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ESSENTIAL NORMS OF THE GENERALIZED VOLTERRA COMPOSITION OPERATORS

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Abstract: In this paper, we introduce a generalized Volterra composition operator on $H(\mathbb{D})$ by $J_{\varphi,g}^{(n)}f(z)=\int_0^z (f^{(n)}\circ\varphi)(\xi)(g\circ\varphi)'(\xi)d\xi$. By using the maps φ and g, we characterize the boundedness and compactness of this operator from Bergman-type space to weighted Zygmund space and weighted Bloch space. We also obtain an asymptotic expression of the essential norm for this operator.

Keywords: Bergman-type space; weighted Zygmund space; generalized Volterra composition operator; essential norm

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. If u is a positive continuous function on $[0,1)$ and there exist positive numbers $\delta\in[0,1)$, s and t, $0<s<t$, such that $u(r)/(1-r)^s$ is decreasing on $[\delta,1)$ and $\lim\limits_{r\to1^-}u(r)/(1-r)^s=0$; $u(r)/(1-r)^t$ is increasing on $[\delta,1)$ and $\lim\limits_{r\to1^-}u(r)/(1-r)^t=\infty$, then u is called a normal weight function (see [1]). For such normal weights, one can consider the following examples (see [2])

$$u(r)=(1-r^2)^\alpha,\ \ \alpha\in(0,\infty),$$
$$u(r)=(1-r^2)^\alpha\{\log 2(1-r^2)^{-1}\}^\beta,\ \ \alpha\in(0,1),\ \ \beta\in\Big[\frac{\alpha-1}{2}\log 2,0\Big]$$

and

$$u(r)=(1-r^2)^\alpha\{\log\log e^2(1-r^2)^{-1}\}^\gamma,\ \ \alpha\in(0,1),\ \ \gamma\in\Big[\frac{\alpha-1}{2}\log 2,0\Big].$$

For $0<p<\infty$ and the normal weight function u, the Bergman-type space A_u^p on $\mathbb{D}$ is defined by

$$A_u^p=\Big\{f\in H(\mathbb{D}):\|f\|^p=\int_{\mathbb{D}}|f(z)|^p\frac{u(|z|)^p}{1-|z|}dA(z)<\infty\Big\}.$$

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When $1 \le p < \infty$, A_u^p is a Banach space with the norm $\|\cdot\|$. When $0 < p < 1$, it is a Fréchet space with the translation invariant metric

$$d(f,g) = \|f-g\|^p.$$

For this space and some operators, see, e.g., [2] and [3].

For $0 < \beta < \infty$, the weighted Bloch space $\mathcal{B}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(z)| < \infty.$$

It is a Banach space under the norm

$$\|f\|_{\mathcal{B}_\beta} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(z)|.$$

Similar to the weighted Bloch space, the weighted Zygmund space $\mathcal{Z}_\beta$ is defined by

$$\mathcal{Z}_\beta = \big\{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f''(z)| < \infty\big\}.$$

The norm on this space is

$$\|f\|_{\mathcal{Z}_\beta} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f''(z)|,$$

and under this norm it is a Banach space. There are a lot of papers to study weighted Bloch spaces, weighted Zygmund spaces and operators between weighted Bloch spaces or weighted Zygmund spaces and some other analytic function spaces. We refer the readers to see, e.g., [4–8] and the references therein.

Let φ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. The Volterra composition operator $J_{\varphi,g}$ on $H(\mathbb{D})$ is

$$J_{\varphi,g}f(z) = \int_0^z (f\circ\varphi)(\xi)(g\circ\varphi)'(\xi)d\xi,\ \ z \in \mathbb{D}.$$

In the beginning, people studied this operator for the case $\varphi(z) = z$ (see [1, 9–12]). For this case, it is called the integral operator or the Volterra operator. People pay a lot of attention to this operator on analytic function spaces, due to it's relation with other branches of complex analysis such as, Bekollé-Bonami weights, univalent functions, Littlewood-Paley type formula, conformal invariance spaces and Carleson measures (see [1, 9, 10, 13–15]). Recently, Li characterized the bounded and compact Volterra composition operators between weighted Bergman spaces and Bloch type spaces in [16]. Wolf characterized the bounded and compact Volterra composition operators between weighted Bergman spaces and weighted Bloch type spaces in [17]. The present author studied the bounded and compact Volterra composition operators from Bergman-type spaces to Bloch-type spaces and obtained an asymptotic expression of the essential norm in [2].

