T;m,M)=\left\{(\rho,u)(t,x)\;\middle|\;\begin{array}{c}\rho(t,x)-\bar\rho\in C(0,T;H^4(\mathbb{R}))\cap C^1(0,T;H^2(\mathbb{R}))\\ u(t,x)\in C(0,T;H^3(\mathbb{R}))\cap C^1(0,T;H^1(\mathbb{R}))\\ (\rho_x,u_x)(t,x)\in L^2(0,T;H^4(\mathbb{R})\times H^3(\mathbb{R}))\\ m\le\rho(t,x)\le M\end{array}\right\},$$

where $M\ge m>0$ and $T>0$ are some positive constants.

Under the assumptions of Theorem 1.1, we have the following local existence result.

Proposition 2.1 (Local existence) Under the assumptions of Theorem 1.1, there exists a sufficiently small positive constant t_1 depending only on m_0,m_1, $\|\rho_0-\bar\rho\|_4$ and $\|u_0\|_3$ such that the Cauchy problems (1.1)–(1.2) admits a unique smooth solution $(\rho,u)(t,x)\in X(0,t_1;\frac{m_0}{2},2m_1)$ and

$$\sup_{[0,t_1]}\{\|(\rho-\bar\rho)(t)\|_4+\|u(t)\|_3\}\le b\{\|\rho_0-\bar\rho\|_4+\|u_0\|_3\},$$

where $b>1$ is a positive constant depending only on m_0,m_1.

The proof of Proposition 2.1 can be done by using the dual argument and iteration technique, which is similar to that of Theorem 1.1 in [10] and thus omitted here for brevity. Suppose that the local solution $(\rho,u)(t,x)$ obtained in Proposition 2.1 has been extended to the time step $t=T\ge t_1$ for some positive constant $T>0$. To prove Theorem 1.1, one needs only to show the following a priori estimates.

Proposition 2.2 (A priori estimates) Under the assumptions of Theorem 1.1, suppose that $(\rho,u)(t,x)\in X(0,T;M_0,M_1)$ is a solution of the Cauchy problem (1.1)–(1.2) for some positive constants T and $M_0, M_1 > 0$. Then there exist two positive constants C_1 and C_2 which are independent of T, M_0, M_1 such that the following estimates hold:

$$C_1^{-1} \le \rho(t,x) \le C_1, \quad \forall\,(t,x) \in [0,T]\times\mathbb{R}, \tag{2.1}$$

$$\|(\rho-\bar\rho)(t)\|_4^2 + \|u(t)\|_3^2 + \int_0^t \left(\|\rho_x(s)\|_4^2 + \|u_x(s)\|_3^2\right) ds \le C_2\left(\|\rho_0-\bar\rho\|_4^2 + \|u_0\|_3^2\right),$$
$$\forall\, t\in[0,T]. \tag{2.2}$$

Proposition 2.2 can be obtained by a series of lemmas below. We first give the following key lemma.

Lemma2.1 (Basic energy estimates) Under the assumptions of Proposition 2.2, it holds that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{1}{2}\rho u^2 + G(\rho,\bar\rho) + \frac{1}{2}\kappa(\rho)\rho_x^2\right)dx + \int_0^t\!\!\int_{\mathbb{R}} \mu(\rho)u_x^2 dx d\tau\\ =&\int_{\mathbb{R}}\left(\frac{1}{2}\rho_0 u_0^2 + G(\rho_0,\bar\rho) + \frac{1}{2}\kappa(\rho_0)\rho_{0x}\right)dx\end{aligned} \tag{2.3}$$

for all $t\in[0,T]$, where the function $G(\rho,\bar\rho)$ is defined by

$$G(\rho,\bar\rho) = \rho\int_{\bar\rho}^{\rho}\frac{s^\gamma - \bar\rho^\gamma}{s^2}ds = \frac{\rho^\gamma}{\gamma-1} - \frac{(\bar\rho)^\gamma}{\gamma-1} - \frac{\gamma}{\gamma-1}(\bar\rho)^{\gamma-1}(\rho-\bar\rho). \tag{2.4}$$

Proof In view of the continuity equation $(1.1)_1$, we have

$$\begin{aligned}\left(\frac{1}{2}\rho u^2 + G(\rho,\bar\rho)\right)_t &= \frac{1}{2}\rho_t u^2 + \rho u u_t + G_\rho(\rho,\bar\rho)\rho_t\\ &= \frac{1}{2}u^2(-\rho u)_x + \rho u u_t + G_\rho(\rho,\bar\rho)(-\rho u)_x.\end{aligned} \tag{2.5}$$

