

# GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR NEWTONIAN FILTRATION EQUATIONS COUPLED WITH BOUNDARY CONDITIONS

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**Abstract:** In this paper, we deal with the large time behavior of solutions to the multi-dimensional Newtonian filtration equations coupled via the nonlinear boundary conditions. By constructing various kinds of upper and lower solutions, we obtain the critical global existence curve and the critical Fujita curve.

**Keywords:** global existence; blow up; Newtonian filtration equation; critical curves

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## 1 Introduction

In this paper, we deal with Newtonian filtration equations on the exterior domain of the unit ball in  $\mathbb{R}^N$ , i.e.,

$$\frac{\partial u_i}{\partial t} = \Delta u_i^{m_i}, \quad x \in \mathbb{R}^N \setminus B_1(0), t > 0, i = 1, 2, \dots, n, \quad (1.1)$$

$$\nabla u_i^{m_i} \cdot \vec{\nu} = u_{i+1}^{p_i}(x, t), \quad x \in \partial B_1(0), t > 0, i = 1, 2, \dots, n, \quad (1.2)$$

$$u_i(x, 0) = u_{0,i}(x), \quad x \in \mathbb{R}^N \setminus B_1(0), i = 1, 2, \dots, n, \quad (1.3)$$

where  $m_i > 1$ ,  $p_i \geq 0$ ,  $u_{n+1} = u_1$ ,  $B_1(0)$  is the unit ball in  $\mathbb{R}^N$  with boundary  $\partial B_1(0)$ ,  $\vec{\nu}$  is the inward normal vector on  $\partial B_1(0)$ , and  $u_{0,i}(x)$  are nonnegative, suitably smooth and bounded functions with compact support that satisfy the appropriate compatibility conditions.

The equation in (1.1) was intensively used in the models of chemical reactions, population dynamics, heat transfer, and so on. For problems (1.1)–(1.3), the local existence and the comparison principle of the weak solutions can be established, see [1, 2]. In this paper, we investigate the large time behavior of solutions to systems (1.1)–(1.3), such as global existence and blow-up in a finite time.

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**Biography:** Wang Zejia (1979–), female, born at Anda, Heilongjiang, professor, major in partial differential equations.

Since the beginning work on critical exponent done by Fujita [3] in 1966, lots of Fujita type results were established for various of problems, see the survey papers [4–6] and the references therein. It was Glalaktionov and Levine who first discussed the critical exponents for the one-dimensional nonlinear diffusion equations with boundary sources in [7]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}, \quad x > 0, t > 0, \quad (1.4)$$

$$\frac{\partial u^m}{\partial x}(0, t) = u^\alpha(0, t), \quad t > 0, \quad (1.5)$$

$$u(x, 0) = u_0(x), \quad x \in (0, +\infty), \quad (1.6)$$

here  $m > 1, \alpha \geq 0$ . For problems (1.4)–(1.6), they proved that  $\alpha_0 = (m + 1)/2, \alpha_c = m + 1$ . Here, we call  $\alpha_0$  as the critical global existence exponent and  $\alpha_c$  as the critical Fujita exponent respectively,

- (i) if  $0 < \alpha < \alpha_0$ , then every nontrivial nonnegative solution is global in time;
- (ii) if  $\alpha_0 < \alpha < \alpha_c$ , then the nontrivial nonnegative solutions blow up in a finite time;
- (iii) if  $\alpha > \alpha_c$ , then the solutions exist globally for the small initial data and blow up in a finite time for the large initial data.

