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EQUIVALENCE BETWEEN TIME AND NORM OPTIMAL CONTROL PROBLEMS OF THE HEAT EQUATION WITH POINTWISE CONTROL CONSTRAINTS

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Abstract: In this paper, we study the problem of the equivalence of the heat equation with pointwise control constraints. By making use of the uniqueness of time optimal control, controllability properties and the characterization of norm optimal controls through variational methods, we establish the equivalence between time and norm optimal control problems of the heat equation with pointwise control constraints, and extend the results in the related literature.

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1 Introduction

There are the following two distinct versions of time optimal control problems [1]:

(i) to reach the target set at a fixed time while delaying initiation of active control as late as possible;

(ii) immediate activation of the control to reach the target set in the shortest time.

In this paper, we shall establish the equivalence between the above two versions of time optimal control problems for an internally controlled heat equation with pointwise control constraints, and their corresponding norm optimal control problems. Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 1$, with a sufficiently smooth boundary $\partial\Omega$. Let ω be an open subset of Ω and $C_0(\Omega) = \{y \in C(\overline{\Omega}) : y = 0 \text{ on } \partial\Omega\ \}$. We formulate time optimal control problems and corresponding norm optimal control problems considered in this paper as follows.

For the first version of time optimal control problems studied in this paper, let $T > 0$ be fixed. Consider the controlled heat equation

$$\begin{cases} \partial_t y - \Delta y = \chi_{(\tau,T)\times\omega} u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $\chi_{(\tau,T)\times\omega}$ is characteristic function of the set $(\tau,T)\times\omega$, $0\le\tau<T$, $y_1\in C_0(\Omega)$ is a given function, and $u(t,x)$ is a control function taken from the set of functions as follows

$$\mathcal{U}_M\equiv\{u:(0,T)\times\Omega\to\mathbb{R}\text{ mesaurable; }|u(t,x)|\le M\text{ for almost all }(t,x)\in(0,T)\times\Omega\},$$

here M is a positive constant. It is well-known that for each $u\in L^\infty((0,T)\times\Omega)$, equation(1.1) has a unique solution, denoted by $y(t,x;y_1,\chi_{(\tau,T)\times\omega}u)$ in $C([0,T];C_0(\Omega))$. In what follows, we write Q_T, $\sum_T$, $Q^\omega_{\tau,T}$, Q^ω_T for the product sets $(0,T)\times\Omega$, $(0,T)\times\partial\Omega$, $(\tau,T)\times\omega$ and $(0,T)\times\omega$, respectively. We shall omit variables t and x for functions of (t,x) and omit the variable x for functions of x, if there is no risk of causing confusion. Now, we are prepared to state the first version of time optimal control problems under consideration

$$(P_M)\qquad \sup\{\tau:\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1,\tau\in[0,T),u\in\mathcal{U}_M\}.$$

Without loss of generality, we assume that $\|y(T,\cdot;y_1,0)\|_{C_0(\Omega)}>1$. We call

$$\tau^*(M)\equiv\sup\{\tau:\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1,\tau\in[0,T),u\in\mathcal{U}_M\}$$

the optimal time for problem (P_M) and $u^*_M\in\mathcal{U}_M$ the associated time-optimal control (or optimal control for simplicity) with corresponding state $y(t,x;y_1,\chi_{Q^\omega_{\tau^*(M),T}}u^*_M)$, solution of (1.1), satisfying $\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*(M),T}}u^*_M)\|_{C_0(\Omega)}\le1$. We call a control $u\in\mathcal{U}_M$ an admissible control for problem (P_M), if there exists some $\tau\in[0,T)$ such that $\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1$. Since the value of the control in $Q_T\backslash Q^\omega_{\tau,T}$ has no effect on the control system (1.1), we consistently assign the control to have the value 0 in $Q_T\backslash Q^\omega_{\tau,T}$.

Let $\tau\in[0,T)$ be fixed. The norm optimal control problem corresponding to (P_M) reads as follows

$$(P^\tau_{nm})\qquad \min\{\|u\|_{L^\infty(Q_T)}:u\in L^\infty(Q_T)\text{ satisfying }\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau,T}}u)\|_{C_0(\Omega)}\le1\}.$$

We denote $N^*_\infty(\tau)=\min(P^\tau_{nm})$.