On the other hand, by using $(1.1)_1$ again, the movement equation $(1.1)_2$ can be rewritten as

$$u_t + uu_x + \frac{P'(\rho)}{\rho}\rho_x = \frac{1}{\rho}(\mu(\rho)u_x)_x + \left(\kappa(\rho)\rho_{xx} + \frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$\begin{aligned}\left(\frac{1}{2}\rho u^2 + G(\rho,\bar\rho)\right)_t =& -\frac{1}{2}u^2(\rho u)_x - \rho u^2 u_x - uP'(\rho)\rho_x + u(\mu(\rho)u_x)_x\\ &+\rho u\left(\kappa(\rho)\rho_{xx} + \frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x - (\rho u)_x G_\rho(\rho,\bar\rho)\\ =& -\frac{1}{2}(\rho u^3)_x - \frac{P'(\rho)}{\rho}\rho_x\rho u - (\rho u)_x G_\rho(\rho,\bar\rho) + (\mu(\rho)uu_x)_x\\ &-\mu(\rho)u_x^2 + \rho u\left(\kappa(\rho)\rho_{xx} + \frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x\\ =&\{\cdots\}_x - \mu(\rho)u_x^2 + \rho u\left(\kappa(\rho)\rho_{xx} + \frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x.\end{aligned} \tag{2.7}$$

Here and hereafter, $\{\cdots\}_x$ denotes the terms which will disappear after integrating with respect to x.

Moreover, it follows from $(1.1)_1$ that

$$\begin{aligned}\rho u\left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x&=\{\cdots\}_x-(\rho u)_x\left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)\\&=\{\cdots\}_x+\rho_t\left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)\\&=\{\cdots\}_x-\left(\frac{\kappa(\rho)}{2}\rho_x^2\right)_t.\end{aligned}\tag{2.8}$$

Combining (2.7) and (2.8), and integrating the resultant equation with respect to t and x over $[0,t]\times\mathbb{R}$, we can get (2.3). This completes the proof of Lemma 2.1.

In order to apply Kanel's method [25] to show the lower and upper bound of the density $\rho(t,x)$, we need to establish the following lemma.

Lemma 2.2 There exists a uniform positive constant c_0 such that

$$G(\rho,\overline{\rho})\geq c_0\frac{(\rho-\overline{\rho})^2}{\rho+\overline{\rho}}.\tag{2.9}$$

Proof Using the L' Hospital rule, we obtain

$$\begin{aligned}\lim_{\rho\to\bar\rho}\frac{G(\rho,\bar\rho)(\rho+\bar\rho)}{(\rho-\bar\rho)^2}&=\lim_{\rho\to\bar\rho}\frac{G_\rho(\rho,\bar\rho)(\rho+\bar\rho)+G(\rho,\bar\rho)}{2(\rho-\bar\rho)}\\&=\lim_{\rho\to\bar\rho}\frac{\left(\frac{\gamma\rho^{\gamma-1}}{\gamma-1}-\frac{\gamma}{\gamma-1}(\bar\rho)^{\gamma-1}\right)(\rho+\bar\rho)+G(\rho,\bar\rho)}{2(\rho-\bar\rho)}\\&=\lim_{\rho\to\bar\rho}\frac{(\rho+\bar\rho)\gamma\rho^{\gamma-2}+2G_\rho(\rho,\bar\rho)}{2}=\gamma(\bar\rho)^{\gamma-1}>0\end{aligned}\tag{2.10}$$

and

$$\lim_{\rho\to+\infty}\frac{G(\rho,\bar\rho)(\rho+\bar\rho)}{(\rho-\bar\rho)^2}=\lim_{\rho\to+\infty}\frac{\left(\frac{\rho^\gamma}{\gamma-1}-\frac{\gamma}{\gamma-1}(\bar\rho)^{\gamma-1}\rho+(\bar\rho)^\gamma\right)(\rho+\bar\rho)}{(\rho-\bar\rho)^2}=+\infty.\tag{2.11}$$

Consequently, there exist a sufficiently small constant δ and a large constant $M(>\bar\rho+\delta)$ such that

$$\frac{G(\rho,\bar\rho)(\rho+\bar\rho)}{(\rho-\bar\rho)^2}>\begin{cases}\frac{\gamma(\bar\rho)^{\gamma-1}}{2}, & \text{if } \rho\in(\bar\rho-\delta,\bar\rho+\delta),\\ 1, & \text{if } \rho>M.\end{cases}\tag{2.12}$$

Note that the function $\frac{G(\rho,\bar\rho)(\rho+\bar\rho)}{(\rho-\bar\rho)^2}$ is continuous on $[0,\bar\rho-\delta]\cup[\bar\rho+\delta,M]$, thus by setting

$$m_0=\min_{\rho\in[0,\bar\rho-\delta]\cup[\bar\rho+\delta,M]}\frac{G(\rho,\bar\rho)(\rho+\bar\rho)}{(\rho-\bar\rho)^2},$$

and $c_0=\min\{m_0,1,\frac{\gamma(\bar\rho)^{\gamma-1}}{2}\}$, we have (2.9) holds. This completes the proof of Lemma 2.2.

Based on Lemmas 2.1–2.2, we now show the lower and upper bounds of $\rho(t,x)$ by using Kanel's method [25].