In fact, instead of critical exponents there exist the critical global existence curve and the critical Fujita curve for the coupled system of diffusion equations, see [8]. For the one-dimensional nonlinear diffusion equations, Quirós and Rossi [9] considered the coupled Newtonian filtration equations as follows

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u^{m_1}}{\partial x^2}, & \frac{\partial v}{\partial t} &= \frac{\partial^2 v^{m_2}}{\partial x^2}, & x > 0, t > 0, \\ -\frac{\partial u^{m_1}}{\partial x}(0, t) &= v^\alpha(0, t), & -\frac{\partial v^{m_2}}{\partial x}(0, t) &= u^\beta(0, t), & t > 0, \\ u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x > 0. \end{aligned}$$

They showed that the critical global existence curve is given by  $\alpha\beta = (m_1 + 1)(m_2 + 1)/4$  and the critical Fujita curve is given by  $\min\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} = 0$ , where

$$\begin{aligned} \alpha_1 &= \frac{2\alpha + m_2 + 1}{(m_1 + 1)(m_2 + 1) - 4\alpha\beta}, & \beta_1 &= \frac{\alpha(m_1 - 1 - 2\beta) + (m_2 + 1)m_1}{(m_1 + 1)(m_2 + 1) - 4\alpha\beta}, \\ \alpha_2 &= \frac{2\beta + m_1 + 1}{(m_1 + 1)(m_2 + 1) - 4\alpha\beta}, & \beta_2 &= \frac{\beta(m_2 - 1 - 2\alpha) + (m_1 + 1)m_2}{(m_1 + 1)(m_2 + 1) - 4\alpha\beta}. \end{aligned}$$

The similar results were established in [10–12].

In the present paper, we consider the critical curves for the multi-dimensional systems (1.1)–(1.3), the case of single equation was studied in [13] and proved that both the critical global existence exponent and the critical Fujita exponent are given by  $p = m$ . We extend the results in [13] to the problem on multiple equations. Furthermore, by virtue of the radial symmetry of the exterior domain of the unit ball, we can extend our results to the following more general equations

$$\frac{\partial}{\partial t}(|x|^{\tilde{\lambda}_i} u_i) = \operatorname{div}(|x|^{\tilde{\lambda}_i} \nabla u_i^{m_i}), \quad x \in \mathbb{R}^N \setminus B_1(0), t > 0 \quad (1.7)$$

with  $\tilde{\lambda}_i > 2 - N$ ,  $N \geq 1$ .

The rest of this paper is organized as follows. Section 2 is devoted to the large time behavior of solutions to the nonlinear boundary problem for the Newtonian filtration equations, namely, (1.1)–(1.3) and (1.7), (1.2), (1.3).

## 2 Main Results and Their Proofs

In this section, we first introduce our results on the system of Newtonian filtration equations coupled with boundary conditions, then we give the proofs.

**Theorem 2.1** The critical global existence curve and the critical Fujita curve for systems (1.1)–(1.3) are given by

$$\prod_{i=1}^n p_i = \prod_{i=1}^n m_i.$$

Namely, if  $\prod_{i=1}^n p_i \leq \prod_{i=1}^n m_i$ , then all nonnegative nontrivial solutions to systems (1.1)–(1.3) exist globally in time; while if  $\prod_{i=1}^n p_i > \prod_{i=1}^n m_i$ , then the nonnegative solutions to systems (1.1)–(1.3) blow up in finite time for large initial data and exist globally for small initial data.

**Theorem 2.2** Assume  $\tilde{\lambda}_i > 2 - N$ ,  $N \geq 1$ . For equation (1.7) with the initial and boundary conditions (1.2), (1.3), the critical global existence curve and the critical Fujita curve both are given by  $\prod_{i=1}^n p_i = \prod_{i=1}^n m_i$ .

