Vol. 36 (2016)
No. 3

数 学 杂 志
J. of Math. (PRC)

ON COMLETE SHRINKING RICCI-HARMONIC SOLITONS

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Abstract: In this paper, we study the geometry of shrinking Ricci-harmonic solitons. By utilizing the method of Manola, Gabriele and Carlo [4] under the Ricci soliton, we prove the result that every compact shrinking Ricci-harmonic soliton is a gradient one, which extends the result in the case of Ricci solition. Moreover, by utilizing the method of Zhang [14], we prove a more precise volume growth estimate than that of at most Euclidean growth for the complete non-compact gradient shrinking Ricci-harmonic soliton, which extends the result of Zhang [14] in the case of Ricci solition.

Keywords: shrinking Ricci-harmonic soliton; gradient; volume growth

2010 MR Subject Classification: 53C25; 53C43

Document code: A **Article ID:** 0255-7797(2016)03-0494-07

1 Introduction

Let (M^n, g) be a complete smooth Riemannian manifold, the metric g is called a Ricci-harmonic soliton if there exists a vector field X and a constant λ, such that

$$\begin{cases} R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \dfrac{1}{2}\mathcal{L}_X g = \lambda g_{ij}, \\ \tau_g\phi = \nabla_X\phi, \end{cases} \tag{1.1}$$

where $\phi : (M^n, g) \to (N^m, h)$ is a map between the Remannian manifolds (M^n, g) and (N^m, h), Rc is the Ricci curvature of (M, g), $\tau_g\phi = \mathrm{trace}\nabla d\phi$ and α is a nonnegative constant.

We call the Ricci-harmonic soliton (1.1) a shrinking, steady, expanding Ricci-harmonic soliton if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$. If X is a gradient of some function f, then $\mathcal{L}_X g = \nabla^2 f$, we call the Ricci-harmonic soliton a gradient Ricci-harmonic soliton with potential function f.

Similar to the Ricci soliton, the Ricci-harmonic soliton is a self-similar solution to the Ricci-harmonic flow,

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2Rc + 2\alpha(t)\nabla\phi\otimes\nabla\phi, \\ \dfrac{\partial}{\partial t}\phi = \tau_g\phi, \end{cases} \tag{1.2}$$

* **Received date:** 2015-03-01 **Accepted date:** 2015-06-02

Foundation item: Supported by Tian Yuan Special Funds of the National Natural Science Foundation of China (11326076).

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Perelman [9] proved the result that every compact Ricci soliton is a gradient one. Manola, Gabriele and Carlo [4] gave another proof by Perelman's work [10] and previous others, see Hamilton [6] (dimension 2) and Ivey [7] (dimension 3). Aquino, Barros and Ribeiro [1] showed that the potential function in the compact Ricci soliton equals to the Hodge-de Rham potential. Müller [8] proved the result that a shrinking Ricci-harmonic breather is a gradient soliton. Yang and Shen [13] obtained a monotone volume formula for a general geometric flow by utilizing Perelman's method under the Ricci flow.

Cao and Zhou [2] proved that a complete noncompact gradient shrinking Ricci soliton has at most Euclidean volume growth by estimating the bounds for the potential function. Zhang [14] proved a more precise estimate, $Vol(B(o,r)) \leq C(R+1)^{n-2\delta}$, when the scalar curvature is bounded below by a positive constant δ. Yang and Shen [12] found that the complete noncompact gradient shrinking Ricci-harmonic soliton still has at most Euclidean volume growth by proving $R - \alpha|\nabla\phi|^2$ is nonnegative and estimating the bounds of the potential function f.

In Section 2, we prove the result that every compact shrinking Ricci-harmonic soliton is a gradient one. The method is inspired by Manola, Gabriele and Carlo's work [4] and different from that in [8]. In Section 3, we extend Zhang's work [14] to the case of complete noncompact Ricci-harmonic soliton.

Our main theorems in this paper are below.

Theorem 1.1 Every compact shrinking Ricci-harmonic soliton is a gradient one.