Lemma 2.3 (Lower and upper bounds of $\rho(t,x)$ for the cases (ii) (a) of Theorem 1.1) Under the assumptions of Proposition 2.2, if the capillarity coefficient $\kappa(\rho)$ satisfies the condition (ii) (a) of Theorem 1.1, then there exists a positive constant C_3 depending only on $m_0, m_1, \|\rho_0-\bar{\rho}\|, \|u_0\|$ and $\|\rho_{0x}\|$ such that

$$C_3^{-1} \le \rho(t,x) \le C_3 \tag{2.13}$$

for all $(t,x) \in [0,T]\times\mathbb{R}$.

Proof Let

$$\Psi(\rho)=\frac{(\rho-\bar{\rho})^2}{\rho+\bar{\rho}}, \Upsilon(\rho)=\int_1^\rho \sqrt{\Psi(s)}\sqrt{\kappa(s)}ds,$$

then under the condition (ii)(a) of Theorem 1.1, we have

$$\Upsilon(\rho)\to\begin{cases} -\infty, & \rho\to 0,\\ +\infty, & \rho\to+\infty.\end{cases} \tag{2.14}$$

On the other hand, we deduce from Lemmas 2.1–2.2 that

$$\begin{aligned}|\Upsilon(\rho)| &= \left|\int_{-\infty}^x (\Upsilon(\rho))_y \mathrm{d}y\right| = \left|\int_{-\infty}^x \sqrt{\Psi(\rho)}\sqrt{\kappa(\rho)}\rho_y \mathrm{d}y\right| \\ &\le \int_{-\infty}^{+\infty}\left|\sqrt{\Psi(\rho)}\right|\cdot\left|\sqrt{\kappa(\rho)}\rho_x\right|dx \le \left\|\sqrt{\Psi(\rho)}(t)\right\|\cdot\left\|\sqrt{\kappa(\rho)}\rho_x(t)\right\| \\ &\le C\left\|\sqrt{G(\rho,\bar{\rho})}(t)\right\|\cdot\left\|\sqrt{\kappa(\rho)}\rho_x(t)\right\| \le C(m_0,m_1)\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2.\end{aligned} \tag{2.15}$$

(2.13) thus follows from (2.14) and (2.15) immediately. This completes the proof of Lemma 2.3.

Next, we give the estimate on

$$\int_{\mathbb{R}}\frac{\mu^2(\rho)}{2\rho^3}\rho_x^2dx.$$

Lemma 2.4 Let condition (i) of Theorem 1.1 holds and

$$f(\rho)=-\frac{2}{3}\left(\sqrt{\frac{\mu(\rho)\kappa(\rho)}{\rho}}\left(\sqrt{\frac{\mu(\rho)\kappa(\rho)}{\rho}}\right)'\right)'+\left(\left(\sqrt{\frac{\mu(\rho)\kappa(\rho)}{\rho}}\right)'\right)^2+\frac{\kappa(\rho)}{3}\left(\frac{\mu(\rho)}{\rho}\right)''+\frac{\mu(\rho)\kappa''(\rho)}{6\rho}.$$

Then if $f(\rho)\le 0$, there exists a positive constant C_4 depending only on m_0, m_1 and $\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|$ such that for $0\le t\le T$,

$$\begin{aligned}&\int_{\mathbb{R}}\frac{\mu^2(\rho)}{\rho^3}\rho_x^2dx+\int_0^t\int_{\mathbb{R}}\frac{P'(\rho)\mu(\rho)}{\rho^2}\rho_x^2dxd\tau+\int_0^t\int_{\mathbb{R}}\left[\left(\sqrt{\frac{\mu\kappa}{\rho}}\rho_x\right)_x\right]^2dxd\tau \\ \le& C_4\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2.\end{aligned} \tag{2.16}$$

Proof First, by the continuity equation $(1.1)_1$, we have

$$\begin{aligned}\frac{1}{\rho}(\mu(\rho)u_x)_x &= -\frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}(\rho_t+\rho_x u)\right)_x = -\frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_t\right)_x - \frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x u\right)_x \\ &= -\frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x\right)_t - \frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x u\right)_x,\end{aligned} \tag{2.17}$$

where we have used the fact that

$$\left(\frac{\mu(\rho)}{\rho}\rho_t\right)_x = \left(\frac{\mu(\rho)}{\rho}\rho_x\right)_t.$$

Putting (2.17) into (2.6), and multiplying the resultant equation by $\frac{\mu(\rho)}{\rho}\rho_x$ yields

$$\begin{aligned}&u_t\frac{\mu(\rho)}{\rho}\rho_x + uu_x\frac{\mu(\rho)}{\rho}\rho_x + \frac{P'(\rho)\mu(\rho)}{\rho^2}\rho_x^2 \\ =&-\frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x\right)_t\frac{\mu(\rho)}{\rho}\rho_x - \frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x u\right)_x\frac{\mu(\rho)}{\rho}\rho_x + \left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x\frac{\mu(\rho)}{\rho}\rho_x.\end{aligned} \tag{2.18}$$