For the second version of time optimal control problems studied in this paper, we consider the following controlled heat equation

$$\begin{cases}\partial_t y-\Delta y=\chi_\omega(x)v & \text{in }(0,+\infty)\times\Omega,\\ y=0 & \text{on }(0,+\infty)\times\partial\Omega,\\ y(0,x)=y_2(x) & \text{in }\Omega,\end{cases}\tag{1.2}$$

where $y_2\in C_0(\Omega)$ is a given function, and $v(t,x)$ is a control function taken from the set of functions as follows:

$$\mathcal{V}_M\equiv\{v:(0,+\infty)\times\Omega\to\mathbb{R}\text{ measurable; }|v(t,x)|\le M\text{ for almost all }(t,x)\in(0,+\infty)\times\Omega\},$$

here M is a positive constant. For each $v\in L^\infty((0,+\infty)\times\Omega)$, we denote the unique solution of (1.2) by $y(t,x;y_2,v)$. Now, we state the second version of time optimal control problems under consideration

$$(\widetilde{P}_M)\qquad \inf\{T:||y(T,\cdot;y_2,v)||_{C_0(\Omega)}\le1,T\in(0,+\infty),v\in\mathcal{V}_M\}.$$

Without loss of generality, we assume that $\|y_2(\cdot)\|_{C_0(\Omega)} > 1$. We call

$$T^*(M) \equiv \inf\{T : \|y(T,\cdot;y_2,v)\|_{C_0(\Omega)} \leq 1, T \in (0,+\infty), v \in \mathcal{V}_M\}$$

the optimal (minimal) time for problem $(\widetilde{P}_M)$ and $\widetilde{v}_M^* \in \mathcal{V}_M$ the associated time-optimal control (or optimal control for simplicity) with corresponding state $y(t,x;y_2,\widetilde{v}_M^*)$, solution of (1.2), satisfying $\|y(T^*(M),\cdot;y_2,\widetilde{v}_M^*)\|_{C_0(\Omega)} \leq 1$. We call a control $v \in \mathcal{V}_M$ an admissible control for problem $(\widetilde{P}_M)$, if there exists some $T > 0$ such that $\|y(T,\cdot;y_2,v)\|_{C_0(\Omega)} \leq 1$. The value of the control in $((0,+\infty)\times\Omega)\setminus Q_T^\omega$ has no effect on the control system (1.2), and therefore we consistently assign the control to have the value 0 in $((0,+\infty)\times\Omega)\setminus Q_T^\omega$.

The norm optimal control problem corresponding to $(\widetilde{P}_M)$ reads as follows

$$(\widetilde{P}_{nmT}) \qquad \min\{\|v\|_{L^\infty(Q_T)} : v \in L^\infty(Q_T) \text{ satisfying } \|y(T,\cdot;y_2,v)\|_{C_0(\Omega)} \leq 1\}.$$

We denote $\widetilde{N}_\infty^*(T) = \min(\widetilde{P}_{nmT})$.

To the best of our knowledge, there are few works about equivalence between time and norm optimal control problems for parabolic equations, see [2, 3]. In [2], the equivalence of time optimal control and the norm optimal control was established for abstract equations in Banach spaces. The main differences between [2] and our paper are as follows

(i) The time optimal control problem in [2] is of the second version, while we consider two versions of time optimal control problems.

(ii) The methods for the study of the equivalence between time and norm optimal control problems are different. In [2], necessary and sufficient conditions for both time and norm optimal controls were obtained, using the argument of separation of target sets from attainable sets. Since those sufficient and necessary conditions have the same form, the equivalence between time and norm optimal controls follows. In our paper, we derive the equivalence directly by making use of the uniqueness of time optimal control, the well known controllability properties and the characterization of norm optimal controls through variational methods as in [4].

(iii) The paper [2] developed an abstract theory whose applications are limited to the case where the control is distributed in the whole domain which corresponds to the case of $\omega = \Omega$ in our study. In our paper, ω is an arbitrarily open subset of Ω. The idea of our paper utilizes the approach from [3]. However, there are some main differences between [3] and our paper

(i) The time optimal control problem in [3] was of the second version, while we consider two versions of time optimal control problems.

(ii) The procedure for the study of the equivalence between time and norm optimal control problem is different. We start by researching the optimal norm as a function of time, i.e., the functions $N_\infty^*(\cdot)$ and $\widetilde{N}_\infty^*(\cdot)$. While in [3], they began with study of the optimal time as a function of control bound.

(iii) In our paper, the control constraint is in pointwise form and the target set is a closed ball in $C_0(\Omega)$, while in [3], the control constraint is in integral form and the target set was a closed ball in $L^2(\Omega)$ or 0. Recently, in [5], the equivalence of optimal target control problems, optimal time control problems and optimal norm control problems were discussed for the heat equation with internal controls.

The main results of this paper are as follows.

Theorem 1.1 Let $\tau \in [0,T)$. Then $\tau = \tau^*(N_\infty^*(\tau))$. Furthermore, (P_{nm}^τ) has a unique solution and this solution is also the optimal control to $(P_{N_\infty^*(\tau)})$. Conversely, for each $M \in [N_\infty^*(0), +\infty)$, the optimal control to (P_M) is also the solution to $(P_{nm}^{\tau^*(M)})$.