Let n be a nonnegative integer, φ an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. We define a generalized Volterra composition operator $J_{\varphi,g}^{(n)}$ on $H(\mathbb{D})$ by

$$J_{\varphi,g}^{(n)} f(z) = \int_0^z (f^{(n)} \circ \varphi)(\xi)(g \circ \varphi)'(\xi) d\xi,$$

where $f^{(0)} = f$. It is easy to see that this operator is a generalization of the Volterra composition operator. A natural problem is how to provide a function theoretic characterization of φ and g when they induce the bounded and compact operator between analytic function spaces in the unit disk. In this paper, we consider this problem for operator $J_{\varphi,g}^{(n)}$ from Bergman-type spaces to weighted Zygmund spaces and weighted Bloch spaces. We obtain the following Theorems 3.1, 3.3.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \leq b \leq Ca$.

2 Prerequisites

First, we have the following lemma. Since the proof is standard, it is omitted (see Proposition 3.11 in [13]).

Lemma 2.1 Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $Y = \mathcal{B}_\beta$ or $\mathcal{Z}_\beta$, and the operator $J_{\varphi,g}^{(n)} : A_u^p \to Y$ is bounded, then the operator $J_{\varphi,g}^{(n)} : A_u^p \to Y$ is compact if and only if for bounded sequence $\{f_j\}$ in A_u^p such that $f_j \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that

$$\lim_{j\to\infty} \|J_{\varphi,g}^{(n)} f_j\|_Y = 0.$$

For the cases of $j = 0$ and $j = 1$, the next lemma was proved in [4]. For the general positive integer j, it can be proved similarly, and it is omitted here.

Lemma 2.2 For $j \in \mathbb{N}_+$, there is a positive constant C independent of $f \in A_u^p$ such that for every $z \in \mathbb{D}$ the following inequality holds

$$|f^{(j)}(z)| \leq C \frac{\|f\|}{u(|z|)(1-|z|^2)^{j+\frac{1}{p}}}.$$

The next Lemma 2.3 can be found in [4].

Lemma 2.3 Suppose that $w \in \mathbb{D}$, then for $t \geq 0$ the function

$$f_{w,t}(z) = \frac{(1-|w|^2)^t}{u(|w|)(1-\overline{w}z)^{t+\frac{1}{p}}}$$

belongs to A_u^p and $\|f_{w,t}\| \asymp 1$.

Using the function $f_{w,t}$, we have

Lemma 2.4 Suppose that $w \in \mathbb{D}$, then for each fixed $j \in \{1, 2, \cdots, n\}$ there exist constants c_1, c_2, $\cdots$, c_n such that the function

$$f(z) = \sum_{i=1}^{n} c_i f_{\varphi(w), n-2+i}(z)$$

satisfying

$$f^{(j)}(\varphi(w)) = C \frac{\overline{\varphi(w)}^j}{u(|\varphi(w)|)(1-|\varphi(w)|^2)^{j+\frac{1}{p}}} \text{ and } f^{(k)}(\varphi(w)) = 0 \tag{2.1}$$

for each $k \in \{1, 2, \cdots, n\} \setminus \{j\}$, where C is a constant related to j.