A direct calculation yields that

$$\begin{aligned}u_t\frac{\mu(\rho)}{\rho}\rho_x &= \left(u\frac{\mu(\rho)}{\rho}\rho_x\right)_t - u\left(\frac{\mu(\rho)}{\rho}\rho_x\right)_t = \left(u\frac{\mu(\rho)}{\rho}\rho_x\right)_t - u\left(\frac{\mu(\rho)}{\rho}\rho_t\right)_x \\ =&\left(u\frac{\mu(\rho)}{\rho}\rho_x\right)_t + u_x\frac{\mu(\rho)}{\rho}(-\rho u)_x + \{\cdots\}_x \\ =&\left(u\frac{\mu(\rho)}{\rho}\rho_x\right)_t - \mu(\rho)u_x^2 - uu_x\frac{\mu(\rho)}{\rho}\rho_x + \{\cdots\}_x,\end{aligned} \tag{2.19}$$

$$\begin{aligned}&-\frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x\right)_t\frac{\mu(\rho)}{\rho}\rho_x - \frac{1}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x u\right)_x\frac{\mu(\rho)}{\rho}\rho_x \\ =&-\left(\frac{\mu^2(\rho)}{2\rho^3}\rho_x^2\right)_t + \left(\frac{1}{\rho}\right)_t\frac{\mu^2(\rho)}{2\rho^2}\rho_x^2 - \frac{u}{\rho}\left(\frac{\mu(\rho)}{\rho}\rho_x\right)_x\frac{\mu(\rho)}{\rho}\rho_x - \frac{1}{\rho}\frac{\mu^2(\rho)}{\rho^2}\rho_x^2 u_x \\ =&-\left(\frac{\mu^2(\rho)}{2\rho^3}\rho_x^2\right)_t - \frac{\rho_t}{\rho^2}\cdot\frac{\mu^2(\rho)}{2\rho^2}\rho_x^2 + \left(\frac{u}{\rho}\right)_x\frac{\mu^2(\rho)}{2\rho^2}\rho_x^2 - \frac{1}{\rho}\frac{\mu^2(\rho)}{\rho^2}\rho_x^2 u_x - \{\cdots\}_x \\ =&-\left(\frac{\mu^2(\rho)}{2\rho^3}\rho_x^2\right)_t - \{\cdots\}_x.\end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), and integrating the resultant equation in t and x over $[0,t]\times\mathbb{R}$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\frac{\mu^2(\rho)}{2\rho^3}\rho_x^2dx + \int_0^t\int_{\mathbb{R}}\frac{P'(\rho)\mu(\rho)}{\rho^2}\rho_x^2dxd\tau \\ \le& C\left(\int_{\mathbb{R}}u_0\frac{\mu(\rho_0)}{\rho_0}\rho_{0x}^2dx + \int_{\mathbb{R}}\frac{\mu^2(\rho_0)}{2\rho_0^3}\rho_{0x}^2dx\right) \\ &+C\left\{\int_0^t\int_{\mathbb{R}}\mu(\rho)u_x^2dxd\tau + \int_0^t\int_{\mathbb{R}}\left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x\frac{\mu(\rho)}{\rho}\rho_xdxd\tau\right\},\end{aligned} \tag{2.21}$$

where we have used the fact that

$$\int_{\mathbb{R}} u\frac{\mu(\rho)}{\rho}\rho_x dx \leq \frac{1}{4}\int_{\mathbb{R}}\frac{\mu^2(\rho)}{\rho^3}\rho_x^2 dx + C\int_{\mathbb{R}}\rho u^2 dx.$$

