Vol. 36 (2016)
No. 1

数 学 杂 志
J. of Math. (PRC)

HIGH ORDER SCHWARZ BOUNDARY VALUE PROBLEM IN A SECTOR

WANG Ying[1], YE Liu-qing[2], TONG Li-zhen[1]

(*1. School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, China*)

(*2. Dept. of Math., Jiaozuo Teachers College, Jiaozuo 454002, China*)

Abstract: In this paper, we give a general discussion of Schwarz problem for polyanalytic equations in a sector with angle $\theta = \frac{\pi}{\alpha},\ \alpha \geq 1/2$. By constructing proper Poly-Schwarz and Pompeiu operators, the explicit solvability expressions for Schwarz problem are obtained, which extends the related result to higher order case and enriches the development of boundary problems in a sector.

Keywords: Schwarz problem; Poly-Schwarz operator; Pompeiu operator

2010 MR Subject Classification: 30E25; 35G35

Document code: A **Article ID:** 0255-7797(2016)01-0001-05

1 Introduction

In recent years, many investigators paid much attention to boundary value problems for complex differential equations in different domains, such as unit disc, half plane, upper unit disc, ring, sector and even high dimension space [2–9]. In [6], the authors studied Schwarz problem in half disc and half ring, and Wang extended boundary conditions, discussing high order Schwarz problem for polyanalytic equation in half unit disc and a triangle [7, 8]. In [9], we also investigated a Schwarz problem for Cauchy-Riemann equation in a sector with angle $\theta = \frac{\pi}{\alpha},\ \alpha \geq 1$. In this article, we study a high order Schwarz problem for polyanalytic equation in a general sector with angle $\theta = \frac{\pi}{\alpha},\ \alpha \geq 1/2$, giving explicit solvability expression.

Let Ω be a sector with angle $\theta = \frac{\pi}{\alpha}$ $(\alpha \geq 1/2)$, that is, $\Omega = \{|z| < 1,\ 0 < \arg z < \frac{\pi}{\alpha},\ \alpha \geq 1/2\}$. Its boundary $\partial\Omega = [0,1] \cup \Gamma \cup [\varpi, 0]$ is oriented counter-clockwise, where $0,\ 1,\ \varpi = e^{i\theta}$ are three corner points and the oriented circular arc Γ is given by $\Gamma :\ \tau \longmapsto e^{i\tau},\ \ \tau \in \left[0,\ \frac{\pi}{\alpha}\right]$.

Lemma 1.1 (see [9]) The Schwarz problem for Cauchy-Riemann equation in Ω

$$w_{\overline{z}} = f \text{ in } \Omega, \quad \mathrm{Re} w = \gamma \text{ on } \partial\Omega,$$

* **Received date:** 2013-07-09 **Accepted date:** 2013-09-04

Foundation item: Supported by Tianyuan Foundation of China (11326087); the Fundamental Research Funds for the Central Universities; the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

Biography: Wang Ying (1984–), female, born at Zhumadian, Henan, lecturer, Ph.D., major in complex analysis and application.

$$\frac{\alpha}{\pi}\int_0^{\frac{\pi}{\alpha}} \mathrm{Im}w(e^{i\varphi})\mathrm{d}\varphi = c, \quad c\in\mathbb{R}$$

for $f\in L_p(\Omega;\mathbb{C}),\ p>2,\quad \gamma\in C(\partial\Omega;\mathbb{R})$, is uniquely solvable by

$$\begin{aligned} w(z)= &\frac{\alpha}{2\pi i}\int\limits_{\Gamma}\gamma(\zeta)\left(\frac{\zeta^\alpha+z^\alpha}{\zeta^\alpha-z^\alpha}-\frac{\overline{\zeta^\alpha}+z^\alpha}{\overline{\zeta^\alpha}-z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta}+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]\cup[0,1]}\gamma(\zeta)\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\mathrm{d}\zeta \\ &-\frac{\alpha}{\pi}\int\limits_{\Omega}\left[f(\zeta)\Big(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\Big)\zeta^{\alpha-1}-\overline{f(\zeta)}\Big(\frac{1}{\overline{\zeta^\alpha}-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\overline{\zeta^\alpha}}\Big)\overline{\zeta^{\alpha-1}}\right]\mathrm{d}\xi\mathrm{d}\eta+ic. \end{aligned}$$