Proof For a fixed $w \in \mathbb{D}$ and arbitrary constants d_1, d_2, $\cdots$, d_n, define the function

$$f(z) = \sum_{i=1}^{n} \frac{d_i}{n-2+i+\frac{1}{p}} f_{\varphi(w), n-2+i}(z).$$

To finish the proof, we only need to show that there exist constants d_1, d_2, $\cdots$, d_n such that f satisfying assertion (2.1). By calculating $f^{(k)}$ for each $k \in \{1, \cdots, n\}$, we obtain the following linear system

$$\begin{cases} \sum\limits_{i=1}^{n} d_i = 0, \\ \sum\limits_{i=1}^{n} (n+i-1+\frac{1}{p}) d_i = 0, \\ \qquad \cdots\cdots\cdots\cdots \\ \sum\limits_{j=1}^{n} \prod\limits_{i=0}^{k-2} (n+i+j-1+\frac{1}{p}) d_i = 1, \\ \qquad \cdots\cdots\cdots\cdots \\ \sum\limits_{j=1}^{n} \prod\limits_{i=0}^{n-2} (n+i+j-1+\frac{1}{p}) d_i = 0. \end{cases}$$

If we can show that the determinant of this linear system is different from zero, the problem will be solved. Applying Lemma 2.3 in [18] with $a = n + 1/p > 0$, we see that it is different from zero. Let $c_i = d_i/(n-2+i+\frac{1}{p})$, and then proof is end.

The following lemma is also a folklore.

Lemma 2.5 $H^\infty \subseteq A_u^p$. In particular, every polynomial function belongs to A_u^p.

3 Main Results

We first give the conditions for $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{Z}_\beta$ to be a bounded operator.

Theorem 3.1 Suppose that φ is an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$, then the operator $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{Z}_\beta$ is bounded if and only if the following conditions are satisfied

(i) $M_0 := \sup\limits_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}} |g'(\varphi(z))||\varphi'(z)|^2 < \infty$;

(ii) $M_1 := \sup\limits_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}} \left|g'(\varphi(z))\varphi''(z) + g''(\varphi(z))\varphi'(z)^2\right| < \infty$.

Moreover, if the operator $J_{\varphi,g}^{(n)}: A_u^p \to \mathcal{Z}_{\beta,0}$ is bounded, then

$$\|J_{\varphi,g}^{(n)}\|_{A_u^p \to \mathcal{Z}_{\beta,0}} \asymp M_0 + M_1,$$

where $\mathcal{Z}_{\beta,0} = \{f \in \mathcal{Z}_\beta : f(0) = 0\}$ is the closed subspace of $\mathcal{Z}_\beta$.

Proof First suppose that the operator $J_{\varphi,g}^{(n)}: A_u^p \to \mathcal{Z}_\beta$ is bounded. Take the function $f(z) = z^n/n!$. Then from Lemma 2.5 it follows that $f \in A_u^p$. Since the operator $J_{\varphi,g}^{(n)}: A_u^p \to \mathcal{Z}_\beta$ is bounded, we have

$$\|g \circ \varphi\|_{\mathcal{Z}_\beta} = \|J_{\varphi,g}^{(n)} f\|_{\mathcal{Z}_\beta} \le C\|J_{\varphi,g}^{(n)}\|,$$

that is

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(g\circ\varphi)''(z)| \le C\|J_{\varphi,g}^{(n)}\|. \tag{3.1}$$

Considering $f(z) = z^{n+1}/(n+1)!$, by the boundedness of the operator $J_{\varphi,g}^{(n)}: A_u^p \to \mathcal{Z}_\beta$ we have

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(\varphi(z))\varphi'(z)^2 + \varphi(z)(g\circ\varphi)''(z)| \le \|J_{\varphi,g}^{(n)} f\|_{\mathcal{Z}_\beta} \le C\|J_{\varphi,g}^{(n)}\|. \tag{3.2}$$

From (3.1), (3.2) and the fact $|\varphi(z)| < 1$ on $\mathbb{D}$, it follows that

$$\begin{aligned}
\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|^2 &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \Big| -\varphi(z)(g\circ\varphi)''(z) \\
&\quad + g'(\varphi(z))\varphi'(z)^2 + \varphi(z)(g\circ\varphi)''(z)\Big| \\
&\le \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|(g\circ\varphi)''(z)\big| \\
&\quad + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|g'(\varphi(z))\varphi'(z)^2 + \varphi(z)(g\circ\varphi)''(z)\big| \\
&\le C\|J_{\varphi,g}^{(n)}\|.
\end{aligned} \tag{3.3}$$

