

INTERPOLATING SEQUENCES IN Q_K SPACES

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Abstract: In this paper, we investigate the interpolating question in Q_K spaces. Using some skills of complex analysis and harmonic analysis, we obtain a necessary and sufficient condition for a sequence $\{z_n\}$ in the unit disc to be an interpolating sequence for the analytic function space $Q_K \cap H^\infty$, which extends part of results in Q_p spaces.

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1 Introduction

Throughout this paper, the unit disc is denoted by \mathbb{D} . The Möbius transformation of \mathbb{D} is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad a \in \mathbb{D}.$$

Definition 1.1 Let $K : [0, \infty) \rightarrow [0, \infty)$ be a right-continuous and nondecreasing function. An analytic function f on the unit disc is said to belong to the space Q_K if

$$\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K \left(\log \frac{1}{|\varphi_a(z)|} \right) dA(z) < \infty, \quad (1.1)$$

where $dA(z)$ is the Euclidean area element on \mathbb{D} so that $A(\mathbb{D}) = 1$.

Equipped with the norm $|f(0)| + \|f\|_{Q_K}$, the space Q_K is Banach. We know that the space Q_K is Möbius invariant in the sense that $\|f \circ \varphi_a\|_{Q_K} = \|f\|_{Q_K}$ for any $a \in \mathbb{D}$. See [5] and [6] for a general theory of Q_K spaces. By [5] we know that Q_K spaces are contained in the Bloch space. Recall that an analytic function f on the unit disc is said to belong to the Bloch space, denoted by \mathcal{B} , if

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The Hardy space H^∞ consists of all analytic f on the unit disk \mathbb{D} for which $\sup_{z \in \mathbb{D}} |f(z)| < \infty$. A sequence $\{z_n\} \subset \mathbb{D}$ is called an interpolating sequence for $Q_K \cap H^\infty$ if each bounded

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sequence $\{w_n\}$ of complex numbers there exists $f \in Q_K \cap H^\infty$ such that $f(z_n) = w_n$ for all n .

In this paper the weight function K satisfies

$$\int_0^{\frac{1}{e}} K\left(\log \frac{1}{r}\right) dr < \infty.$$

Otherwise, the space Q_K only contains constant functions. By Theorem 2.1 in [5] we may assume that K is defined on $[0,1]$ and extend its domain to $[0, \infty)$ by setting $K(t) = K(1)$ for $t > 1$.

We need the following two conditions on K :

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty \tag{1.2}$$

and

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty, \tag{1.3}$$

where

$$\varphi_K(s) = \sup_{0 < t \leq 1} K(st)/K(t).$$

For a subarc $I \subset \partial\mathbb{D}$, let θ be the midpoint of I and denote the Carleson box

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| < |z| < 1, |\theta - \arg z| < \frac{|I|}{2} \right\}$$

for $|I| \leq 1$ and $S(I) = \mathbb{D}$ for $|I| > 1$.

Definition 1.2 A positive measure $d\mu$ is said to be a K -Carleson measure on \mathbb{D} provided

$$\|\mu\|_K = \sup_{I \subset \mathbb{D}} \int_{S(I)} K\left(\frac{1 - |z|}{|I|}\right) d\mu(z) < \infty. \tag{1.4}$$

The interpolation problems for different analytic function spaces were discussed in many paper (see [2, 3, 10, 13, 14]). A well-known result of Carleson (see [3]) asserts that $\{z_n\}$ is an interpolating sequence for H^∞ if and only if $d\mu = \sum \delta_{z_n}$ is a Carleson measure and $\{z_n\}$ is separated, that is

$$\inf_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_n z_m} \right| = \delta > 0. \tag{1.5}$$

In this paper, we study the interpolation problem in the analytic function space $Q_K \cap H^\infty$. The main result is the following:

Theorem 1.1 Let K satisfy conditions (1.2) and (1.3). Then $\{z_n\}$ is an interpolating sequence for $Q_K \cap H^\infty$ if and only if $\{z_n\}$ is separated and $d\mu = \sum \delta_{z_n}$ is a K -Carleson measure.

In addition, we say that $f \lesssim g$ (for two functions f and g) if there exists a constant C (independent of f and g) such that $f(t) \leq Cg(t), t > 0$. We say $f \approx g$ (that is, f is comparable with g) whenever $f \lesssim g \lesssim f$.

2 Some Auxiliary Lemma

We give some auxiliary lemmas that will be needed in the proof of the main result.

Lemma 2.1 If conditions (1.2) and (1.3) hold, then the weight K has the following proposition:

- (1) $K(t)/t^c$ is non-decreasing for some $c > 0$.
- (2) $K(t)/t^p$ is non-increasing for some $0 < p < 1$.
- (3) $K(2t) \approx K(t)$ for any $0 < t < \infty$.