Before we give the proof of Theorem 2.1 and Theorem 2.2, we consider the problem

$$\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_i^{m_i}}{\partial r^2} + \frac{\lambda_i}{r} \frac{\partial u_i^{m_i}}{\partial r}, \quad r > 1, t > 0, i = 1, 2, \dots, n, \quad (2.1)$$

$$-\frac{\partial u_i^{m_i}}{\partial r}(1, t) = u_{i+1}^{p_i}(1, t), \quad t > 0, i = 1, 2, \dots, n, \quad (2.2)$$

$$u_i(r, 0) = u_{0,i}(r), \quad r > 1, i = 1, 2, \dots, n, \quad (2.3)$$

where  $r = |x|$ ,  $m_i > 1$ ,  $p_i \geq 0$ ,  $N \geq 1$ ,  $\lambda_i > 1$ , and  $u_{0,1}(r), u_{0,2}(r), \dots, u_{0,n}(r)$  are nonnegative, nontrivial functions with compact supports. Clearly, the solution  $(u_1, u_2, \dots, u_n)$  of the system (2.1)–(2.3) with  $\lambda_i = N - 1$  is also the solution of systems (1.1)–(1.3) if  $u_{0,1}(x), u_{0,2}(x), \dots, u_{0,n}(x)$  are radially symmetrical. If  $\lambda_i = \tilde{\lambda}_i + N - 1$ , the same facts also hold valid between systems (2.1)–(2.3) with systems (1.7), (1.2), (1.3). In order to obtain Theorem 2.1, Theorem 2.2, we firstly show the following results on systems (2.1)–(2.3).

**Proposition 2.1** If  $\prod_{i=1}^n p_i < \prod_{i=1}^n m_i$ , then all nonnegative nontrivial solutions to systems (2.1)–(2.3) exist globally in time.

**Proposition 2.2** If  $\prod_{i=1}^n p_i > \prod_{i=1}^n m_i$ , then the nonnegative solutions with large initial data to systems (2.1)–(2.3) blow up in a finite time.

**Remark 1** It can be seen from Proposition 2.1 and Proposition 2.2 that the critical global existence curve for systems (2.1)–(2.3) is  $\prod_{i=1}^n p_i = \prod_{i=1}^n m_i$ .

**Proposition 2.3** If  $\prod_{i=1}^n p_i \neq \prod_{i=1}^n m_i$ , then every nonnegative nontrivial solution with small initial data to problems (2.1)–(2.3) exists globally.

**Remark 2** From Proposition 2.1–Proposition 2.3, we have the critical Fujita curve for systems (2.1)–(2.3) is given by  $\prod_{i=1}^n p_i = \prod_{i=1}^n m_i$ .

Now, we prove Proposition 2.1 – Proposition 2.3.

**Proof of Proposition 2.1.** We prove this proposition by constructing a kind of global supersolution in the following form

$$\tilde{u}_i(r, t) = (T + t)^{k_i} h_i(\xi_i), \quad \xi_i = \frac{r - 1}{(T + t)^{l_i}}, \quad r > 1, t > 0, i = 1, 2, \dots, n, \quad (2.4)$$

where  $T > 0$ , and  $m_{n+1} = m_1, k_{i+1} = k_i \gamma_i = k_1 \prod_{j=1}^i \gamma_j, \gamma_i < \frac{m_i}{p_i}, i = 1, 2, \dots, n - 1$  with  $k_1$  being the positive constants to be determined and

$$l_i = \frac{1 + k_i(m_i - 1)}{2}, \quad i = 1, 2, \dots, n.$$

Fix  $\xi_i > 0$  for any  $i \in \{1, 2, \dots, n\}$ , we take

$$h_i(\xi_i) = \left( \frac{m_i - 1}{m_i} (1 - \xi_i)_+ \right)^{1/(m_i - 1)}, \quad i = 1, 2, \dots, n.$$

Denote

$$\bar{u}_i(r, t) = r^{-\alpha_i/m_i} \tilde{u}_i(r, t), \quad r > 1, t > 0, i = 1, 2, \dots, n, \quad (2.5)$$

where  $\alpha_i$  are given by the following

$$\alpha_i = \begin{cases} \lambda_i - 1, & 1 < \lambda_i < 2, \\ \frac{1}{2} \lambda_i, & \lambda_i \geq 2. \end{cases}$$