For $w \in \mathbb{D}$, taking the function f in Lemma 2.4, we know that there exist constants c_1, c_2, $\cdots$, c_{n+1} such that

$$f(z) = \sum_{j=1}^{n+1} c_j f_{\varphi(w),n-1+j}(z)$$

satisfying

$$f^{(n+1)}(\varphi(w)) = C\frac{\overline{\varphi(w)}^{n+1}}{u(|\varphi(w)|)(1-|\varphi(w)|^2)^{n+1+\frac{1}{p}}}$$

and $f^{(k)}(\varphi(w)) = 0$ for each $k \in \{1, 2, \cdots, n\}$. Then by the boundedness of the operator $J_{\varphi,g}^{(n)}: A_u^p \to \mathcal{Z}_\beta$, we get

$$(1-|w|^2)^\beta \big|(J_{\varphi,g}^{(n)} f)''(w)\big| = \frac{(1-|w|^2)^\beta |g'(\varphi(w))||\varphi'(w)|^2|\varphi(w)|^{n+1}}{u(|\varphi(w)|)(1-|\varphi(w)|^2)^{n+1+\frac{1}{p}}} \le C\|J_{\varphi,g}^{(n)}\|,$$

i.e., we have

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|^2|\varphi(z)|^{n+1}}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}} \le C\|J_{\varphi,g}^{(n)}\|.$$

So for $\delta \in (0,1)$,

$$\sup_{\{z:|\varphi(z)|>\delta\}} \frac{(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|^2}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}} \le C\delta^{-(n+1)}\|J_{\varphi,g}^{(n)}\|, \tag{3.4}$$

and by (3.3),

$$\begin{aligned}\sup_{\{z:|\varphi(z)|\le\delta\}} \frac{(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|^2}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}} &\le C\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|^2 \\ &\le C\|J_{\varphi,g}^{(n)}\|, \end{aligned}\tag{3.5}$$

where the first constant

$$C = \frac{1}{\min\limits_{0\le r\le\delta} u(r)(1-\delta^2)^{n+1+\frac{1}{p}}}.$$

From (3.4) and (3.5), assertion (i) follows.

Once again choose the function f in Lemma 2.4 which satisfies

$$f^{(n)}(\varphi(w)) = C\frac{\overline{\varphi(w)}^n}{u(|\varphi(w)|)(1-|\varphi(w)|^2)^{n+\frac{1}{p}}}$$

and $f^{(k)}(\varphi(w)) = 0$ for each $k \in \{1,2,\cdots,n-1,n+1\}$. Then by the similar method, we can prove that assertion (ii) is true, and the proof is omitted here.

Conversely, suppose that assertions (i) and (ii) hold. For $f \in A_u^p$, by Lemma 2.2 we have

$$\begin{aligned}\|J_{\varphi,g}^{(n)}f\|_{\mathcal{Z}_\beta} &= \left|(f^{(n)}\circ\varphi)(0)(g\circ\varphi)'(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|\left[(f^{(n)}\circ\varphi)\cdot(g\circ\varphi)'\right]'(z)\right| \\ &\le C\|f\| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|f^{(n)}(\varphi(z))\Big(g''(\varphi(z))\varphi'(z)^2 + g'(\varphi(z))\varphi''(z)\Big) \\ &\quad + f^{(n+1)}(\varphi(z))g'(\varphi(z))\varphi'(z)^2\Big| \\ &\le C\|f\|\Big(1+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}}\left|g'(\varphi(z))\varphi''(z)+g''(\varphi(z))\varphi'(z)^2\right| \\ &\quad + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}}\left|g'(\varphi(z))\right|\left|\varphi'(z)\right|^2\Big). \end{aligned}\tag{3.6}$$

From assertions (i), (ii) and (3.6), it follows that $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{Z}_\beta$ is bounded. From the above inequalities, we also obtain the asymptotic expression $\|J_{\varphi,g}^{(n)}\|_{A_u^p\to\mathcal{Z}_{\beta,0}} \asymp M_0 + M_1$.

The following result can be similarly proved.