The above result can be found in [6].

Remark 2.2 If conditions (1.2) and (1.3) hold, then the space Q_K is a subset of D_p for some $0 < p < 1$ by Lemma 2.1, where the space D_p consists of all analytic function f on \mathbb{D} such that

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^p dA(z) < \infty.$$

Lemma 2.3 Let K satisfy conditions (1.2) and (1.3). There is a constant C (independent of w) such that

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^q}{K(1 - |z|^2)|1 - \bar{w}z|^{b+q}} dA(z) \leq \frac{C}{(1 - |w|^2)^{b-2}K(1 - |w|^2)}, \quad b \geq 2 \tag{2.1}$$

for any given $w \in \mathbb{D}$ and $q \geq 0$.

Proof In our proof we borrowed a technique from [15]. Without loss of generality, we may assume $\text{Im}(w) = 0$. Set $\alpha = 1 - w$, then α belongs to $(0, 1)$. We divide the unit disk into $S_1 \cup S_2 \cup S_3$, where

$$S_1 = \{z : 0 < 1 - |z| \leq \alpha, |\arg z| \leq \alpha/2\},$$

$$S_2 = \{z : \alpha < 1 - |z| \leq 1, |\arg z| \leq \alpha/2\},$$

and

$$S_3 = \{z : 0 < 1 - |z| \leq 1, |\arg z| > \alpha/2\}.$$

Then

$$\begin{aligned} \int_{S_1} \frac{(1 - |z|^2)^q}{K(1 - |z|^2)|1 - \bar{w}z|^{b+q}} dA(z) &\lesssim \alpha \int_0^\alpha \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b+q}} \\ &\leq \int_0^\alpha \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} \end{aligned}$$

and

$$\begin{aligned} \int_{S_2} \frac{(1 - |z|^2)^q}{K(1 - |z|^2)|1 - \bar{w}z|^{b+q}} dA(z) &\lesssim \alpha \int_\alpha^1 \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b+q}} \\ &\leq \int_\alpha^1 \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{S_3} \frac{(1 - |z|)^q}{K(1 - |z|^2)|1 - \bar{w}z|^{b+q}} dA(z) \\ & \leq 2 \int_0^1 \frac{t^q}{K(t)} \left(\int_\alpha^\pi \frac{d\theta}{[(\alpha + t(1 - \alpha))^2 + \sin^2(\theta/2)]^{\frac{b+q}{2}}} \right) dt \\ & \lesssim \int_0^1 \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} \\ & \leq \int_0^\alpha \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} + \int_\alpha^1 \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}}. \end{aligned}$$

By Lemma 2.1, $K(t)/t^p$ is non-increasing for some $0 < p < 1$. This gives

$$\begin{aligned} \int_0^\alpha \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} & \leq \frac{\alpha^p}{K(\alpha)} \int_0^\alpha \frac{t^{q-p}}{(\alpha + t(1 - \alpha))^{b-1+q}} dt \\ & \leq \frac{1}{K(\alpha)\alpha^{b-1+q-p}} \int_0^\alpha t^{q-p} dt \\ & \lesssim \frac{1}{K(\alpha)\alpha^{b-2}}. \end{aligned}$$

Applying Lemma 2.1 again we obtain that $K(t)/t^c$ is non-decreasing for some small enough $c > 0$. Note that $c < p$ and $b \geq 2$. Then we have

$$\begin{aligned} \int_\alpha^1 \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} & \leq \frac{\alpha^c}{K(\alpha)} \int_\alpha^1 \frac{t^{q-c}}{(\alpha + t(1 - \alpha))^{b-1+q}} dt \\ & \leq \frac{\alpha^c}{K(\alpha)} \int_\alpha^1 \frac{dt}{t^{b-1+c}} \\ & \lesssim \frac{1}{K(\alpha)\alpha^{b-2}}. \end{aligned}$$

The above estimates give

$$\begin{aligned} & \int_{\mathbb{D}} \frac{(1 - |z|^2)^q}{K(1 - |z|^2)|1 - \bar{w}z|^{b+q}} dA(z) = \sum_{j=1}^3 \int_{S_j} \frac{(1 - |z|^2)^q}{K(1 - |z|^2)|1 - \bar{w}z|^{b+q}} dA(z) \\ & \lesssim \int_0^\alpha \frac{t^q}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} dt + \int_\alpha^1 \frac{t^q dt}{K(t)(\alpha + t(1 - \alpha))^{b-1+q}} \\ & \lesssim \frac{1}{K(\alpha)\alpha^{b-2}}. \end{aligned}$$

Hence (2.1) holds.