Theorem 1.2 $T = T^*(\widetilde{N}_\infty^*(T))$, $\forall T \in (0, T_0]$, where $T_0 = \inf\{T : \|y(T,\cdot;y_2,0)\|_{C_0(\Omega)} \leq 1, T > 0\}$. Furthermore, $(\widetilde{P}_{nmT})$ has a unique solution and this solution, when it is extended over $\mathbb{R}^+$ by taking zero value over $(T,+\infty)$, is also the optimal control to $(\widetilde{P}_{\widetilde{N}_\infty^*(T)})$. Conversely, for each $M \in [0,+\infty)$, the optimal control to $(\widetilde{P}_M)$, when it is restricted over $(0, T^*(M))$, is also the solution to $(\widetilde{P}_{nmT^*(M)})$.

It should be pointed out that in [6], by establishing the connections between (P_M) and $(P_{nm}^{\tau^*(M)})$, $(\widetilde{P}_M)$ and $(\widetilde{P}_{nmT^*(M)})$, as well as strict monotonicity of $N_\infty^*(\cdot)$ and $\widetilde{N}_\infty^*(\cdot)$, necessary and sufficient conditions for optimal time and optimal control of (P_M) and $(\widetilde{P}_M)$ were obtained in [6]. However, the equivalence between time and norm optimal control problems is not proved.

The rest of this paper is organized as follows. In Section 2 and Section 3, we shall give the proofs of Theorem 1.1 and Theorem 1.2, respectively.

2 Equivalence between (P_M) and (P_{nm}^τ)

In this section, we shall prove Theorem 1.1. To this end, we first cite the following lemmas (see [6]).

Lemma 2.1 (i) Assume that $\tau^*(M)$ is the optimal time for (P_M). Then (P_M) has a unique solution, denoted by u_M^*. Moreover, $|u_M^*(t,x)| = M$ for almost all $(t,x) \in Q_{\tau^*(M),T}^\omega$.

(ii) Problem (P_{nm}^τ) has at least one solution. The function $N_\infty^*(\cdot) : [0,T) \to (0,+\infty)$ is strictly increasing, continuous and $\lim\limits_{\tau \to T^-} (N_\infty^*(\tau)) = +\infty$.

(iii) Assume that $\tau^*(M)$ is the optimal time for (P_M). Then $(P_{nm}^{\tau^*(M)})$ has a unique solution and this solution is the optimal control for (P_M).

Now, we give some properties about the function $\tau^*(\cdot)$.

Lemma 2.2 $\tau^*(\cdot) : [N_\infty^*(0), +\infty) \to [0,T)$ is strictly increasing, continuous, $\tau^*(N_\infty^*(0)) = 0$ and $\lim\limits_{M \to +\infty} \tau^*(M) = T$.

Proof The proof is split into five steps.

Step 1 $\tau^*(N_\infty^*(0)) = 0$.

It suffices to show that if there exists a control u with $\|u\|_{L^\infty(Q_T)} \leq N_\infty^*(0)$ such that $\|y(T,\cdot;y_1,\chi_{Q_{\tau,T}^\omega} u)\|_{C(\overline{\Omega})} \leq 1$ for a certain $\tau \in [0,T)$, then $\tau = 0$. By contradiction, $\tau > 0$. Then on one hand, by (ii) in Lemma 2.1, we have that

$$N_\infty^*(0) < N_\infty^*(\tau). \tag{2.1}$$

On the other hand, by the definition of $N^*_\infty(\tau)$, we get that $N^*_\infty(\tau) \le \|u\|_{L^\infty(Q_T)}$. This together with (2.1) implies that $N^*_\infty(0) < \|u\|_{L^\infty(Q_T)} \le N^*_\infty(0)$, which leads to a contradiction and completes the proof.

Step 2 $\tau^*(\cdot) : [N^*_\infty(0), +\infty) \to [0, T)$ is strictly increasing.