We claim that  $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n)$  is an upper solution to problems (2.1)–(2.3). First, it needs to verify the boundary conditions

$$-\frac{\partial \bar{u}_i^{m_i}}{\partial r}(1, t) \geq \bar{u}_{i+1}^{p_i}(1, t), \quad i = 1, 2, \dots, n - 1, \quad (2.6)$$

$$-\frac{\partial \bar{u}_n^{m_n}}{\partial r}(1, t) \geq \bar{u}_1^{p_n}(1, t). \quad (2.7)$$

Hence, due to that

$$\frac{\partial \tilde{u}_i^{m_i}}{\partial r} = (T + t)^{m_i k_i - l_i} (h_i^{m_i})'(\xi_i) \leq 0, \quad i = 1, 2, \dots, n,$$

we have

$$\begin{aligned} -\frac{\partial \bar{u}_i^{m_i}}{\partial r}(1, t) &= \alpha_i \tilde{u}_i^{m_i}(1, t) - \frac{\partial \tilde{u}_i^{m_i}}{\partial r}(1, t) \\ &\geq \alpha_i \tilde{u}_i^{m_i}(1, t) = \alpha_i (T+t)^{k_i m_i} \left( \frac{m_i - 1}{m_i} \right)^{m_i / (m_i - 1)}. \end{aligned}$$

Similarly, it is clear that

$$\bar{u}_{i+1}^{p_i}(1, t) = (T+t)^{k_{i+1} p_i} \left( \frac{m_{i+1} - 1}{m_{i+1}} \right)^{p_i / (m_{i+1} - 1)}.$$

Thus, the inequalities in (2.6) and (2.7) are valid if

$$\alpha_i \left( \frac{m_i - 1}{m_i} \right)^{m_i / (m_i - 1)} (T+t)^{k_i m_i} \geq \left( \frac{m_{i+1} - 1}{m_{i+1}} \right)^{p_i / (m_{i+1} - 1)} (T+t)^{k_{i+1} p_i}, \quad (2.8)$$

$$\alpha_n \left( \frac{m_n - 1}{m_n} \right)^{m_n / (m_n - 1)} (T+t)^{k_n m_n} \geq \left( \frac{m_1 - 1}{m_1} \right)^{p_n / (m_1 - 1)} (T+t)^{k_1 p_n}. \quad (2.9)$$

Noticing that  $\prod_{i=1}^n p_i < \prod_{i=1}^n m_i$ , there exists constant  $\gamma_i > 0, i = 1, 2, \dots, n-1$ , such that

$\prod_{i=1}^{n-1} \gamma_i > \frac{p_n}{m_n}$ . For any fixed  $k_1 > 0$  and  $k_{i+1} = k_i \gamma_i = k_1 \prod_{j=1}^i \gamma_j, \gamma_i < \frac{m_i}{p_i}, i = 1, 2, \dots, n-1$ ,

(2.8) and (2.9) hold for the large enough  $T$ .

Second, we verify that  $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n)$  is the upper solutions to the equations in (2.1).

A simple computation yields

$$(h_i^{m_i})''(\xi_i) + \xi_i h_i'(\xi_i) - \frac{1}{m_i - 1} h_i(\xi_i) = 0, \quad \xi_i > 0, i = 1, 2, \dots, n.$$

Then

$$\begin{aligned} \frac{\partial \tilde{u}_i}{\partial t} - \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} &= (T+t)^{k_i - 1} \left( k_i h_i(\xi_i) - l_i \xi_i h_i'(\xi_i) - (h_i^{m_i})''(\xi_i) \right) \\ &= (T+t)^{k_i - 1} \left[ \left( k_i - \frac{1}{m_i - 1} \right) h_i(\xi_i) - (l_i - 1) \xi_i h_i'(\xi_i) \right]. \end{aligned}$$