Remark 2.4 If $K(t) = t^s, 0 < s < 1$ and $p \geq 0$, Lemma 2.3 gives the result

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^{q-s}}{|1 - \bar{w}z|^{b+q}} dA(z) \leq \frac{C}{(1 - |w|^2)^{b-2+s}}, \quad b \geq 2.$$

This case was proved in [18].

Given finitely many complex numbers w_1, \dots, w_n , consider the 2^n possible sumes $\sum_{j=1}^n \pm w_j$ obtained as the plus-minus signs vary in the 2^n possible ways. For $p > 0$, denote by $\mathbb{E} \left(\left| \sum_{j=1}^n \pm w_j \right|^p \right)$ the average value of $\left| \sum_{j=1}^n \pm w_j \right|^p$ over the 2^n choices of sign. The following Lemma is called Khinchin's inequality (see [7]).

Lemma 2.5 If $0 < q < \infty$, then

$$\mathbb{E} \left(\left| \sum_{j=1}^n \pm w_j \right|^q \right) \leq C_q \left(\sum_{j=1}^n |w_j|^2 \right)^{q/2}, \tag{2.2}$$

where C_q is a constant that does not depend on n .

3 Interpolating Theorem

In this section we will prove Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.1, we know that Q_K is a subset of D_p for some $0 < p < 1$. By the reproducing formula of Rochberg and Wu (see [12] or p.75, [18]), we have

$$f(z) = f(0) + \int_{\mathbb{D}} f'(w)B(z, w)(1 - |w|^2)^p dA(w), \quad z \in \mathbb{D}, \tag{3.1}$$

where

$$B(z, w) = \frac{1 - (1 - z\bar{w})^{1+p}}{\bar{w}(1 - z\bar{w})^{1+p}}.$$

We now assume that $\{z_n\}$ is an interpolating sequence for $Q_K \cap H^\infty$. Then for $\epsilon_k^{(j)} = \pm 1, j = 1, \dots, 2^n, k = 1, \dots, n$, there are $f_j \in Q_K \cap H^\infty$ such that $f_j(z_k) = \epsilon_k^{(j)}, k = 1, \dots, n$ and

$$\|f_j\|_{H^\infty} + \|f_j\|_{Q_K} \lesssim 1.$$

Replaced f by $f_j \circ \varphi_a$ in (3.1) we have

$$f_j \circ \varphi_a(z) = f_j \circ \varphi_a(0) + \int_{\mathbb{D}} (f_j \circ \varphi_a)'(w)B(z, w)(1 - |w|^2)^p dA(w).$$

Let $z = \varphi_a(z_k), k = 1, 2, \dots, n$. Then

$$f_j(z_k) = f_j \circ \varphi_a(0) + \int_{\mathbb{D}} (f_j \circ \varphi_a)'(w)B(\varphi_a(z_k), w)(1 - |w|^2)^p dA(w),$$

and we have

$$\begin{aligned} & \sum_{k=1}^n K(1 - |\varphi_a(z_k)|^2) = \sum_{k=1}^n \epsilon_k^{(j)} f_j(z_k) K(1 - |\varphi_a(z_k)|^2) \\ &= f_j(a) \sum_{k=1}^n \epsilon_k^{(j)} K(1 - |\varphi_a(z_k)|^2) \\ & \quad + \sum_{k=1}^n \epsilon_k^{(j)} K(1 - |\varphi_a(z_k)|^2) \int_{\mathbb{D}} (f_j \circ \varphi_a)'(w)B(\varphi_a(z_k), w)(1 - |w|^2)^p dA(w) \\ &= A + B. \end{aligned}$$

We now compute the expectation of both side of the equality. Observe that by (2.2) with $q = 1$, we have

$$\mathbb{E}(A) \lesssim \left(\sum_{k=1}^n (K(1 - |\varphi_a(z_k)|^2))^2 \right)^{1/2} \lesssim \left(\sum_{k=1}^n K(1 - |\varphi_a(z_k)|^2) \right)^{1/2}.$$

There exists a constant C such that

$$\sup_{a, z_k, w \in \mathbb{D}} \left| \frac{1 - (1 - \varphi_a(z_k)\bar{w})^{1+p}}{\bar{w}} \right| \leq C. \quad (3.2)$$