By employing integrations by parts, we obtain

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)_x\frac{\mu(\rho)}{\rho}\rho_x dxd\tau\\
= & -\int_0^t\int_{\mathbb{R}}\left(\kappa(\rho)\rho_{xx}+\frac{1}{2}\kappa'(\rho)\rho_x^2\right)\left(\frac{\mu(\rho)}{\rho}\rho_x\right)_x dxd\tau\\
= & -\int_0^t\int_{\mathbb{R}}\left\{\frac{\kappa(\rho)\mu(\rho)}{\rho}\rho_{xx}^2+\frac{1}{2}\kappa'(\rho)\left(\frac{\mu(\rho)}{\rho}\right)'\rho_x^4+\kappa(\rho)\left(\frac{\mu(\rho)}{\rho}\right)'\rho_x^2\rho_{xx}\right.\\
&\left.+\frac{1}{2}\kappa'(\rho)\frac{\mu(\rho)}{\rho}\rho_x^2\rho_{xx}\right\}dxd\tau\\
= & -\int_0^t\int_{\mathbb{R}}\left\{\frac{\kappa(\rho)\mu(\rho)}{\rho}\rho_{xx}^2+\frac{1}{2}\kappa'(\rho)\left(\frac{\mu(\rho)}{\rho}\right)'\rho_x^4+\kappa(\rho)\left(\frac{\mu(\rho)}{\rho}\right)'\left(\frac{\rho_x^3}{3}\right)_x\right.\\
&\left.+\frac{1}{2}\kappa'(\rho)\frac{\mu(\rho)}{\rho}\left(\frac{\rho_x^3}{3}\right)_x\right\}dxd\tau\\
= & -\int_0^t\int_{\mathbb{R}}\frac{\kappa(\rho)\mu(\rho)}{\rho}\rho_{xx}^2 dxd\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{1}{3}\kappa(\rho)(\frac{\mu(\rho)}{\rho})''+\frac{1}{6}\kappa''(\rho)\frac{\mu(\rho)}{\rho}\right)\rho_x^4 dxd\tau
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
&-\int_0^t\int_{\mathbb{R}}\frac{\kappa(\rho)\mu(\rho)}{\rho}\rho_{xx}^2 dxd\tau\\
&=-\int_0^t\int_{\mathbb{R}}\left\{\left[\left(\sqrt{\frac{\kappa\mu}{\rho}}\rho_x\right)_x\right]^2-\frac{2}{3}\left(\rho_x^3\right)_x\sqrt{\frac{\mu\kappa}{\rho}}\left(\sqrt{\frac{\mu\kappa}{\rho}}\right)'-\rho_x^4\left[\left(\sqrt{\frac{\mu\kappa}{\rho}}\right)'\right]^2\right\}dxd\tau\\
&=-\int_0^t\int_{\mathbb{R}}\left[\left(\sqrt{\frac{\kappa\mu}{\rho}}\rho_x\right)_x\right]^2 dxd\tau+\int_0^t\int_{\mathbb{R}}\left\{-\frac{2}{3}\left(\sqrt{\frac{\mu\kappa}{\rho}}\left(\sqrt{\frac{\mu\kappa}{\rho}}\right)'\right)'+\left[\left(\sqrt{\frac{\mu\kappa}{\rho}}\right)'\right]^2\right\}\rho_x^4 dxd\tau.
\end{aligned} \tag{2.23}$$

Inserting (2.22)–(2.23) into (2.21), and using (2.3), we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{\mu^2(\rho)}{2\rho^3}\rho_x^2 dx+\int_0^t\int_{\mathbb{R}}\frac{P'(\rho)\mu(\rho)}{\rho^2}\rho_x^2 dxd\tau+\int_0^t\int_{\mathbb{R}}\left[\left(\sqrt{\frac{\kappa\mu}{\rho}}\rho_x\right)_x\right]^2 dxd\tau\\
&\leq C\|(\rho_0-\bar{\rho},u_0,u_{0x})\|^2+\int_0^t\int_{\mathbb{R}}f(\rho)\rho_x^4 dxd\tau.
\end{aligned} \tag{2.24}$$

(2.24) together with the assumption that $f(\rho)\leq 0$ implies (2.16) immediately. This completes the proof of Lemma 2.4.

Lemma 2.5 Let conditions (i) and (ii) (b) of Theorem 1.1 hold and $f(\rho)\leq 0$, then there exists a positive constant C_5 depending only on m_0, m_1 and $\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|$ such that

$$C_5^{-1}\leq\rho(t,x)\leq C_5 \tag{2.25}$$

for all $(t,x)\in[0,T]\times\mathbb{R}$.

Proof Set

$$\bar{\Psi}(\rho)=\int_1^\rho\sqrt{\Psi(s)}\frac{\mu(s)}{s^{3/2}}ds,$$

then it follows from assumption (ii) (b) of Theorem 1.1 that

$$\bar{\Psi}(\rho)\to\begin{cases}-\infty, & \rho\to 0,\\ +\infty, & \rho\to+\infty.\end{cases}\tag{2.26}$$

On the other hand, Lemmas 2.1 and 2.4 imply that

$$\begin{aligned}|\bar{\Psi}(\rho)|&=\left|\int_{-\infty}^x(\bar{\Psi}(\rho))_y dy\right|=\left|\int_{-\infty}^x\sqrt{\bar{\Psi}(\rho)}\frac{\mu(\rho)}{s^{3/2}}\rho_y dy\right|\\&=\int_{-\infty}^{+\infty}\left|\sqrt{\bar{\Psi}(\rho)}\frac{\mu(\rho)}{s^{3/2}}\rho_y\right|dy=\left\|\sqrt{\bar{\Psi}(\rho)}(t)\right\|\cdot\left\|\frac{\mu(\rho)}{s^{3/2}}\rho_x(t)\right\|\le C.\end{aligned}\tag{2.27}$$

From (2.26) and (2.27), we have (2.25) at once. This completes the proof Lemma 2.5.

As a consequence of Lemmas 2.3–2.5, we have

Corollary 2.1 Under the assumptions of Lemmas 2.3–2.5, it holds that for $0\le t\le T$,

$$\|(\rho-\bar{\rho},u)(t)\|^2+\|\rho_x(t)\|^2+\int_0^t\|u_x(\tau)\|^2d\tau\le C_6\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2,\tag{2.28}$$

where $C_6>0$ is a constant depending only on m_0, m_1 and $\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|$.