Theorem 1.2 Let (M^n, g) be the complete noncompact shrinking gradient Ricci-harmonic soliton structure (3.1), if there exists a nonnegative constant δ such that $R - \alpha|\nabla\phi|^2 \geq \delta$, then there is a constant $C < +\infty$ depending only on g and x_0 such that

$$Vol(B_{x_0}(r)) \leq C(r+1)^{n-2\delta} \tag{1.3}$$

for all $r > r_0$, where $B_{x_0}(r)$ is a geodesic ball with radius r and r_0 is a positive constant.

Remark 1.1 The condition $R - \alpha|\nabla\phi|^2 \geq \delta$ added in Theorem 1.3 is reasonable for Yang and Shen [12] proved $R - \alpha|\nabla\phi|^2 \geq 0$.

2 The Compact Case

Lemma 2.1 (Log Sobolev inequality, see [3]) Let (M^n, g) be a compact Riemannian manifold. For any $a > 0$, there exists a constant $C(a, g)$ such that if $\varphi > 0$ satisfies $\int_M \varphi^2 dVol = 1$, then

$$\int_M \varphi^2 \log\varphi dVol \leq a\int_M |\nabla\varphi|^2 dVol + C(a,g). \tag{2.1}$$

Lemma 2.2 Let (M^n, g) be a compact Riemannian manifold, $F: M \to \mathbb{R}$ be a smooth function and λ be a positive constant, then there exists a smooth function $f: M \to \mathbb{R}$ satisfies the equation

$$F + 2\triangle f - |\nabla f|^2 + 2\lambda f = \text{Const.}. \tag{2.2}$$

Proof Define a functional W

$$W(g,f)=\int_M (F+2\triangle f-|\nabla f|^2+2\lambda f)e^{-f}dVol$$

and

$$\mu(g)=\inf\{W(g,f): f\in C^\infty(M) \text{ with } \int_M e^{-f}dVol=1\}.$$

Let $\omega=e^{-\frac{f}{2}}$, we have

$$\int_M \omega^2 dVol=1$$

and

$$\begin{aligned} W(g,f) &= \int_M (F+|\nabla f|^2+2\lambda f)e^{-f}dVol \\ &= \int_M [(F-4\lambda\log\omega)\omega^2+4|\nabla\omega|^2]dVol \\ &:= H(g,\omega). \end{aligned}$$

Since F is bounded below on M and from Log Sobolev inequality (Lemma 2.1), there exist a constant $C<+\infty$ such that

$$\int_M \omega^2\log\omega dVol\le\frac{1}{\lambda}\int_M|\nabla\omega|^2+C.$$

Then the positive minimizer ω_1 realizing $\mu(g)$ is the lowest positive eigenvalue of the nonlinear operator

$$\Theta(\omega):=-4\Delta\omega+(F-4\lambda\log\omega)\omega=\mu(g)\omega.$$

Choose ω such that $H(g,\omega)\le C_1$, then $\int_M\omega^2dVol=1$, and there exists a positive constant C_2 with

$$C_1\ge H(g,\omega)\ge 2\int_M|\nabla\omega|^2dVol-C_2.$$

Hence any minimizing sequence for $H(g,\cdot)$ is bounded in $W^{1,2}(M)$. We get a minimizer $\omega_1\in W^{1,2}(M)$ and ω_1 is a weak solution to

$$-4\Delta\omega+(F-4\lambda\log\omega)\omega=\mu(g)\omega.$$

By elliptic regularity theory (see Gilbarg and Trudinger [5]), we have $\omega_1\in C^\infty$. It's easy to verify that $\omega_1>0$. Then there exists a smooth function $f_1=-2\log\omega_1$ realizing $\mu(g)$, i.e.,

$$F+2\triangle f_1-|\nabla f_1|^2+2\lambda f_1=\mu(g)$$

for $\lambda>0$.