Applying the Cauchy-Schwarz inequality, (2.1), (2.2) with $q = 2$ and (3.2) we have

$$\begin{aligned} \mathbb{E}(B) &\leq \left(\sum_{k=1}^n \left| \int_{\mathbb{D}} (f_j \circ \varphi_a)'(w) \frac{B(\varphi_a(z_k), w)(1 - |w|^2)^p}{(K(1 - |\varphi_a(z_k)|^2))^{-1}} dA(w) \right|^2 \right)^{1/2} \\ &\leq \sup_j \|f\|_{Q_K} \left(\sum_{k=1}^n \int_{\mathbb{D}} \left(\frac{B(\varphi_a(z_k), w)}{K(1 - |\varphi_a(z_k)|^2)^{-1}} \frac{(1 - |w|^2)^p}{\sqrt{K(1 - |w|^2)}} \right)^2 dA(w) \right)^{1/2} \\ &\lesssim \left(\sum_{k=1}^n (K(1 - |\varphi_a(z_k)|^2))^2 \int_{\mathbb{D}} \frac{(1 - |w|^2)^{2p} dA(w)}{|1 - \varphi_a(z_k)\bar{w}|^{2(1+p)} K(1 - |w|^2)} \right)^{1/2} \\ &\lesssim \left(\sum_{k=1}^n K(1 - |\varphi_a(z_k)|^2) \right)^{1/2}. \end{aligned}$$

The estimates involving $\mathbb{E}(A)$ and $\mathbb{E}(B)$ indicate that $d\mu = \sum \delta_{z_n}$ is a K -Carleson measure by Corollary 3.1 in [6]. Since $\{z_n\}$ is an interpolating sequence sequence for H^∞ , then $\{z_n\}$ is separated.

We need the following result to prove sufficiency, which dues to Earl (see [7]).

Lemma 3.1 Let $\{z_n\}$ be a sequence in the unit disc such that (1.5) holds. Then there is a constant C such that whenever $\{a_n\} \in l^\infty$, there exists $f \in H^\infty$ such that

$$f(z_n) = a_n, \quad n = 1, 2, \dots \quad (3.3)$$

and such that

$$f(z) = C \left(\sup_n |a_n| \right) B(z),$$

where $B(z)$ is a Blaschke product. The zeros ξ_n of $B(z)$ satisfy $|\varphi_{\xi_n}(z_n)| \leq \delta/3$ and $\{\xi_j\}$ is separated.

For any given $\{a_n\} \in l^\infty$, by Lemma 3.1 there exists a function $f \in H^\infty$ such that (3.3) holds. We now show that $d\mu(z) = \sum \delta_{\xi_n}$ is a K -Carleson measure. For any $a \in \mathbb{D}$, by

Lemma 1.4 in [7], we have

$$\begin{aligned} 1 - |\varphi_a(z_n)|^2 &\geq 1 - |\varphi_a(z_n)| \geq 1 - \frac{|\varphi_a(\xi_n)| + |\varphi_{z_n}(\xi_n)|}{1 + |\varphi_a(\xi_n)|\varphi_{z_n}(\xi_n)} \\ &\geq 1 - \frac{\delta/3 + |\varphi_{z_n}(\xi_n)|}{1 + \delta|\varphi_{z_n}(\xi_n)|/3} \geq \frac{1 - \delta/3}{1 + \delta/3}(1 - |\varphi_a(\xi_n)|) \\ &\geq \frac{1 - \delta/3}{2(1 + \delta/3)}(1 - |\varphi_a(\xi_n)|^2). \end{aligned}$$

Note that $K(t) \approx K(2t)$ and $d\mu = \sum \delta_{z_n}$ is a K -Carleson measure. Then we have

$$\sum_{n=1}^{\infty} K(1 - |\varphi_a(\xi_n)|^2) \lesssim \sum_{n=1}^{\infty} K(1 - |\varphi_a(z_n)|^2) < \infty.$$

This shows that $d\mu(z) = \sum \delta_{z_n}$ is a K -Carleson measure by Corollary 3.1 in [6]. So we have $f \in Q_K$ by Theorem 5.1 in [6]. The proof is completed.

If we choose $K(t) = t^p$, $0 < p < 1$, then the space Q_K is just a Q_p space. Note that the conditions (1.2) and (1.3) hold. This gives the following result which appeared in [10].

Corollary 3.2 The sequence $\{z_n\}$ is an interpolating sequence for $Q_p \cap H^\infty$ if and only if $\{z_n\}$ is separated and $d\mu = \sum \delta_{z_n}$ is a p -Carleson measure.

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Q_K 空间的插值序列

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摘要: 本文研究了 Q_K 空间的插值问题. 利用复分析和调和分析的方法, 获得了单位圆盘上的一个序列 $\{z_n\}$ 是 $Q_K \cap H^\infty$ 空间的插值序列的一个充分必要条件, 推广了 Q_p 空间的部分结果.

关键词: Q_K 空间; K -Carleson 测度; 插值

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