The next lemma gives an estimate on

$$\int_0^t\|\rho_x(\tau)\|_1^2d\tau.$$

Lemma 2.6 There exists a positive constant C_7 depending only on m_0, m_1 and $\|(\rho_0-\bar{\rho},u_0,\rho_{0x}\|$ such that for $0\le t\le T$,

$$\|\rho_x(t)\|^2+\int_0^t\|\rho_x(\tau)\|_1^2d\tau\le C_7\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2.\tag{2.29}$$

Proof We derive from Lemmas 2.3–2.5 that

$$\|\rho_x(t)\|^2+\int_0^t\|\rho_x(\tau)\|^2d\tau\le C\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2.\tag{2.30}$$

On the other hand, Lemmas 2.3–2.5 also imply that

$$\|\rho_x(t)\|^2+\int_0^t\|\rho_x(\tau)\|_1^2d\tau\le C\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2+C\int_0^t\int_{\mathbb{R}}\left|\rho_x^2\rho_{xx}\right|dxd\tau.\tag{2.31}$$

From the Cauchy equality and (2.30), we infer that

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} \left|\rho_x^2\rho_{xx}\right| dxd\tau &\le \frac{1}{2}\int_0^t \|\rho_{xx}(\tau)\|^2 d\tau + C\int_0^t\int_{\mathbb{R}} \rho_x^4 dxd\tau \\
&\le \frac{1}{2}\int_0^t \|\rho_{xx}(\tau)\|^2 d\tau + C\int_0^t \|\rho_x(\tau)\|^3\cdot\|\rho_{xx}(\tau)\| d\tau \\
&\le \frac{3}{4}\int_0^t \|\rho_{xx}(\tau)\|^2 d\tau + C\int_0^t \sup_{0\le\tau\le t}\{\|\rho_x(\tau)\|^4\}\cdot\|\rho_x(\tau)\|^2 d\tau \\
&\le \frac{3}{4}\int_0^t \|\rho_{xx}(\tau)\|^2 d\tau + C\|(\rho_0-\bar{\rho}, u_0, \rho_{0x})\|^6.
\end{aligned} \tag{2.32}
$$

Then (2.29) follows from (2.31) and (2.32) immediately. This completes the proof of Lemma 2.6.

For the estimate on $\|u_x(t)\|^2$, we have

Lemma 2.7 There exists a positive constant C_8 depending only on m_0, m_1 and $\|(\rho_0-\bar{\rho}, u_0, \rho_{0x})\|$ such that for $0\le t\le T$,

$$
\|u_x(t)\|^2 + \|\rho_{xx}(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \le C_8(\|\rho_0-\bar{\rho}\|_2^2 + \|u_0\|_1^2). \tag{2.33}
$$

Proof Multiplying (2.6) by $-u_{xx}$, and using the continuity equation $(1.1)_1$, we have

$$
\begin{aligned}
&\left(\frac{u_x^2}{2} + \frac{\kappa(\rho)}{2\rho}\rho_{xx}^2\right)_t + \frac{\mu(\rho)}{\rho}u_{xx}^2 \\
=& uu_xu_{xx} - \frac{\mu'(\rho)}{\rho}\rho_xu_xu_{xx} + \frac{P'(\rho)}{\rho}\rho_xu_{xx} - \frac{1}{2}\kappa''(\rho)\rho_x^3u_{xx} \\
&- \kappa'(\rho)\rho_x\rho_{xx}u_{xx} - \frac{\kappa'(\rho)}{2}\rho_{xx}^2u_x - \frac{2\kappa(\rho)}{\rho}u_x\rho_{xx}^2 - \frac{3\kappa(\rho)}{\rho}\rho_x\rho_{xx}u_{xx} + \{\cdots\}_x.
\end{aligned} \tag{2.34}
$$

Integrating (2.34) in t and x over $[0,t]\times\mathbb{R}$ gives

$$
\frac{1}{2}\int_{\mathbb{R}} u_x^2 dx + \int_{\mathbb{R}} \frac{\kappa(\rho)}{2\rho}\rho_{xx}^2 dx + \int_0^t\int_{\mathbb{R}} \frac{\mu(\rho)}{\rho}u_{xx}^2 dxd\tau = \frac{1}{2}\int_{\mathbb{R}} u_{0x}^2 dx + \int_{\mathbb{R}} \frac{\kappa(\rho_0)}{2\rho}\rho_{0xx}^2 dx + \sum_{i=1}^2 I_i, \tag{2.35}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^t\int_{\mathbb{R}} \left\{uu_xu_{xx} - \frac{\mu'(\rho)}{\rho}\rho_xu_xu_{xx} + \frac{P'(\rho)}{\rho}\rho_xu_{xx}\right\} dxd\tau, \\
I_2 &= \int_0^t\int_{\mathbb{R}} \left\{-\frac{1}{2}\kappa''(\rho)\rho_x^3u_{xx} - \left(\kappa'(\rho) + \frac{3\kappa(\rho)}{\rho}\right)\rho_x\rho_{xx}u_{xx} - \left(\frac{\kappa'(\rho)}{2} + \frac{2\kappa(\rho)}{\rho}\right)\rho_{xx}^2u_x\right\} dxd\tau.
\end{aligned}
$$