Theorem 3.2 Suppose that φ is an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$, then the operator $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{B}_\beta$ is bounded if and only if

$$M_2 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}} |g'(\varphi(z))||\varphi'(z)| < \infty.$$

Moreover, if the operator $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{B}_{\beta,0}$ is bounded, then $\|J_{\varphi,g}^{(n)}\|_{A_u^p\to\mathcal{B}_{\beta,0}} \asymp M_2$, where $\mathcal{B}_{\beta,0} = \{f \in \mathcal{B}_\beta : f(0) = 0\}$ is the closed subspace of $\mathcal{B}_\beta$.

We begin to estimate the essential norm of the generalized Voterra composition operators. Let us recall the definition of the essential norm of the bounded linear operators. Suppose that X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, then the essential norm of the operator $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\| : K \in \mathcal{K}\},$$

where $\mathcal{K}$ denotes the set of all compact linear operators from X to Y. By the definition, it is clear that the bounded linear operator $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

Theorem 3.3 Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{Z}_\beta$ is bounded, then

$$\begin{aligned}\|J_{\varphi,g}^{(n)}\|_{e,A_u^p\to\mathcal{Z}_\beta} \asymp &\limsup_{j\to\infty} \frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+\frac{1}{p}}} \left|g'(\varphi(z_j))\varphi''(z_j) + g''(\varphi(z_j))\varphi'(z_j)^2\right| \\ &+ \limsup_{j\to\infty} \frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+1+\frac{1}{p}}} \left|g'(\varphi(z_j))\right|\left|\varphi'(z_j)\right|^2,\end{aligned}$$

where the sequence $\{z_j\}$ satisfies $|\varphi(z_j)| \to 1^-$ as $j \to \infty$.

Proof Suppose that $\{z_j\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1^-$ as $j \to \infty$. For each $\varphi(z_j)$, taking f_j the function in the proof of Theorem 3.1, we have seen that $\max_{j\in\mathbb{N}} \|f_j\| \le C$, and it is obvious that $f_j \to 0$ uniformly on compacta of $\mathbb{D}$ as $j \to \infty$. Hence for every compact operator $K : A_u^p \to \mathcal{Z}_\beta$, we have $\|Kf_j\| \to 0$ as $j \to \infty$. Thus it follows that

$$\begin{aligned}\|J_{\varphi,g}^{(n)} - K\| &= \sup_{\|f\|=1} \|(J_{\varphi,g}^{(n)} - K)f\|_{\mathcal{Z}_\beta} \\ &\ge \limsup_{j\to\infty} \frac{\|(J_{\varphi,g}^{(n)} - K)f_j\|_{\mathcal{Z}_\beta}}{\|f_j\|} \\ &\ge \limsup_{j\to\infty} \frac{\|J_{\varphi,g}^{(n)} f_j\|_{\mathcal{Z}_\beta} - \|Kf_j\|_{\mathcal{Z}_\beta}}{\|f_j\|} \\ &\ge C^{-1} \limsup_{j\to\infty} \frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+1+\frac{1}{p}}} |g'(\varphi(z_j))||\varphi'(z_j)|^2. \qquad (3.7)\end{aligned}$$

By taking the infimum in (3.7) over the set of all compact operators $K : A_u^p \to \mathcal{Z}_\beta$, we obtain

$$\|J_{\varphi,g}^{(n)}\|_{e,A_u^p\to\mathcal{Z}_\beta} \ge C^{-1} \limsup_{j\to\infty} \frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+1+\frac{1}{p}}} |g'(\varphi(z_j))||\varphi'(z_j)|^2. \qquad (3.8)$$

Using the similar method, we also have

$$\|J^{(n)}_{\varphi,g}\|_{e,A^p_u\to\mathcal{Z}_\beta}\geq C^{-1}\limsup_{j\to\infty}\frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+\frac{1}{p}}}\big|g'(\varphi(z_j))\varphi''(z_j)+g''(\varphi(z_j))\varphi'(z_j)^2\big|.$$