2 High Order Schwarz Problem

Let

$$H(z,\zeta)=\left[\frac{1}{\zeta^\alpha-z^\alpha}-\frac{1}{\zeta^\alpha-\overline{z^\alpha}}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}+\frac{\overline{z^\alpha}}{1-\overline{z^\alpha}\zeta^\alpha}\right]\zeta^{\alpha-1},\quad z,\zeta\in\Omega \tag{2.1}$$

and a poly-Schwarz operator for Ω is

$$\begin{aligned} &S_n[\gamma_0,\gamma_1,\cdots,\gamma_{n-1}](z) \\ =&\sum_{l=0}^{n-1}\frac{(-1)^l}{l!}\Bigg\{\frac{\alpha}{2\pi i}\int\limits_{\Gamma}\gamma_l(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)^l\left(\frac{\zeta^\alpha+z^\alpha}{\zeta^\alpha-z^\alpha}-\frac{\overline{\zeta^\alpha}+z^\alpha}{\overline{\zeta^\alpha}-z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]\cup[0,1]}\gamma_l(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)^l\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\mathrm{d}\zeta\Bigg\},\ z\in\Omega. \end{aligned}\tag{2.2}$$

Then we have the following result.

Lemma 2.1 For $\gamma_0,\gamma_1,\cdots,\gamma_{n-1}\in C(\partial\Omega,\mathrm{R})$, then

$$\left\{\mathrm{Re}\frac{\partial^k S_n[\gamma_0,\gamma_1,\cdots,\gamma_{n-1}]}{\partial\overline{z}^k}\right\}^+(t)=\gamma_k(t)\quad\text{for}\quad t\in\partial\Omega;\quad k=0,1,2,\cdots,n-1$$

and

$$\frac{\partial^n S_n[\gamma_0,\gamma_1,\cdots,\gamma_{n-1}](z)}{\partial\overline{z}^n}=0.$$

Proof For $k=0,1,2,\cdots,n-1$, by (2.2),

$$\begin{aligned} &\frac{\partial^k S_n[\gamma_0,\gamma_1,\cdots,\gamma_{n-1}](z)}{\partial\overline{z}^k} \\ =&\sum_{l=k}^{n-1}\frac{(-1)^{l-k}}{(l-k)!}\Bigg\{\frac{\alpha}{2\pi i}\int\limits_{\Gamma}\gamma_l(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)^{l-k}\left(\frac{\zeta^\alpha+z^\alpha}{\zeta^\alpha-z^\alpha}-\frac{\overline{\zeta^\alpha}+z^\alpha}{\overline{\zeta^\alpha}-z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]\cup[0,1]}\gamma_l(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)^{l-k}\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\mathrm{d}\zeta\Bigg\} \\ =&\sum_{l=k}^{n-1}\sum_{j=0}^{l-k}\frac{(-1)^{l-k}}{j!(l-k-j)!}(-z-\overline{z})^{l-k-j}\Bigg\{\frac{\alpha}{2\pi i}\int\limits_{\Gamma}\gamma_l(\zeta)\left(\zeta+\overline{\zeta}\right)^j\left(\frac{\zeta^\alpha+z^\alpha}{\zeta^\alpha-z^\alpha}-\frac{\overline{\zeta^\alpha}+z^\alpha}{\overline{\zeta^\alpha}-z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]\cup[0,1]}\gamma_l(\zeta)\left(\zeta+\overline{\zeta}\right)^j\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\mathrm{d}\zeta\Bigg\}. \end{aligned}\tag{2.3}$$

Then we have

$$\mathrm{Re}\left\{\frac{\partial^k S_n[\gamma_0,\gamma_1,\cdots,\gamma_{n-1}](z)}{\partial\overline{z}^k}\right\}=\sum_{l=k}^{n-1}\sum_{j=0}^{l-k}\frac{(-1)^{l-k}(-z-\overline{z})^{l-k-j}}{j!(l-k-j)!}\frac{\alpha}{2\pi i}\int_{\partial\Omega}\gamma_l(\zeta)\left(\zeta+\overline{\zeta}\right)^j H(z,\zeta)\mathrm{d}\zeta,$$

where $H(z,\zeta)$ is given by (2.1). From the proof in [9],

$$\lim_{z\to t,t\in\partial\Omega}\frac{\alpha}{2\pi i}\int_{\partial\Omega}\gamma_l(\zeta)\left(\zeta+\overline{\zeta}\right)^j H(z,\zeta)\mathrm{d}\zeta=\gamma_l(t)\left(t+\overline{t}\right)^j.$$