Recall that  $k_{i+1} = k_i \gamma_i = k_1 \prod_{j=1}^i \gamma_j, i = 1, 2, \dots, n-1$ , we choose  $k_1 > 0$  such that

$$k_1 > \max \left\{ \frac{1}{m_1 - 1}, \frac{1}{\gamma_1(m_2 - 1)}, \frac{1}{\gamma_1 \gamma_2(m_3 - 1)}, \dots, \prod_{i=1}^{n-1} \frac{1}{\gamma_i(m_n - 1)} \right\}$$

implies  $l_i > 1, k_i > \frac{1}{m_i - 1}$ . Combing with that  $h_i'(\xi_i) \leq 0$ , we get

$$\frac{\partial \tilde{u}_i}{\partial t} \geq \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} \geq \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} + \frac{\tilde{\alpha}_i}{r} \frac{\partial \tilde{u}_i^{m_i}}{\partial r} \quad (2.10)$$

with

$$\tilde{\alpha}_i = \begin{cases} 2 - \lambda_i, & 1 < \lambda_i < 2, \\ 0, & \lambda_i \geq 2. \end{cases}$$

Note that

$$\begin{aligned}\frac{\partial \tilde{u}_i}{\partial t} &= r^{\alpha_i/m_i} \frac{\partial \tilde{u}_i}{\partial t}, \\ \frac{\partial \tilde{u}_i^{m_i}}{\partial r} &= \alpha_i r^{\alpha_i-1} \tilde{u}_i^{m_i} + r^{\alpha_i} \frac{\partial \tilde{u}_i^{m_i}}{\partial r}, \\ \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} &= r^{\alpha_i} \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} + 2\alpha_i r^{\alpha_i-1} \frac{\partial \tilde{u}_i^{m_i}}{\partial r} + \alpha_i(\alpha_i-1)r^{\alpha_i-2} \tilde{u}_i^{m_i}.\end{aligned}$$

Substituting the above equalities into (2.10), we obtain that

$$\begin{aligned}r^{\alpha_i/m_i} \frac{\partial \tilde{u}_i}{\partial t} &\geq r^{\alpha_i} \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} + (2\alpha_i + \tilde{\alpha}_i)r^{\alpha_i-1} \frac{\partial \tilde{u}_i^{m_i}}{\partial r} + \alpha_i(\alpha_i-1 + \tilde{\alpha}_i)r^{\alpha_i-2} \tilde{u}_i^{m_i} \\ &\geq r^{\alpha_i} \left( \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} + \frac{\lambda_i}{r} \frac{\partial \tilde{u}_i^{m_i}}{\partial r} \right),\end{aligned}$$

due to that  $\alpha_i - 1 + \tilde{\alpha}_i \geq 0$  and  $2\alpha_i + \tilde{\alpha}_i = \lambda_i$ . The above inequality implies that for  $r > 1$ ,

$$\frac{\partial \tilde{u}_i}{\partial t} \geq r^{\alpha_i(1-1/m_i)} \left( \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} + \frac{\lambda_i}{r} \frac{\partial \tilde{u}_i^{m_i}}{\partial r} \right) \geq \frac{\partial^2 \tilde{u}_i^{m_i}}{\partial r^2} + \frac{\lambda_i}{r} \frac{\partial \tilde{u}_i^{m_i}}{\partial r}.$$

Finally, we verify the initial conditions that

$$u_i(r, 0) \geq u_{0,i}(r), \quad i = 1, 2, \dots, n. \quad (2.11)$$

Denote

$$M_i = \max_{r>1} u_{0,i}(r), \quad \text{supp} u_{0,i} = [1, R_i], \quad i = 1, 2, \dots, n.$$

Then (2.11) holds provided with

$$\tilde{u}_i(R_i, 0) \geq M_i, \quad i = 1, 2, \dots, n, \quad (2.12)$$

since that  $\tilde{u}_i(r, 0)$  are decreasing with respect to  $r$ . In fact, we can choose  $T$  to be large enough to satisfy that

$$R_i^{-\alpha_i/m_i} T^{k_i} \left( \frac{m_i-1}{m_i} \left( 1 - \frac{R_i-1}{T^{l_i}} \right)_+ \right)^{1/(m_i-1)} \geq M_i, \quad i = 1, 2, \dots, n.$$

From the above, for large  $T$  satisfying (2.8) and (2.12), it is seen that  $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)$  is a supersolution of systems (2.1)–(2.3). Therefore, we have the solution of problems (2.1)–(2.3) exists globally by the comparison principle. The proof is completed.