Combing these inequalities, we have got

$$\begin{aligned}\|J^{(n)}_{\varphi,g}\|_{e,A^p_u\to\mathcal{Z}_\beta}\geq&C\Big(\limsup_{j\to\infty}\frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+\frac{1}{p}}}\big|g'(\varphi(z_j))\varphi''(z_j)+g''(\varphi(z_j))\varphi'(z_j)^2\big|\\&+\limsup_{j\to\infty}\frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+1+\frac{1}{p}}}\big|g'(\varphi(z_j))\big|\big|\varphi'(z_j)\big|^2\Big).\end{aligned}$$

Now suppose that $\{r_j\}$ is a positive sequence which increasingly converges to 1. For every r_j, we define the operator by

$$J^{(n)}_{r_j\varphi,g}f(z)=\int_0^z(f^{(n)}\circ r_j\varphi)(\xi)(g\circ\varphi)'(\xi)d\xi,\ z\in\mathbb{D}.$$

Since $J^{(n)}_{\varphi,g}:A^p_u\to\mathcal{Z}_\beta$ is bounded, by Theorem 3.1 one can check that the operator $J^{(n)}_{r_j\varphi,g}:A^p_u\to\mathcal{Z}_\beta$ is bounded. Since $|r_j\varphi(z)|\leq r_j<1$, by Lemma 2.1, the operator $J^{(n)}_{r_j\varphi,g}:A^p_u\to\mathcal{Z}_\beta$ is also compact. Since $\big|g\circ\varphi(0)\big|\big|f^{(n)}\circ\varphi(0)-f^{(n)}\circ r_j\varphi(0)\big|\to0$ as $j\to\infty$, we omit this part in the calculating $\|J^{(n)}_{\varphi,g}-J^{(n)}_{r_j\varphi,g}\|$. Hence we have

$$\begin{aligned}&\|J^{(n)}_{\varphi,g}-J^{(n)}_{r_j\varphi,g}\|=\sup_{\|f\|=1}\|(J^{(n)}_{\varphi,g}-J^{(n)}_{r_j\varphi,g})f\|_{\mathcal{Z}_\beta}\\=&\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\big((f^{(n)}\circ r_j\varphi)(z)(g\circ\varphi)'(z)\big)'-\big((f^{(n)}\circ\varphi)(z)(g\circ\varphi)'(z)\big)'\big|\\=&\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|f^{(n+1)}(r_j\varphi(z))r_j\varphi'(z)^2g'(\varphi(z))+f^{(n)}(r_j\varphi(z))(g\circ\varphi)''(z)\\&-f^{(n+1)}(\varphi(z))\varphi'(z)^2g'(\varphi(z))-f^{(n)}(\varphi(z))(g\circ\varphi)''(z)\Big|\\\leq&\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|^2\big|f^{(n+1)}(\varphi(z))-r_jf^{(n+1)}(r_j\varphi(z))\big|\\&+\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(g\circ\varphi)''(z)|\big|f^{(n)}(\varphi(z))-f^{(n)}(r_j\varphi(z))\big|\\\leq&\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|^2\big|f^{(n+1)}(\varphi(z))-f^{(n+1)}(r_j\varphi(z))\big|\\&+(1-r_j)\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|^2\big|f^{(n+1)}(r_j\varphi(z))\big|\\&+\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(g\circ\varphi)''(z)|\big|f^{(n)}(\varphi(z))-f^{(n)}(r_j\varphi(z))\big|\\\leq&\sup_{\|f\|=1}\sup_{|\varphi(z)|\leq\delta}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|^2\big|f^{(n+1)}(\varphi(z))-f^{(n+1)}(r_j\varphi(z))\big|\\&+\sup_{\|f\|=1}\sup_{|\varphi(z)|>\delta}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|^2\big|f^{(n+1)}(\varphi(z))-f^{(n+1)}(r_j\varphi(z))\big|\\&+C(1-r_j)\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{n+1+\frac{1}{p}}}|g'(\varphi(z))||\varphi'(z)|^2\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta|(g\circ\varphi)''(z)|\big|f^{(n)}(\varphi(z))-f^{(n)}(r_j\varphi(z))\big|\\
&+\sup_{\|f\|=1}\sup_{|\varphi(z)|>\delta}(1-|z|^2)^\beta|(g\circ\varphi)''(z)|\big|f^{(n)}(\varphi(z))-f^{(n)}(r_j\varphi(z))\big|\\
\le\ &\sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|^2\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}\big|f^{(n+1)}(\varphi(z))-f^{(n+1)}(r_j\varphi(z))\big|\\
&+C\sup_{|\varphi(z)|>\delta}\frac{(1-|z|^2)^\beta}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{n+1+\frac{1}{p}}}|g'(\varphi(z))||\varphi'(z)|^2\\
&+C\sup_{|\varphi(z)|>\delta}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}}|g'(\varphi(z))||\varphi'(z)|^2\\
&+C(1-r_j)\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{n+1+\frac{1}{p}}}|g'(\varphi(z))||\varphi'(z)|^2\\
&+\|g\circ\varphi\|_{\mathcal{Z}_\beta}\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}\big|f^{(n)}(\varphi(z))-f^{(n)}(r_j\varphi(z))\big|\\
&+C\sup_{|\varphi(z)|>\delta}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}}\big|g''(\varphi(z))\varphi'(z)^2+g'(\varphi(z))\varphi''(z)\big|\\
&+C\sup_{|\varphi(z)|>\delta}\frac{(1-|z|^2)^\beta}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{n+\frac{1}{p}}}\big|g''(\varphi(z))\varphi'(z)^2+g'(\varphi(z))\varphi''(z)\big|. \qquad (3.9)
\end{aligned}
$$