Let $M_1 > M_2 \ge N^*_\infty(0)$. Let $u^*_{M_2}$ be the optimal control to (P_{M_2}). Then

$$\|u^*_{M_2}\|_{L^\infty(Q_T)} \le M_2 < M_1 \text{ and } \|y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*(M_2),T}} u^*_{M_2})\|_{C_0(\Omega)} \le 1. \tag{2.2}$$

Hence $u^*_{M_2} \in \mathcal{U}_{M_1}$. By the optimality of $\tau^*(M_1)$ to problem (P_{M_1}), we see that $\tau^*(M_1) \ge \tau^*(M_2)$. Next we show that the above inequality is strict. By contradiction, we would have that $\tau^*(M_1) = \tau^*(M_2)$. Then

$$\|y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*(M_1),T}} u^*_{M_2})\|_{C_0(\Omega)} = \|y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*(M_2),T}} u^*_{M_2})\|_{C_0(\Omega)} \le 1,$$

which, combined with the fact that $u^*_{M_2} \in \mathcal{U}_{M_1}$, indicates that $u^*_{M_2}$ is the optimal control to (P_{M_1}). Hence, by (i) in Lemma 2.1, we get that $\|u^*_{M_2}\|_{L^\infty(Q_T)} = M_1$. This contradicts with the first inequality in (2.2).

Step 3 $\tau^*(\cdot) : [N^*_\infty(0), +\infty) \to [0, T)$ is right continuous.

Let $M^* \in [N^*_\infty(0), +\infty)$ be fixed. By Step 2, we infer that $\lim\limits_{M \downarrow M^*} \tau^*(M)$ exists. We claim that this limit is equal to $\tau^*(M^*)$. If not, there would exist a sequence $M_n \downarrow M^*$ such that

$$\lim_{n\to\infty} \tau^*(M_n) = \tau^*(M^*) + \delta \text{ for some } \delta \in (0, T - \tau^*(M^*)). \tag{2.3}$$

Write $u^*_{M_n}$ for the optimal control to (P_{M_n}). Then by (i) in Lemma 2.1, we have

$$\|u^*_{M_n}\|_{L^\infty(Q_T)} = M_n \to M^*.$$

Hence there exist a subsequence of $\{n\}_{n=1}^\infty$, still denoted by itself, and $\tilde{u} \in L^\infty(Q_T)$, such that

$$u^*_{M_n} \to \tilde{u} \text{ weakly star in } L^\infty(Q_T) \tag{2.4}$$

and

$$\|\tilde{u}\|_{L^\infty(Q_T)} \le M^*. \tag{2.5}$$

It follows from (2.3) and (2.4) that

$$\chi_{Q^\omega_{\tau^*(Mn),T}} u^*_{M_n} \to \chi_{Q^\omega_{\tau^*(M^*)+\delta,T}} \tilde{u} \text{ weakly star in } L^\infty(Q_T). \tag{2.6}$$

Then by (2.6), L^p-estimate for parabolic equation and embedding theorem (see Theorem 1.14 of Chapter 1 in [7] and Theorem 1.4.1 in [8]), we infer that there exists a subsequence of $\{n\}_{n=1}^\infty$, still denoted by itself, such that

$$\|y(\cdot, \cdot; y_1, \chi_{Q^\omega_{\tau^*(Mn),T}} u^*_{M_n}) - y(\cdot, \cdot; y_1, \chi_{Q^\omega_{\tau^*(M^*)+\delta,T}} \tilde{u})\|_{C(\bar{Q}_T)} \to 0,$$

which, together with the fact that $\|y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*(Mn),T}} u^*_{M_n})\|_{C_0(\Omega)} \le 1$, indicates

$$\|y(T, \cdot; y_1, \chi_{Q^\omega_{\tau^*(M^*)+\delta,T}} \tilde{u})\|_{C_0(\Omega)} \le 1.$$

Hence by (2.5) and the above inequality, we get $\tau^*(M^*) + \delta \le \tau^*(M^*)$. This leads to a contradiction.

Step 4 $\tau^*(\cdot) : [N^*_\infty(0), +\infty) \to [0, T)$ is left continuous.

Let $M^* \in (N^*_\infty(0), +\infty)$ be fixed and $M_n \uparrow M^*$. It suffices to show that

$$\lim_{n\to\infty} \tau^*(M_n) = \tau^*(M^*). \tag{2.7}$$

Consider the following equation

$$\begin{cases} \partial_t z - \Delta z = \chi_{Q^\omega_{\tau^*(M^*)-\delta n, \tau^*(M^*)}} u & \text{in } (0, \tau^*(M^*)) \times \Omega, \\ z(t,x) = 0 & \text{on } (0, \tau^*(M^*)) \times \partial\Omega, \\ z(0,x) = (1 - M_n M^{*-1}) y_1(x) & \text{in } \Omega. \end{cases} \tag{2.8}$$