Proof of Theorem 1.1 Considering the compact shrinking Ricci-harmonic soliton

$$\begin{cases} R_{ij}-\alpha\nabla_i\phi\nabla_j\phi+\frac{1}{2}\mathcal{L}_Xg=\lambda g_{ij}, \\ \tau_g\phi=\nabla_X\phi. \end{cases} \tag{2.3}$$

From Lemma 2.2, there exists a smooth function $f : M \to \mathbb{R}$ satisfying

$$R - \alpha|\nabla\phi|^2 + 2\triangle f - |\nabla f|^2 + 2\lambda f = \text{Const.},$$

we have

$$\begin{aligned}
&\nabla_j[2(R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij})e^{-f}]\\
=&(\nabla_i R - 2\alpha\nabla_j\nabla_i\phi\nabla_j\phi - 2\alpha\nabla_i\phi\tau_g\phi + 2\triangle\nabla_i f)e^{-f}\\
&- 2\nabla_j f(R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij})e^{-f}\\
=&(\nabla_i R - \alpha\nabla_i|\nabla\phi|^2 - 2\alpha\nabla_i\phi\nabla_j\phi X^j + 2\nabla_i\triangle f + 2R_{ij}\nabla_j f)e^{-f}\\
&- (2R_{ij}\nabla_j f - 2\alpha\nabla_i\phi\nabla_j\phi\nabla_j f + \nabla_i|\nabla f|^2 - 2\lambda\nabla_i f)e^{-f}\\
=&\nabla_i(R - \alpha|\nabla\phi|^2 + 2\triangle f - |\nabla f|^2 + 2\lambda f)e^{-f} + 2\alpha\nabla_i\phi\nabla_j\phi(\nabla_j f - X^j)e^{-f}\\
=&2\alpha\nabla_i\phi\nabla_j\phi(\nabla_j f - X^j)e^{-f}.
\end{aligned}$$

As $\nabla_i X^j + \nabla_j X^i = -2R_{ij} + 2\alpha\nabla_i\phi\nabla_j\phi + 2\lambda g_{ij}$ and $\alpha \geq 0$, we have

$$\begin{aligned}
&\nabla_i[(\nabla_j f - X^j)(R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij})e^{-f}]\\
=&(\nabla_i\nabla_j f - \nabla_i X^j)(R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij})e^{-f} + 2\alpha|\nabla_{\nabla f - X}\phi|^2 e^{-f}\\
=&\frac{1}{2}(2\nabla_i\nabla_j f - \nabla_i X^j - \nabla_j X^i)(R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij})e^{-f} + 2\alpha|\nabla_{\nabla f - X}\phi|^2 e^{-f}\\
=&|R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij}|^2 e^{-f} + 2\alpha|\nabla_{\nabla f - X}\phi|^2 e^{-f}\\
\geq&0.
\end{aligned}$$

Denote $|R_{ij} - \alpha\nabla_i\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij}|^2 e^{-f} + 2\alpha|\nabla_{\nabla f - X}\phi|^2 e^{-f}$ by Q, we conclude that

$$0 \leq Q = \text{div}[(\nabla_j f - X^j)(R_{ij} - \alpha\nabla_i\nabla_j\phi + \nabla_i\nabla_j f - \lambda g_{ij})e^{-f}]. \tag{2.4}$$

Integrating Q, we have $Q \equiv 0$ by Stokes's theorem and the compactness of M^n. This implies compact shrinking Ricci-harmonic soliton (2.3) is a gradient Ricci-harmonic soliton with $X = \nabla f$.

Similar to the proof of Theorem 1.1, we have the direct corollary.

Corollary 2.1 Every compact steady Ricci-harmonic soliton is a gradient one.

Proposition 2.1 For the compact shrinking Ricci-harmonic soliton (2.3), the potential function f equals a Hodge-de Rham potential up to a constant.

Proof By the Hodge-de Rham decomposition theorem, there exists a divergence-free vector field Y and a function b on M^n, such that

$$X = Y + \nabla b, \tag{2.5}$$

we deduce $\text{div}X = \triangle b$. By Theorem 1.1, we can find a potential function f to (M, g, X) satisfying $X = \nabla f$, then $\text{div}X = \triangle f$.

We conclude that $f = b + \text{Const.}$ for $\triangle(f - b) = 0$ and M is compact.