**Proof of Proposition 2.2** The proposition is proved by constructing a kind of lower blow-up lower solution of systems (2.1)–(2.3). For  $r > 1, 0 < t < T$ , set

$$\underline{u}_i(r, t) = (T-t)^{-\mu_i} f_i(\eta), \quad \eta = (r-1)(T+t), \quad i = 1, 2, \dots, n,$$

where  $T > 0, \mu_1 > 0$  is a given constant,

$$\begin{aligned}\mu_1 &> \max \left\{ \frac{1}{m_1-1}, \prod_{j=1}^i \frac{p_j}{m_j(m_{i+1}-1)} \right\}, \quad i = 1, 2, \dots, n-1, \\ -\mu_i m_i + \mu_{i+1} p_i &= 0, \quad i = 1, 2, \dots, n-1.\end{aligned}$$

Assume that  $f_i$  satisfy  $f_i \geq 0, f_i' \leq 0, (f_i^{m_i})'' \geq 0, i = 1, 2, \dots, n$ , by a direct calculation, we can see that  $(\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n)$  with  $\underline{u}_i(r, 0) \leq u_{0,i}(r)$  is a subsolution to systems (2.1)–(2.3), if the following inequalities hold

$$T^2(f_i^{m_i})'' + 2T\frac{\lambda_i}{r}(f_i^{m_i})' - \mu_i(T-t)^{\mu_i m_i - \mu_i - 1} f_i(\eta) \geq 0, \quad i = 1, 2, \dots, n, \quad (2.13)$$

$$-2T(T-t)^{-\mu_i m_i + \mu_{i+1} p_i} (f_i^{m_i})'(0) \leq f_{i+1}^{p_i}(0), \quad i = 1, 2, \dots, n. \quad (2.14)$$

Note that  $r > 1$  and

$$\begin{aligned} \mu_i m_i - \mu_i - 1 &> 0, \quad i = 1, 2, \dots, n, \\ -\mu_i m_i + \mu_{i+1} p_i &= 0, \quad i = 1, 2, \dots, n-1. \end{aligned}$$

Thus (2.13) and (2.14) hold if

$$T^2(f_i^{m_i})'' + 2T\lambda_i(f_i^{m_i})' - \mu_i T^{\mu_i m_i - \mu_i - 1} f_i(\eta) \geq 0, \quad i = 1, 2, \dots, n, \quad (2.15)$$

$$-2T(f_i^{m_i})'(0) \leq f_{i+1}^{p_i}(0), \quad i = 1, 2, \dots, n \quad (2.16)$$

with  $f_{n+1} = f_1$ . For any  $i \in \{1, 2, \dots, n\}$ , namely,

$$f_i(\eta) = A_i(B_i - \eta)_+^{1/(m_i - 1)}, \quad \eta > 0, \quad (2.17)$$

where  $A_i$  are positive constants to be determined, and

$$B_i = \min \left\{ \frac{T}{4\lambda_i(m_i - 1)}, \frac{m_i T^{3 + \mu_i - \mu_i m_i}}{2\mu_i(m_i - 1)^2} \right\}.$$