Here $\delta_n \in (0, \tau^*(M^*))$ will be determined later. It is obvious that (2.8) can be rewritten as

$$\begin{cases} \partial_t z_1 - \Delta z_1 = 0 & \text{in } (0, \tau^*(M^*) - \delta_n) \times \Omega, \\ z_1(t,x) = 0 & \text{on } (0, \tau^*(M^*) - \delta_n) \times \partial\Omega, \\ z_1(0,x) = (1 - M_n M^{*-1}) y_1(x) & \text{in } \Omega \end{cases} \tag{2.9}$$

and

$$\begin{cases} \partial_t z_2 - \Delta z_2 = \chi_\omega u & \text{in } (\tau^*(M^*) - \delta_n, \tau^*(M^*)) \times \Omega, \\ z_2(t,x) = 0 & \text{on } (\tau^*(M^*) - \delta_n, \tau^*(M^*)) \times \partial\Omega, \\ z_2(\tau^*(M^*) - \delta_n, x) = z_1(\tau^*(M^*) - \delta_n, x) & \text{in } \Omega, \end{cases} \tag{2.10}$$

where

$$z(t,\cdot) = \begin{cases} z_1(t,\cdot), & t \in [0, \tau^*(M^*) - \delta_n), \\ z_2(t,\cdot), & t \in [\tau^*(M^*) - \delta_n, \tau^*(M^*)]. \end{cases} \tag{2.11}$$

It follows from Theorem 3.1 in [9] and (2.8)–(2.11) that there exists a control, denoted by u_n, such that the solution of (2.8) corresponding to u_n, denoted by z_n, satisfies

$$z_n(\tau^*(M^*), \cdot) = 0. \tag{2.12}$$

Moreover,

$$\|u_n\|_{L^\infty((0,\tau^*(M^*)-\delta_n)\times\Omega)} = 0 \tag{2.13}$$

and

$$\begin{aligned} \|u_n\|_{L^\infty((\tau^*(M^*)-\delta_n, \tau^*(M^*))\times\Omega)} &\le e^{c_1(1+\delta_n+\delta_n^{-1})} \|z_n(\tau^*(M^*) - \delta_n, \cdot)\|_{L^2(\Omega)} \\ &\le e^{c_1(1+\delta_n+\delta_n^{-1})} \|z_n(0,\cdot)\|_{L^2(\Omega)} \\ &\le e^{c_2(1+\delta_n^{-1})} (1 - M_n M^{*-1}), \end{aligned} \tag{2.14}$$

where c_1 and c_2 are positive constants independent of n.

Write $u^*_{M^*}$ for the optimal control to (P_{M^*}) and $y^*(t,x) = y(t,x;y_1,\chi_{Q^\omega_{\tau^*(M^*),T}}u^*_{M^*})$. Then by (2.8) and (2.12), we have that

$$\begin{cases} \partial_t(z_n + M_nM^{*-1}y^*) - \Delta(z_n + M_nM^{*-1}y^*) = \chi_{Q^\omega_{\tau^*(M^*)-\delta n,\tau^*(M^*)}}u_n & \text{in } Q_{\tau^*(M^*)}, \\ (z_n + M_nM^{*-1}y^*)(t,x) = 0 & \text{on } \Sigma_{\tau^*(M^*)}, \\ (z_n + M_nM^{*-1}y^*)(0,x) = y_1(x) & \text{in } \Omega \end{cases} \tag{2.15}$$

and

$$(z_n + M_nM^{*-1}y^*)(\tau^*(M^*),\cdot) = M_nM^{*-1}y^*(\tau^*(M^*),\cdot). \tag{2.16}$$

Denote

$$\widehat{u}_n = \begin{cases} u_n, & \text{in } (0,\tau^*(M^*))\times\Omega, \\ M_nM^{*-1}u^*_{M^*}, & \text{in } [\tau^*(M^*),T)\times\Omega. \end{cases} \tag{2.17}$$

It follows from (2.15)–(2.17) that

$$y(t,\cdot;y_1,\chi_{Q^\omega_{\tau^*(M^*)-\delta n,T}}\widehat{u}_n) = \begin{cases} (z_n + M_nM^{*-1}y^*)(t,\cdot), & t\in[0,\tau^*(M^*)), \\ M_nM^{*-1}y^*(t,\cdot), & t\in[\tau^*(M^*),T]. \end{cases} \tag{2.18}$$

Take

$$\delta_n = (\frac{1}{c_2}\ln\frac{M_n}{1-M_nM^{*-1}} - 1)^{-1}. \tag{2.19}$$

This, together with (2.13), (2.14), (2.17) and (2.18), indicates

$$\|\widehat{u}_n\|_{L^\infty(Q_T)} \le M_n \tag{2.20}$$

and

$$\|y(T,\cdot;y_1,\chi_{Q^\omega_{\tau^*(M^*)-\delta n,T}}\widehat{u}_n)\|_{C_0(\Omega)} = \|M_nM^{*-1}y^*(T,\cdot)\|_{C_0(\Omega)} \le 1. \tag{2.21}$$

By (2.20) and (2.21), we infer that $\tau^*(M^*) - \delta_n \le \tau^*(M_n)$, which, combined with the fact that $\tau^*(M_n) \le \tau^*(M^*)$ and (2.19), implies (2.7).