We consider $I_j^{(n)}:=\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}\big|f^{(n)}(\varphi(z))-f^{(n)}(r_j\varphi(z))\big|$. By using the mean value theorem, the subharmonicity of $f^{(n)}$ and Lemma 2.2, we have

$$
I_j^{(n)}\le\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}(1-r_j)|\varphi(z)|\sup_{|z|\le\delta}|f^{(n+1)}(z)|\le C\frac{1-r_j}{\min\limits_{0\le r\le\delta}u(r)(1-\delta^2)^{n+1+\frac{1}{p}}}. \qquad (3.10)
$$

By (3.10), we obtain that $I_j^{(n)}\to 0$ as $j\to\infty$. Using the same method, we also have that $I_j^{(n+1)}\to 0$ as $j\to\infty$. Let $j\to\infty$ in (3.9), by the above discussions and the boundedness of $J_{\varphi,g}^{(n)}:A_u^p\to\mathcal{Z}_\beta$, we obtain

$$
\begin{aligned}
\|J_{\varphi,g}^{(n)}-J_{r_j\varphi,g}^{(n)}\| \le\ & 2C\sup_{|\varphi(z)|>\delta}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}}\big|g''(\varphi(z))\varphi'(z)^2+g'(\varphi(z))\varphi''(z)\big|\\
&+2C\sup_{|\varphi(z)|>\delta}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}}|g'(\varphi(z))||\varphi'(z)|^2
\end{aligned}
$$

as $j\to\infty$. Since $\|J_{\varphi,g}^{(n)}\|_{e,A_u^p\to\mathcal{Z}_\beta}\le\|J_{\varphi,g}^{(n)}-J_{r_j\varphi,g}^{(n)}\|$, we end the proof.

By Theorem 3.3, we obtain the characterization of the compact operator $J_{\varphi,g}^{(n)}:A_u^p\to\mathcal{Z}_\beta$.

Corollary 3.4 Suppose that φ is an analytic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$ and $J_{\varphi,g}^{(n)}:A_u^p\to\mathcal{Z}_\beta$ is bounded, then $J_{\varphi,g}^{(n)}:A_u^p\to\mathcal{Z}_\beta$ is compact if and only if the following conditions are satisfied

(i) $\lim\limits_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}}\big|g'(\varphi(z))\varphi''(z)+g''(\varphi(z))\varphi'(z)^2\big|=0,$

(ii) $\lim\limits_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+\frac{1}{p}}}\big|g'(\varphi(z))\big|\big|\varphi'(z)\big|^2=0.$

Similar to the proof of Theorem 3.3, we can also prove the next result.