Remark 2.1 Proposition 2.1 provides another way to find the potential function f to the generic compact shrinking Ricci-harmonic soliton structure (M^n, g, X). Normalizing f, we can replace f by the Hodge-de Rham potential to vector field X.

3 The Complete Noncompact Case

In this section, we consider the complete noncompact gradient shrinking Ricci-harmonic soliton

$$\begin{cases} R_{ij} - \alpha\nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j f = \frac{1}{2}g_{ij}, \\ \tau_g\phi = \langle\nabla\phi, \nabla f\rangle. \end{cases} \tag{3.1}$$

Lemma 3.1 Let (M^n, g) be the complete noncompact gradient shrinking Ricci-harmonic soliton structure (3.1), we have the following four equalities

$$R - \alpha|\nabla\phi|^2 + \triangle f = \frac{n}{2}, \tag{3.2}$$

$$R - \alpha|\nabla\phi|^2 + |\nabla f|^2 - f = 0. \tag{3.3}$$

Proof Taking trace of the first equation of (3.1), (3.2) is obtained.

Taking covariant derivatives and using the commutation formula for the covariant derivatives, we have

$$\nabla_i R_{jk} - \nabla_j R_{ik} - \alpha(\nabla_j\phi\nabla_i\nabla_k\phi - \nabla_i\phi\nabla_j\nabla_k\phi) + R_{ijkl}\nabla_l f = 0.$$

Taking the trace on j and k, we have

$$\nabla_i R - \nabla_j R_{ij} - \alpha\nabla_j\phi\nabla_i\nabla_j\phi + \alpha\nabla_i\phi\tau_g\phi - R_{il}\nabla_l f = 0.$$

Using the fact that

$$\nabla_i R - 2\nabla_j R_{ij} = 0,$$
$$\nabla_i\nabla_j\phi\nabla_j\phi = \frac{1}{2}\nabla_i(\nabla_j\phi\nabla_j\phi) = \frac{1}{2}\nabla_i|\nabla\phi|^2$$

and

$$\alpha\nabla_i\phi\tau_g\phi - R_{il}\nabla_l f = \nabla_i\nabla_j f\nabla_j f - \frac{1}{2}g_{ij}\nabla_j f = \frac{1}{2}\nabla_i(|\nabla f|^2 - f),$$

we obtain $\frac{1}{2}\nabla(R - \alpha|\nabla\phi|^2 + |\nabla f|^2 - f) = 0$, hence $R - \alpha|\nabla\phi|^2 + |\nabla f|^2 - f = \text{Const.}$. Normalizing f by by adding a constant, (3.3) follows.

Lemma 3.2 Let (M^n, g) be the complete noncompact shrinking Ricci-harmonic soliton structure (3.1) and $\mathbf{V}(r) := \int_{\{f<r\}} dV$ and $\mathbf{V}_R(r) := \int_{\{f<r\}} R - \alpha|\nabla\phi|^2 dV$, we have

$$n\mathbf{V}(r) - 2r\mathbf{V}'(r) = 2\mathbf{V}_R(r) - 2\mathbf{V}_R'(r). \tag{3.4}$$

Proof Integrating by parts and using eq. (3.2),

$$\frac{n}{2}\mathbf{V}(r) - \mathbf{V}_R(r) = \int_{\{f<r\}} \triangle f dV = \int_{\{f=r\}} \nabla f \cdot \frac{\nabla f}{|\nabla f|} dA = \int_{\{f=r\}} |\nabla f| dA, \tag{3.5}$$

which implies

$$\frac{n}{2}\mathbf{V}(r) \geq \mathbf{V}_R(r). \tag{3.6}$$

By co-Area formula (see [11]), we have

$$\mathbf{V}'(r) = \int_{\{f=r\}} \frac{1}{|\nabla f|} dA \tag{3.7}$$

and

$$\mathbf{V}_R'(r) = \int_{\{f=r\}} \frac{R - \alpha|\nabla f|^2}{|\nabla f|} dA. \tag{3.8}$$

Using (3.3) and combining (3.5), (3.7) and (3.8), we obtain

$$\frac{n}{2}\mathbf{V}(r) - \mathbf{V}_R(r) = r\mathbf{V}'(r) - \mathbf{V}_R'(r).$$