It follows from the Cauchy inequality, the Sobolev inequality, the Young inequality, Lemmas 2.3 and 2.5, and Corollary 2.1 that

$$
\begin{aligned}
I_1 &\leq C\int_0^t\int_{\mathbb{R}}(|uu_xu_{xx}|+|\rho_xu_xu_{xx}|+|\rho_xu_{xx}|)\,dxd\tau\\
&\leq \frac{1}{4}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t\int_{\mathbb{R}}\left(|uu_x|^2+|\rho_xu_x|^2+|\rho_x|^2\right)dxd\tau\\
&\leq \frac{1}{4}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t\left(\|u_x(\tau)\|\cdot\|u_{xx}(\tau)\|\cdot\|u(\tau)\|^2\right.\\
&\quad\left.+\|u_x(\tau)\|\cdot\|u_{xx}(\tau)\|\cdot\|\rho_x(\tau)\|^2+\|\rho_x(\tau)\|^2\right)d\tau\\
&\leq \frac{1}{2}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t\left(\sup_{0\leq\tau\leq t}\left\{\|(u,\rho_x)(\tau)\|^4\right\}\cdot\|u_x(\tau)\|^2+\|\rho_x(\tau)\|^2\right)d\tau\\
&\leq \frac{1}{2}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2, \qquad (2.36)\\
I_2 &\leq C\int_0^t\int_{\mathbb{R}}\left(|\rho_x^3u_{xx}|+|\rho_x\rho_{xx}u_{xx}|+|\rho_{xx}^2u_x|\right)dxd\tau\\
&\leq \frac{1}{8}\int_0^t\|u_{xx}(\tau)\|^2d\tau+\int_0^t\int_{\mathbb{R}}\left(|\rho_x|^6+|\rho_x\rho_{xx}|^2+|\rho_{xx}^2u_x|\right)dxd\tau\\
&\leq \frac{1}{8}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t(\|\rho_x(\tau)\|^4\|\rho_{xx}(\tau)\|^2+\|\rho_x(\tau)\|\cdot\|\rho_{xx}(\tau)\|^3\\
&\quad+\|u_x(\tau)\|^{\frac{1}{2}}\|u_{xx}(\tau)\|^{\frac{1}{2}}\|\rho_{xx}(\tau)\|^2)d\tau\\
&\leq \frac{1}{4}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t(\|\rho_x(\tau)\|^4\|\rho_{xx}(\tau)\|^2+\|\rho_{xx}(\tau)\|^4+\|\rho_x(\tau)\|^4\\
&\quad+\|u_x(\tau)\|^{\frac{2}{3}}\|\rho_{xx}(\tau)\|^{\frac{8}{3}})d\tau\\
&\leq \frac{1}{4}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t(\sup_{0\leq\tau\leq t}\{\|\rho_x(\tau)\|^4\}\|\rho_{xx}(\tau)\|^2+\|\rho_{xx}(\tau)\|^4\\
&\quad+\|u_x(\tau)\|^2+\sup_{0\leq\tau\leq t}\{\|\rho_x(\tau)\|^2\}\|\rho_x(\tau)\|^2)d\tau\\
&\leq \frac{1}{4}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t\left(\|\rho_x(\tau)\|_1^2+\|\rho_{xx}(\tau)\|^4+\|u_x(\tau)\|^2\right)d\tau\\
&\leq \frac{1}{4}\int_0^t\|u_{xx}(\tau)\|^2d\tau+C\int_0^t\|\rho_{xx}(\tau)\|^4d\tau+C\|(\rho_0-\bar{\rho},u_0,\rho_{0x})\|^2. \qquad (2.37)
\end{aligned}
$$

Putting (2.36)–(2.37) into (2.35), and using Growwall's equality, we obtain (2.33). This completes the proof of Lemma 2.7.

Finally, we estimate the term

$$\int_0^t \| \rho_{xx}(\tau) \|_1^2 .$$

Lemma 2.8 There exists a positive constant C_9 depending only on m_0, m_1, $\|\rho_0-\bar{\rho}\|_2$ and $\|u_0\|_1$ such that for $t\in[0,T]$,

$$\|\rho_{xx}(t)\|^2+\int_0^t\|\rho_{xx}(\tau)\|_1^2\leq C_9\left(\|\rho_0-\bar{\rho}\|_2^2+\|u_0\|_1^2\right). \qquad (2.38)$$