Theorem 3.5 Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{B}_\beta$ is bounded, then

$$\|J_{\varphi,g}^{(n)}\|_{e,A_u^p \to \mathcal{B}_\beta} \asymp \limsup_{j\to\infty} \frac{(1-|z_j|^2)^\beta}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{n+\frac{1}{p}}} |g'(\varphi(z_j))||\varphi'(z_j)|,$$

where the sequence $\{z_j\}$ satisfies $|\varphi(z_j)| \to 1^-$ as $j \to \infty$.

So we have

Corollary 3.6 Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{B}_\beta$ is bounded, then $J_{\varphi,g}^{(n)} : A_u^p \to \mathcal{B}_\beta$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)^\beta}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+\frac{1}{p}}} |g'(\varphi(z))||\varphi'(z)| = 0.$$

References

[1] Aleman A, Cima J A. An integral operator on H^p and Hardy's inequality[J]. J. Anal. Math., 2001, 85: 157–176.

[2] Jiang Z J. On Volterra compositon operators from Bergman-type space to Bloch-type space[J]. Czech. Math. J., 2011, 61(136): 993–1005.

[3] Li X S, Stević S. Weighted composition operators from Bergman-type spaces into Bloch spaces[J]. Proc. Indian Acad. Sci., 2007, 117: 371–385.

[4] Jiang Z J. On a class of opertors from weighted Bergman spaces to some spaces of analytic functions[J]. Taiwan. J. Math. Soc., 2011, 15(5): 2095–2121.

[5] Li X S, Stević S. Weighted composition operators from Zygmund spaces into Bloch spaces[J]. Appl. Math. Comput., 2008, 206(2): 825–831.

[6] Ohno S. Weighted composition operators between H^∞ and the Bloch space[J]. Taiwan. J. Math. Soc., 2001, 5(3): 555–563.

[7] Ohno S. Products of composition and differentiation on Bloch spaces[J]. Bull. Korean Math. Soc., 2009, 46(6): 1135–1140.

[8] Yang W. Weighted composition operators from Bloch-type spaces to weighted-type spaces[J]. Ars. Combin., 2009, 93: 265–274.

[9] Aleman A, Siskakis A G. An integral operator on H^p[J]. Compl. Var. The. Appl., 1995, 28: 149–158.

[10] Aleman A, Siskakis A G. Integration operators on Bergman spaces[J]. Indiana Univ. Math. J., 1997, 46: 337–356.

[11] Siskakis A G, Zhao R H. An Volterra type operator on spaces of analytic functions[J]. Funct. Spaces (Edwardville IL), 1998: 299–311.

[12] Xiao J. Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of unit ball[J]. J. London. Math. Soc., 2004, 70(2): 199–214.

[13] Cowen C C, MacCluer B D. Composition operators on spaces of analytic functions[M]. New York: CRC Press, 1995.

[14] Pommerenke Ch. Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation[J]. Comment. Math. Helv., 1977, 52: 591–602.

[15] Shapiro J H. Composition operators and classical function theory[M]. New York: Springer-Verlag, 1993.

[16] Li X S. Volterra composition operators between weighted Bergman spaces and Bloch type spaces[J]. J. Korean Math. Soc., 2008, 45: 229–248.

[17] Wolf E. Volterra composition operators between weighted Bergman spaces and weighted Bloch type spaces[J]. Collect. Math., 2010, 61: 57–63.

[18] Stević S. Composition operators from weighted Bergman spaces to the n-th weighted spaces on the unit disc[J]. Discr. Dyn. Nat. Soc., Vol. 2009, Artical ID 742019, 11 page.

推广的Volterra复合算子的本性范数

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摘要: 本文研究单位圆盘上Bergman型空间到Zygmund型空间上的一类推广的Volterra复合算子. 利用符号函数 φ 和 g 刻画这类算子的有界性、紧性, 并计算其本性范数.

关键词: Bergman型空间; 加权Zygmund空间; 推广的Volterra复合算子; 本性范数

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