Step 5 $\lim\limits_{M\to+\infty}\tau^*(M) = T$.

We use a contradiction argument. If not, we could make use of Step 2 to get a sequence of $\{M_n\}_{n=1}^\infty$, with $M_n \uparrow +\infty$, such that

$$\lim_{n\to\infty}\tau^*(M_n) = T-\delta \text{ for some } \delta\in(0,T).$$

Hence

$$\tau^*(M_n) < T-\delta \text{ for all } n. \tag{2.22}$$

Consider the following equation:

$$\begin{cases} \partial_t y - \Delta y = \chi_{Q^\omega_{T-2^{-1}\delta,T}}u & \text{in } Q_T, \\ y(t,x) = 0 & \text{on } \Sigma_T, \\ y(0,x) = y_1(x) & \text{in } \Omega. \end{cases} \tag{2.23}$$

It follows from Theorem 3.1 in [9] and the same arguments to get (2.12)–(2.14) that there exists a control, denoted by u_δ, such that the solution of (2.23) corresponding to u_δ, denoted by y_δ, satisfies

$$y_\delta(T,\cdot)=0. \tag{2.24}$$

Moreover,

$$\|u_\delta\|_{L^\infty((0,T-2^{-1}\delta)\times\Omega)}=0 \tag{2.25}$$

and

$$\begin{aligned}\|u_\delta\|_{L^\infty((T-2^{-1}\delta,T)\times\Omega)} &\le e^{c_3(1+2^{-1}\delta+2\delta^{-1})}\|y_\delta(T-2^{-1}\delta,\cdot)\|_{L^2(\Omega)}\\ &\le e^{c_3(1+2^{-1}\delta+2\delta^{-1})}\|y_1(\cdot)\|_{L^2(\Omega)},\end{aligned} \tag{2.26}$$

here c_3 is a positive constant. Since $M_n\uparrow+\infty$, by (2.25) and (2.26), we can fix such an n that $\|u_\delta\|_{L^\infty(Q_T)}\le M_n$. This combined with (2.23) and (2.24) implies $\tau^*(M_n)\ge T-2^{-1}\delta$, which contradicts with (2.22).

Then we give the proof of Theorem 1.1.

Proof We first show that

$$\tau=\tau^*(N_\infty^*(\tau)),\qquad \forall\tau\in[0,T). \tag{2.27}$$

Let $\tau\in[0,T)$ and u be a solution to (P_{nm}^τ). Then

$$\|u\|_{L^\infty(Q_T)}=N_\infty^*(\tau)\text{ and }\ \|y(T,\cdot;y_1,\chi_{Q_{\tau,T}^\omega}u)\|_{C_0(\Omega)}\le 1.$$

By the optimality of $\tau^*(N_\infty^*(\tau))$ to $(P_{N_\infty^*(\tau)})$, we have that $\tau\le\tau^*(N_\infty^*(\tau))$. If $\tau^*(N_\infty^*(\tau))>\tau$, then by Lemma 2.2 , we obtain that there exists $M_1\ge N_\infty^*(0)$ such that

$$M_1<N_\infty^*(\tau)\text{ and }\tau^*(M_1)=\tau.$$

Let $u_{M_1}^*$ be the optimal control to (P_{M_1}). By (i) in Lemma 2.1, we have that

$$\|u_{M_1}^*\|_{L^\infty(Q_T)}=M_1<N_\infty^*(\tau) \tag{2.28}$$

and

$$\|y(T,\cdot;y_1,\chi_{Q_{\tau,T}^\omega}u_{M_1}^*)\|_{C_0(\Omega)}=\|y(T,\cdot;y_1,\chi_{Q_{\tau^*(M_1),T}^\omega}u_{M_1}^*)\|_{C_0(\Omega)}\le 1.$$

The latter indicates $\|u_{M_1}^*\|_{L^\infty(Q_T)}\ge N_\infty^*(\tau)$, which contradicts with (2.28). Hence (2.27) holds.

Next, we notice that by (2.27), (P_{nm}^τ) is the same as $(P_{nm}^{\tau^*(N_\infty^*(\tau))})$. Then by (iii) in Lemma 2.1, we deduce that (P_{nm}^τ) has a unique solution and this solution is also the optimal control to $(P_{N_\infty^*(\tau)})$.

Finally, for each $M\in[N_\infty^*(0),+\infty)$, let u_M^* be the optimal control to (P_M). Then by (iii) in Lemma 2.1, we have that u_M^* is the solution to $(P_{nm}^{\tau^*(M)})$.