In the following, we verify that the above  $f_i$  satisfy (2.15) and (2.16). Substituting (2.17) into (2.15) yields that

$$\begin{aligned} A_i^{m_i} \frac{m_i}{(m_i - 1)^2} T^2 - A_i^{m_i} \lambda_i \frac{m_i}{m_i - 1} (2T)(B_i - \eta)_+ - A_i \mu_i T^{\mu_i m_i - \mu_i - 1} (B_i - \eta)_+ &\geq 0, \\ i = 1, 2, \dots, n, \end{aligned}$$

which can be obtained by the following

$$\begin{aligned} A_i^{m_i} \frac{m_i}{(m_i - 1)^2} T^2 &\geq \frac{1}{2} A_i^{m_i} \lambda_i \frac{m_i}{m_i - 1} (2T) B_i, \\ A_i^{m_i} \frac{m_i}{(m_i - 1)^2} T^2 &\geq \frac{1}{2} A_i \mu_i T^{\mu_i m_i - \mu_i - 1} B_i. \end{aligned}$$

The choice of  $B_i$  makes us to conclude that the above inequalities hold for  $A_i > 1$ . This indicated that  $f_i$  satisfy (2.15).

Next, substitute (2.17) into (2.16), we have

$$2T \frac{m_i}{m_i - 1} B_i^{1/(m_i - 1)} A_i^{m_i} \leq B_{i+1}^{p_i/(m_{i+1} - 1)} A_{i+1}^{p_i}, \quad i = 1, 2, \dots, n \quad (2.18)$$

with  $A_{n+1} = A_1$ . Since  $\prod_{i=1}^n p_i > \prod_{i=1}^n m_i$ , there exists a constant  $k > 1$ , such that

$$k^{n-1} < \frac{\prod_{i=1}^n p_i}{\prod_{i=1}^n m_i}. \quad (2.19)$$

For  $i \in \{1, 2, \dots, n-1\}$ , set  $A_{i+1} = A_i^{k\sigma_i}$  with  $\sigma_i = \frac{m_i}{p_i}$ . Then (2.18) can also be written as

$$2T \frac{m_i}{m_i - 1} B_i^{1/(m_i-1)} A_i^{m_i} \leq B_{i+1}^{p_i/(m_{i+1}-1)} A_i^{k m_i}, \quad i = 1, 2, \dots, n. \quad (2.20)$$

Choose  $A_i$  large enough to satisfy (2.20), then we get (2.16).

Therefore, the solution  $(u_1, u_2, \dots, u_n)$  with  $u_{0,i}(r) \geq \underline{u}_i(r, 0)$  ( $i = 1, 2, \dots, n$ ) of problems (2.1)–(2.3) blows up in a finite time. The proof is completed.

**Proof of Proposition 2.3** The proof is completed by constructing the following global upper solution

$$\bar{u}_i(r, t) = (B_i r^{1-\lambda_i})^{1/m_i}, \quad r > 1, t > 0, \quad i = 1, 2, \dots, n,$$

where

$$B_1 = \prod_{i=1}^n (\lambda_i - 1)^{\mu_i}, \quad B_{i+1} = \left( B_i (\lambda_i - 1) \right)^{m_{i+1}/p_i}, \quad i = 1, 2, \dots, n-1$$

with  $\mu_i = \frac{m_1 \prod_{j=1}^{i-1} p_j \prod_{j=i+1}^n m_j}{\prod_{j=1}^n p_j - \prod_{j=1}^n m_j}$ . It can easily be checked that

$$\begin{aligned} \frac{\partial \bar{u}_i}{\partial t} - \frac{\partial^2 \bar{u}_i}{\partial r^2} - \frac{\lambda_i}{r} \frac{\partial \bar{u}_i}{\partial r} &= 0, \quad r > 1, t > 0, i = 1, 2, \dots, n, \\ - \frac{\partial \bar{u}_i}{\partial r}(1, t) &= \bar{u}_{i+1}^{p_i}(1, t), \quad t > 0, i = 1, 2, \dots, n \end{aligned}$$

with  $\bar{u}_{n+1} = \bar{u}_1$ . Furthermore, by the comparison principle, if the initial data

$$(u_{0,1}, u_{0,2}, \dots, u_{0,n})$$

is small enough that

$$u_{0,i}(r) \leq \bar{u}_{0,i}(r, 0), \quad i = 1, 2, \dots, n,$$

the solutions of problems (2.1)–(2.3) exist globally in time.