Hence,

$$\left\{\mathrm{Re}\frac{\partial^k S_n[\gamma_0,\gamma_1,\cdots,\gamma_{n-1}]}{\partial\overline{z}^k}\right\}^+(t)=\sum_{l=k}^{n-1}\sum_{j=0}^{l-k}\frac{(-1)^j(t+\overline{t})^{l-k}\gamma_l(t)}{j!(l-k-j)!}=\gamma_k(t).$$

Obviously, by (2.3), the second equation in Lemma 2.1 is also true.

On the other hand, we define a Pompeiu operator as follows

$$\begin{aligned}T_n[f](z)=\ &\frac{(-1)^n\alpha}{\pi(n-1)!}\int_{\Omega}\left(\zeta-z+\overline{\zeta-z}\right)^{n-1}\left[f(\zeta)\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\right.\\&\left.-\overline{f(\zeta)}\left(\frac{1}{\overline{\zeta^\alpha}-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\overline{\zeta^\alpha}}\right)\overline{\zeta^{\alpha-1}}\right]\mathrm{d}\xi\mathrm{d}\eta,\quad z\in\Omega\end{aligned}\tag{2.4}$$

with $f\in L_p(\Omega;C),\ p>2$, and

$$\begin{aligned}T_1[f](z)=\ &-\frac{\alpha}{\pi}\int_{\Omega}\left[f(\zeta)\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\right.\\&\left.-\overline{f(\zeta)}\left(\frac{1}{\overline{\zeta^\alpha}-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\overline{\zeta^\alpha}}\right)\overline{\zeta^{\alpha-1}}\right]\mathrm{d}\xi\mathrm{d}\eta.\end{aligned}\tag{2.5}$$

Lemma 2.2 For $f\in L_p(\Omega;C),\ p>2$,

$$\frac{\partial^n T_n[f](z)}{\partial\overline{z}^n}=f(z),\quad z\in\Omega;\quad \{\mathrm{Re}T_n[f]\}^+(t)=0,\quad t\in\partial\Omega.$$

Proof Since for $l=1,2,\cdots,n$,

$$T_l[f](z)=\frac{(-1)^{l-1}}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}(-z-\overline{z})^{l-k-1}T_1[(\zeta+\overline{\zeta})^k f](z),\quad z\in\Omega,\tag{2.6}$$

then by $\dfrac{\partial T_1[(\zeta+\overline{\zeta})^k f](z)}{\partial\overline{z}}=(z+\overline{z})^k f(z)$, we obtain

$$\begin{aligned}\frac{\partial T_l[f](z)}{\partial\overline{z}}&=\frac{(-1)^{l-1}}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}\left\{\left(\frac{\partial}{\partial\overline{z}}(-z-\overline{z})^{l-k-1}\right)T_1[(\zeta+\overline{\zeta})^k f](z)\right.\\&\qquad\left.+(-z-\overline{z})^{l-k-1}\frac{\partial T_1[(\zeta+\overline{\zeta})^k f](z)}{\partial\overline{z}}\right\}\\&=\frac{(-1)^{l-2}}{(l-2)!}\sum_{k=0}^{l-2}\binom{l-2}{k}(-z-\overline{z})^{l-k-2}T_1[(\zeta+\overline{\zeta})^k f](z)=T_{l-1}[f](z).\end{aligned}$$

Therefore, $\dfrac{\partial^n T_n[f](z)}{\partial \overline{z}^n} = \dfrac{\partial T_1[f](z)}{\partial \overline{z}} = f(z)$. What's more, from (2.6),

$$\mathrm{Re}\{T_n[f](z)\} = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}(-z-\overline{z})^{n-k-1}\mathrm{Re}\{T_1[(\zeta+\overline{\zeta})^k f](z)\}.$$

By the result in [9], $\mathrm{Re}\{T_1[(\zeta+\overline{\zeta})^k f]\}^+(t) = 0,\ \ t \in \partial\Omega$, thus for $t \in \partial\Omega$, $\{\mathrm{Re}T_n[f]\}^+(t) = 0$.