3 Equivalence between $(\widetilde{P}_M)$ and $(\widetilde{P}_{nmT})$

In this section, we shall give the proof of Theorem 1.2. For this purpose, we first notice that $T_0 < +\infty$ (see Lemma 5.6 in [6]) and cite the following lemmas (see [6]).

Lemma 3.1 (i) Let $T^*(M)$ be the optimal time for $(\widetilde{P}_M)$. Then $(\widetilde{P}_M)$ has a unique solution, denoted by $\widetilde{v}_M^*$. Moreover, $|\widetilde{v}_M^*(t,x)| = M$ for almost all $(t,x) \in Q^{\omega}_{T^*(M)}$.

(ii) For each $T > 0$, problem $(\widetilde{P}_{nmT})$ has at least one solution. The function $\widetilde{N}_\infty^*(\cdot)$: $(0, T_0] \to [0, +\infty)$ is strictly decreasing, continuous, $\widetilde{N}_\infty^*(T) = 0$ for $T \geq T_0$ and

$$\lim_{T\to 0^+} (\widetilde{N}_\infty^*(T)) = +\infty.$$

(iii) Let $T^*(M)$ be the optimal time for $(\widetilde{P}_M)$. Then problem $(\widetilde{P}_{nmT^*(M)})$ has a unique solution. This solution, after being extended to be 0 on $[T^*(M), +\infty) \times \Omega$, is the optimal control for $(\widetilde{P}_M)$.

Now, we give some properties about the function $T^*(\cdot)$.

Lemma 3.2 $T^*(\cdot) : [0, +\infty) \to (0, T_0]$ is strictly decreasing, continuous, $T^*(0) = T_0$ and $\lim\limits_{M\to+\infty} T^*(M) = 0$.

Proof We only show that $T^*(\cdot) : [0, +\infty) \to (0, T_0]$ is left continuous. Proofs of the remainder are similar as those of [3] or Lemma 2.2. Now, fix $M \in (0, +\infty)$ and $M_n \uparrow M$, we claim that

$$\lim_{n\to\infty} T^*(M_n) = T^*(M). \tag{3.1}$$

For this purpose, write $\widetilde{v}_M^*$ for the optimal control to$(\widetilde{P}_M)$, and let

$$y_n(t,x) \equiv y(t,x; y_2, M_n M^{-1} \widetilde{v}_M^*)$$

be the solution to the following equation

$$\begin{cases} \partial_t y_n - \Delta y_n = \chi_\omega M_n M^{-1} \widetilde{v}_M^* & \text{in } (0,+\infty)\times\Omega, \\ y_n(t,x) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ y_n(0,x) = y_2(x) & \text{in } \Omega. \end{cases} \tag{3.2}$$

By (1.2), (3.2), L^p-estimate for parabolic equation and embedding theorem (see Theorem 1.14 of Chapter in [7] and Theorem 1.4.1 in [8]), we obtain that

$$\|y_n(T^*(M),\cdot) - y(T^*(M),\cdot; y_2, \widetilde{v}_M^*)\|_{C_0(\Omega)} \leq \widetilde{c}_1 (M - M_n),$$

where $\widetilde{c}_1$ is a positive constant independent of n. From which, we infer that

$$\|y_n(T^*(M),\cdot)\|_{C_0(\Omega)} \leq 1 + \widetilde{c}_1(M - M_n). \tag{3.3}$$

Consider the following equation

$$\begin{cases} \partial_t z_n - \Delta z_n = 0 & \text{in } (0,T_n)\times\Omega, \\ z_n(t,x) = 0 & \text{on } (0,T_n)\times\partial\Omega, \\ z_n(0,x) = y_n(T^*(M),x) & \text{in } \Omega, \end{cases} \tag{3.4}$$

where $T_n \in (0,1)$ will be determined later. Define

$$\widetilde{z}_n(0,x) = \begin{cases} |y_n(T^*(M),x)| & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.5}$$

and let $\widetilde{z}_n$ satisfy the heat equation $\partial_t \widetilde{z}_n - \Delta \widetilde{z}_n = 0$ for $x \in \mathbb{R}^N,\ t > 0$. Then

$$\widetilde{z}_n(t,x) = \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-s|^2}{4t}} \widetilde{z}_n(0,s)ds, \qquad \forall (t,x) \in (0,+\infty) \times \mathbb{R}^N \tag{3.6}$$

and

$$|z_n(t,x)| \le \widetilde{z}_n(t,x), \qquad \forall (t,x) \in (0,T_n) \times \Omega. \tag{3.7}$$