Proof Differentiating $(1.1)_2$ once with respect to x, then multiplying the resultant equation by ρ_{xx}, and using equation $(1.1)_1$, we have

$$\begin{aligned}
&\left(\frac{\mu(\rho)}{2\rho^2}\rho_{xx}^2+u_x\rho_{xx}\right)_t+\frac{P'(\rho)}{\rho}\rho_{xx}^2+\kappa(\rho)\rho_{xxx}^2\\
=&-u_x^2\rho_{xx}+2\rho_x u_x u_{xx}+\rho u_{xx}^2-\left(\frac{P'(\rho)}{\rho}\right)'\rho_x^2\rho_{xx}+\frac{\mu'(\rho)}{2\rho^2}\rho_{xx}^2\rho_x u\\
&+\frac{\mu'(\rho)}{2\rho}\rho_{xx}^2u_x+\left(\frac{\mu(\rho)}{\rho}\right)'\frac{1}{\rho}\rho_{xxx}\rho_x^2u+\frac{\mu(\rho)}{\rho^2}\rho_{xx}u\rho_{xxx}+\frac{\mu(\rho)}{\rho^2}\rho_x u_x\rho_{xxx}\\
&-\frac{1}{\rho^2}(\frac{\mu(\rho)}{\rho})'\rho_x^3\rho_{xx}u-\frac{\mu(\rho)}{\rho^3}\rho_x\rho_{xx}^2u-\frac{\mu(\rho)}{\rho^3}\rho_x^2\rho_{xx}u_x\\
&+\frac{1}{\rho}\left\{\left(\frac{\mu(\rho)}{\rho}\right)''\rho_x^2(\rho_x u+\rho u_x)+2\left(\frac{\mu(\rho)}{\rho}\right)'\rho_x(\rho_{xx}u+2\rho_x u_x+\rho u_{xx})\right\}\rho_{xx}\\
&-2\kappa'(\rho)\rho_x\rho_{xx}\rho_{xxx}-\frac{1}{2}\kappa''(\rho)\rho_x^3\rho_{xxx}+\{\cdots\}_x.
\end{aligned}\tag{2.39}$$

Integrating (2.39) with respect to t and x over $[0,t]\times\mathbb{R}$, using the Cauchy inequality, the Sobolev inequality, Lemmas 2.3–2.7 and Corollary 2.1, we can get Lemma 2.8, the proof is similar to Lemma 2.7 and thus omitted here. This completes the proof of Lemma 2.8.

It follows from Corollary 2.1, and Lemmas 2.6–2.8 that there exists a positive constant C_{10} depending only on m_0, m_1, $\|\rho_0-\bar\rho\|_2$ and $\|u_0\|_1$ such that for $0\le t\le T$,

$$\|(\rho-\bar\rho)(t)\|_2^2+\|u(t)\|_1^2+\int_0^t\left(\|v_x(\tau)\|_2^2+\|u_x(\tau)\|_1^2\right)d\tau\le C_{10}\left(\|\rho_0-\bar\rho\|_2^2+\|u_0\|_1^2\right).\tag{2.40}$$

Similarly, we can also obtain

$$\|\rho_{xxx}(t)\|_1^2+\|u_{xx}(t)\|_1^2+\int_0^t\left(\|\rho_{xxx}(\tau)\|_2^2+\|u_{xxx}(\tau)\|_1^2\right)d\tau\le C_{11}\left(\|\rho_0\|_2^2+\|u_0\|_1^2\right),\tag{2.41}$$

where C_{11} is a positive constant depending only on m_0, m_1, $\|\rho_0-\bar\rho\|_4$ and $\|u_0\|_3$.

Proof of Proposition 2.2 Proposition 2.2 follows from (2.40) and (2.41) immediately.

Proof of Theorem 1.1 By Propositions 2.1–2.2 and the standard continuity argument, we can extend the local-in-time smooth solution to be a global one (i.e., $T=+\infty$). Thus (1.4) and (1.5) follows from (2.1) and (2.2), respectively. Moreover, estimate (2.2) and system (1.1) imply that

$$\int_0^{+\infty}\left(\|\rho_x(t)\|_3^2+\|u_x(t)\|_2^2+\left|\frac{d}{dt}\left(\|\rho_x(t)\|_3^2+\|u_x(t)\|_2^2\right)\right|\right)dt<\infty,\tag{2.42}$$

which implies that

$$\|\rho_x(t)\|_3+\|u_x(t)\|_2\to 0\quad\text{as}\quad t\to+\infty.\tag{2.43}$$

Furthermore, we deduce from (2.2), (2.43) and the Sobolev inequality that

$$\|(\rho-\bar\rho,u)(t)\|_{L^\infty}\le\|(\rho-\bar\rho,u)(t)\|^{\frac12}\|(\rho_x,u_x)(t)\|^{\frac12}\to 0\quad\text{as}\quad t\to+\infty.\tag{2.44}$$

From (2.43) and (2.44), we have (1.6) at once. This completes the proof of Theorem 1.1.

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一维可压缩Navier-Stokes-Korteweg方程组的大初值整体光滑解

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摘要: 本文研究了当粘性系数和毛细系数是密度函数的一般光滑函数时, 一维等温的可压缩Navier-Stokes-Korteweg方程的Cauchy问题. 利用基本能量方法和Kanel的技巧, 得到了大初值、非真空光滑解的整体存在性与时间渐近行为. 本文结果推广了已有文献中的结论.

关键词: 可压缩Navier-Stokes-Korteweg方程; 整体存在性; 时间渐近行为; 大初值

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