Proof of Theorem 1.2 Calculating directly,

$$\frac{d}{dr}(\log(r^{-\frac{n-2\delta}{2}}\mathbf{V}(r))) = \frac{r\mathbf{V}'(r) - \frac{n-2\delta}{2}\mathbf{V}(r)}{r\mathbf{V}(r)} \leq \frac{\mathbf{V}_R'(r)}{r\mathbf{V}(r)}, \tag{3.9}$$

where the last inequality comes from (3.4) and $\mathbf{V}_R(r) \geq \delta\mathbf{V}(r)$.

Fixed $r_0 > 0$, for any $r_1 > r_0$, integrating (3.9) by parts on $[r_0, r_1]$ yields

$$\begin{aligned} \log\frac{r_1^{-\frac{n-2\delta}{2}}\mathbf{V}(r_1)}{r_0^{-\frac{n-2\delta}{2}}\mathbf{V}(r_0)} &\leq \int_{r_0}^{r_1} \frac{1}{r\mathbf{V}(r)} d\mathbf{V}_R(r) \\ \leq\ & \frac{\mathbf{V}_R(r)}{r\mathbf{V}(r)}\Big|_{r_0}^{r_1} + \int_{r_0}^{r_1} \frac{\mathbf{V}_R(r)}{r^2\mathbf{V}(r)} dr + \int_{r_0}^{r_1} \frac{\mathbf{V}_R(r)\mathbf{V}'(r)}{r\mathbf{V}^2(r)} dr \\ \leq\ & \frac{n}{2r_0} + \frac{n}{2}\left(\frac{\log(\mathbf{V}(r_1))}{r_1} - \frac{\log(\mathbf{V}(r_0))}{r_0}\right) + \frac{n}{2}\int_{r_0}^{r_1} \frac{\log(\mathbf{V}(r))}{r^2} dr. \end{aligned} \tag{3.10}$$

When r_0 is large enough, by the proof of Theorem 1.1 in Yang and Shen [11], we have

$$\mathbf{V}(r) \leq C_1 r^n \tag{3.11}$$

for some positive constant C_1 and for $r \geq r_0$. Plugging inequality (3.11) into (3.10), we have

$$\log\frac{r_1^{-\frac{n-2\delta}{2}}\mathbf{V}(r_1)}{r_0^{-\frac{n-2\delta}{2}}\mathbf{V}(r_0)} < +\infty \tag{3.12}$$

for any $r_1 > r_0$. Moreover, there is a positive constant C_2 such that

$$\mathbf{V}(r_1) \leq C_2 r_1^{\frac{n-2\delta}{2}}. \tag{3.13}$$

For $f(x) \leq \frac{1}{4}(d(x_0, x) + 2\sqrt{f(x_0)})^2$ (see Proposition 4.1 in [12]), $B_{x_0}(r) \subset \{f \leq \frac{1}{4}(r + C_2)^2\}$. We have

$$Vol(B_{x_0}(r)) \leq \mathbf{V}(\frac{1}{4}(d(x_0, x) + 2\sqrt{f(x_0)})^2) \leq C(r+1)^{n-2\delta} \tag{3.14}$$

for some positive constant C depends only on g and x_0.

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关于完备收缩的Ricci-harmonic孤子的研究

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摘要: 本文研究了收缩的Ricci-harmonic孤子的几何性质的问题. 利用文献[4]在Ricci孤子下的方法, 获得了每个紧致Ricci-harmonic孤子是一个梯度孤子的结论, 推广了Perelman等人在Ricci孤子下的结果. 此外, 利用文献[14]在Ricci孤子下的方法, 获得了完备非紧梯度收缩的Ricci-harmonic孤子具有比至多欧氏增长更加精确的体积增长估计的结果, 推广了文献[14]在Ricci孤子下的结果.

关键词: 收缩的Ricci-harmonic孤子; 梯度; 体积增长

MR(2010)**主题分类号:** 53C25; 53C43 **中图分类号:** O186.12