Theorem 2.1 The Schwarz problem for polyanalytic equation in Ω,

$$\begin{cases} \partial_{\overline{z}}^n w(z) = f(z), & z \in \Omega,\ \ f \in L_p(\Omega, \mathrm{C}),\ p > 2; \\ \left\{\mathrm{Re}(\partial_{\overline{z}}^k w)\right\}^+(t) = \gamma_k(t), & t \in \partial\Omega,\ \gamma_k \in C(\partial\Omega,\ \mathrm{R}),\ k = 0, 1, \cdots, n-1 \end{cases} \tag{2.7}$$

is solvable by

$$w(z) = S_n[\gamma_0, \gamma_1, \cdots, \gamma_{n-1}](z) + T_n[f](z) + i\sum_{k=0}^{n-1}(z+\overline{z})^k c_k,$$

where $c_k \in \mathrm{R}$, $S_n,\ T_n$ are given by (2.2) and (2.4), respectively.

Proof Let

$$w_0 = S_n[\gamma_0, \gamma_1, \cdots, \gamma_{n-1}](z) + T_n[f](z).$$

Obviously, from Lemmas 2.1 and 2.2, we know w_0 satisfies the boundary condition (2.7). We write the solution $w(z)$ as

$$w(z) = w_0(z) + U(z),$$

then

$$\begin{cases} \partial_{\overline{z}}^n U(z) = 0, & z \in \Omega; \\ \left\{\mathrm{Re}(\partial_{\overline{z}}^k U)\right\}^+(t) = 0, & t \in \partial\Omega,\ k = 0, 1, \cdots, n-1. \end{cases}$$

From the first equation and [1], the polyanalytic function $U(z)$ in Ω can be expressed as

$$U(z) = \sum_{k=0}^{n-1}(z+\overline{z})^k f_k(z), \quad z \in \Omega, \tag{2.8}$$

where f_k are analytic functions. Putting (2.8) into the second equation, we get $\{\mathrm{Re}f_k\}^+(t) = 0$, $t \in \partial\Omega$. Thus, by Lemma 1.1, $f_k(z) = ic_k$ with c_k being real numbers.Then we complete the proof.

References

[1] Balk B M. Polyanalytic functions[M]. Berlin: Akademie Verlag, 2001.

[2] Abdymanapov A S, Begehr H, Tungatarov B A. Some Schwarz problems in a quarter plane[J]. Eurasian Math. J., 2005, 3: 22–35.

[3] Begehr H. Complex analytic methods for partial differential equations: an introductory text[M]. Singapore: World Scientific, 1994.

[4] Begehr H. Boundary value problems in complex analysis I (II)[J]. Bol. Asoc. Mat. Venezolana, 2005, XII: 65–85.

[5] Begehr H, Schmersau D. The Schwarz problem for polyanalytic functions[J]. ZAA., 2005, 24: 341–351.

[6] Begehr H, Vaitekhovich T. Harmonic boundary value problems in half disc and half ring[J]. Func. App., 2009, 40(2): 251–282.

[7] Wang Yufeng. Schwarz-type boundary value problems for the polyanalytic equation in the half unit disc[J]. Complex Variables and Elliptic Equations, 2012, 57(9): 983–993.

[8] Wang Yufeng, Wang Yanjin. On Schwarz-type boundary-value problems of polyanalytic equation on a triangle[J]. Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica, 2010, IX: 69–78.

[9] Wang Ying. Boundary value problems in a fan-shaped domain[J]. Adv. Pure Appl. Math., 2011, 2: 193–212.

扇形域上高阶Schwarz边值问题

王 莹[1], 叶留青[2], 童丽珍[1]

(1. 中南财经政法大学统计与数学学院, 湖北 武汉 430073)

(2. 焦作师范高等专科学校数学系, 河南 焦作 454002)

摘要: 本文主要讨论一类角度为$\theta = \frac{\pi}{\alpha}$, $\alpha \geq 1/2$的扇形域上高阶多解析方程的Schwarz边值问题. 通过构造适当的高阶-Schwarz算子和Pompeiu算子, 我们给出了详细的解表达式. 本文把边值问题进一步推广到高阶情形, 丰富了扇形域上边值问题的发展.

关键词: Schwarz问题; 高阶-Schwarz算子; Pompeiu算子

MR(2010)**主题分类号**: 30E25; 35G35 **中图分类号**: O174.5; O175.8