Now, we prove the main result for systems (1.1)–(1.3), i.e., Theorem 2.1.

**Proof of Theorem 2.1** Noticing that the functions  $u_{0,1}(x), u_{0,2}(x), \dots, u_{0,n}(x)$  have compact supports, we can choose  $n$  bounded, radially symmetrical functions, denoted by  $u_i(x) = u_i(|x|) \geq u_{0,i}(x), i = 1, 2, \dots, n$ . By using Proposition 2.1 and the comparison principle, we can obtain the global existence of solutions for systems (1.1)–(1.3). However, for

the large and radially symmetric functions  $\underline{u}_1(|x|, 0), \underline{u}_2(|x|, 0), \dots, \underline{u}_n(|x|, 0)$  defined in the proof of Proposition 2.2, if  $(u_{0,1}, u_{0,2}, \dots, u_{0,n})$  is large enough such that  $u_{0,i}(x) \geq \underline{u}_i(|x|, 0)$ ,  $i = 1, 2, \dots, n$ , then the solutions of systems (1.1)–(1.3) with  $\prod_{i=1}^n p_i > \prod_{i=1}^n m_i$  blow up by the comparison principle and Proposition 2.2. This clarifies that the critical global existence curve is  $\prod_{i=1}^n p_i = \prod_{i=1}^n m_i$  for system (1.1)–(1.3).

On the other hand, using the comparison principle again, we conclude that the solution  $(u_1, u_2, \dots, u_n)$  of (1.1)–(1.3) with

$$u_{0,i}(x) \leq (B_i|x|^{2-N})^{1/m_i}, \quad x \in \mathbb{R}^N \setminus B_1(0), i = 1, 2, \dots, n, \tag{2.21}$$

where

$$B_1 = \prod_{i=1}^n (N-2)^{\mu_i}, \quad B_{i+1} = \left( B_i(N-2) \right)^{m_{i+1}/p_i}, \quad i = 1, 2, \dots, n-1$$

with  $\mu_i = \frac{m_1 \prod_{j=1}^{i-1} p_j \prod_{j=i+1}^n m_j}{\prod_{j=1}^n p_j - \prod_{j=1}^n m_j}$ , exists globally for  $\prod_{i=1}^n p_i > \prod_{i=1}^n m_i$  by Proposition 2.3. This combined with Proposition 2.2 indicates that the critical Fujita curve  $\prod_{i=1}^n p_i = \prod_{i=1}^n m_i$  for systems (1.1)–(1.3).

**Proof of Theorem 2.2** By virtue of the same discussion in the proof of Theorem 2.1, if we prove this theorem by taking  $\lambda_i = \tilde{\lambda}_i + N - 1$  in systems (2.1)–(2.3), and replacing (2.21) with

$$u_{0,i}(x) \leq (B_i|x|^{-\tilde{\lambda}_i+2-N})^{1/m_i}, \quad i = 1, 2, \dots, n,$$

where

$$B_1 = \prod_{i=1}^n (\tilde{\lambda}_i + N - 2)^{\mu_i}, \quad B_{i+1} = \left( B_i(\tilde{\lambda}_i + N - 2) \right)^{m_{i+1}/p_i}, \quad i = 1, 2, \dots, n-1$$

with  $\mu_i = \frac{m_1 \prod_{j=1}^{i-1} p_j \prod_{j=i+1}^n m_j}{\prod_{j=1}^n p_j - \prod_{j=1}^n m_j}$ . The proof is completed.

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## 边界耦合的牛顿渗流方程组解的整体存在与爆破

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**摘要:** 本文研究了由边界条件耦合的多维牛顿渗流方程组解的长时间行为. 利用构造的多种上下解, 得到了整体存在临界曲线与Fujita临界曲线.

**关键词:** 整体存在; 爆破; 牛顿渗流方程; 临界曲线

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