It follows from (3.5), (3.6) and (3.7) that

$$\begin{aligned} |z_n(T_n,x)| &\le \int_{\mathbb{R}^N} (4\pi T_n)^{-\frac{N}{2}} e^{-\frac{|x-s|^2}{4T_n}} \widetilde{z}_n(0,s)ds \\ &= \int_{\mathbb{R}^N} (2\pi)^{-\frac{N}{2}} e^{-\frac{|s|^2}{2}} \widetilde{z}_n(0, x + \sqrt{2T_n}s)ds \\ &= \int_{\{s\in\mathbb{R}^N : x+\sqrt{2T_n}s\in\Omega\}} (2\pi)^{-\frac{N}{2}} e^{-\frac{|s|^2}{2}} |y_n(T^*(M), x+\sqrt{2T_n}s)|ds \\ &\le \|y_n(T^*(M),\cdot)\|_{C_0(\Omega)} \int_{\{s\in\mathbb{R}^N : x+\sqrt{2T_n}s\in\Omega\}} (2\pi)^{-\frac{N}{2}} e^{-\frac{|s|^2}{2}} ds, \qquad \forall x \in \Omega. \end{aligned} \tag{3.8}$$

It is easy to check that there exists a positive constant $\widetilde{c}_2$ independent on n such that

$$\{s \in \mathbb{R}^N : x + \sqrt{2T_n}s \in \Omega\} \subset \mathbf{B}(0, \widetilde{c}_2 T_n^{-\frac{1}{2}}), \qquad \forall x \in \Omega, \tag{3.9}$$

here $\mathbf{B}(0, \widetilde{c}_2 T_n^{-\frac{1}{2}})$ denotes a closed ball with center at origin and radius $\widetilde{c}_2 T_n^{-\frac{1}{2}}$. By (3.8), (3.9) and (3.3), we have that

$$\|z_n(T_n,\cdot)\|_{C_0(\Omega)} \le [1 + \widetilde{c}_1(M - M_n)] \int_{\mathbf{B}(0,\widetilde{c}_2 T_n^{-\frac{1}{2}})} (2\pi)^{-\frac{N}{2}} e^{-\frac{|s|^2}{2}} ds. \tag{3.10}$$

Since the function $f(T) = \int_{\mathbf{B}(0,\widetilde{c}_2 T^{-\frac{1}{2}})} (2\pi)^{-\frac{N}{2}} e^{-\frac{|s|^2}{2}} ds$, is a strictly decreasing, continuous function on $(0,+\infty)$, $\lim_{T\to+\infty} f(T) = 0$ and $\lim_{T\to 0^+} f(T) = 1$, we deduce that there exists $T_n \in (0,1)$, such that

$$\int_{\mathbf{B}(0,\widetilde{c}_2 T_n^{-\frac{1}{2}})} (2\pi)^{-\frac{N}{2}} e^{-\frac{|s|^2}{2}} ds = [1 + \widetilde{c}_1(M - M_n)]^{-1}.$$

From the latter equality and (3.10), we infer that

$$\|z_n(T_n,\cdot)\|_{C_0(\Omega)} \le 1 \ \text{ and } \ T_n \downarrow 0. \tag{3.11}$$

Moreover, it follows from (3.2) and (3.4) that

$$y(T^*(M)+t, x; y_2, M_n M^{-1} \widetilde{v}^*_M) = y_n(T^*(M)+t, x) = z_n(t,x), \qquad \forall (t,x) \in (0,T_n) \times \Omega,$$

which, together with (3.11), implies

$$\|y(T^*(M)+T_n,\cdot;y_2,M_nM^{-1}\widetilde{v}^*_M)\|_{C_0(\Omega)}\leq 1,$$

from the latter inequality and the fact that $\|M_nM^{-1}\widetilde{v}^*_M\|_{L^\infty((0,+\infty)\times\Omega)}\leq M_n$, we get $T^*(M_n)\leq T^*(M)+T_n$. Since $T^*(\cdot)$ is strictly decreasing, we have that

$$T^*(M)\leq T^*(M_n)\leq T^*(M)+T_n.$$

This combined with the limit in (3.11) indicates (3.1) and completes the proof.

Then by the similar arguments as those in [3] (or Theorem 1.1) and Lemma 3.2, we can get Theorem 1.2.

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具有点态控制约束热方程的时间与范数最优控制问题的等价性

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摘 要: 本文研究了具有点态控制热方程的等价性问题. 利用变分法分析时间最优控制的唯一性, 能控性以及范数最优控制的特征, 获得了具有点态控制约束热方程的时间与范数最优控制问题之间的等价性, 推广了现有文献的结果.

关键词: bang-bang 性; 时间最优控制; 范数最优控制

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