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LOCAL AUTOMORPHISMS AND LOCAL DERIVATIONS OF UPPER TRIANGULAR MATRIX LIE ALGEBRA OVER A COMMUTATIVE RING

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Abstract: The aim of this paper is to characterize the local automorphisms and local derivations of $T_n(R)$. By using the main result about automorphisms and derivations of $T_n(R)$ and the skill of matrix computation, it is proved that every local automorphism of $T_n(R)$ is an automorphism and that each local derivation of $T_n(R)$ is a derivation, which extend the main result about automorphisms and derivations of $T_n(R)$.

Keywords: local automorphisms; local derivations; upper triangular matrix lie algebra; commutative ring

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1 Introduction

Recently, many scholars paid attention to the significant work has been done in studying the local maps. Larson [1] initially considered local maps in his examination of reflexivity and interpolation for subspaces of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. The notion of local derivations (resp., local automorphisms) was introduced independently by Larson and Sourour [2] and Kadison [3] (resp., Larson and Sourour [2]). Recall that a linear map δ from an algebra $\mathcal{A}$ into itself is called a local derivation (resp., local automorphism) if for every $a \in \mathcal{A}$, there exists a derivation (resp., an automorphism) δ_a of $\mathcal{A}$, depending on a, such that $\delta(a) = \delta_a(a)$. If every local derivation (resp., local automorphism) of an algebra is a derivation (resp., an automorphism), then we can say that the derivations (resp., automorphisms) of those structures are, in a certain sense, completely determined by their local actions.

Local derivations, local automorphisms and other local maps have been studied in a variety of contexts. Larson and Sourour [2] showed that every local derivation (resp., every

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surjective linear local automorphism) on $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a Banach space $\mathcal{H}$, is a derivation (an automorphism). Zhao, Yao and Wang [4] proved that every local Jordan derivation (resp., local Jordan automorphism) of upper triangular matrix algebra is an inner derivation (resp., a Jordan automorphism). Other work on the description of the local derivations or local automorphisms on operator algebras can be found in [5–7]. In those articles all local derivations or local automorphisms are actually global derivations or automorphisms. A nontrivial local derivation on an operator algebra was found by Crist in [8]. Crist [9] showed that any linear local automorphism of a finite dimensional CSL algebra $\mathcal{A}$ is either an automorphism or can be factored as an automorphism and the transpose of a self-adjoint summand of $\mathcal{A}$.

The algebra $T_n(R)$ of all upper triangular matrices over a commutative ring R is an interesting topic for many researchers. Significant research has been done in studying various linear maps of $T_n(R)$. In 1990, Kezlan [10] showed that every R-algebra automorphism of $T_n(R)$ is inner. Cao [11] and Wang and You [12] gave a description of the Lie automorphisms of $T_n(R)$. Tang, Cao and Zhang [13] determined all Jordan isomorphisms of $T_n(R)$. Wang and Yu [14] determined the derivations of any Lie subalgebra of the general linear Lie algebra containing $T_n(R)$. In this paper, we regard $T_n(R)$ as a Lie algebra and we shall study the local automorphisms and local derivations of $T_n(R)$.

Let R be a commutative ring with identity, R^* the group of invertible elements of R. In the following of this paper, we use $T_n(R)$ (resp., $D_n(R)$) denote the Lie algebra of all upper triangular (resp., diagonal) n by n matrices over R, $T_n^*(R)$ the set of all invertible elements in $T_n(R)$. We denote by $\mathbf{n}$ the subalgebra of $T_n(R)$ consisting of all strictly upper triangular matrices. Let e be the identity matrix of $T_n(R)$, $e_{i,j}$ the matrix with 1 at the position (i,j) and zero elsewhere for $1 \le i,j \le n$. For $x \in T_n(R)$, denote by x^t the transpose of x. Let $\mathcal{S}_k = \left\{ \begin{pmatrix} 0 & \\ & x \end{pmatrix} \in T_n(R) \mid x \in T_{n-k}(R) \right\}, k = 1, 2, \cdots, n-1$. Obviously, each $\mathcal{S}_k$ is a subalgebra of $T_n(R)$.

2 Local Automorphisms

Cao [11] and Wang and You [12] gave an explicit description of the automorphisms of $T_n(R)$, respectively. For convenience of the proof of the main result in this section, we give another description of the automorphisms of $T_n(R)$ by the following lemma. Before giving the lemma, let us introduce some standard automorphisms of $T_n(R)$ as follows. In this section, 2 is an unit in R.

(A) Inner automorphisms

Let $a \in T_n^*(R)$, the map $\theta_a : x \mapsto axa^{-1}$ for all $x \in T_n(R)$ is an automorphism of $T_n(R)$, which is called an inner automorphism.

(B) Central automorphisms

Regarding R as an abelian Lie algebra. Let

$$F = \{f \in \ \mathrm{Hom}_R(T_n(R), R) \mid 1 + f(e) \in R^*\}.$$

For $f \in F$, we define a map $\eta_f : x \mapsto x + f(x)e$ for all $x \in T_n(R)$. It can be checked that η_f is an automorphism of $T_n(R)$, which is called a central automorphism. Since $\eta_f(\mathbf{n}) = \mathbf{n}, f(y) = 0$ for any $y \in \mathbf{n}$.

(C) Graph automorphisms

Set $r = e_{1n} + e_{2,n-1} + \cdots + e_{n-1,2} + e_{n1}$. It is clear that $r^2 = e$ and $r^t = r$. Let Υ be the set of all idempotents in R. For $\varepsilon \in \Upsilon$, it is easy to check that the map $w_\varepsilon : x \mapsto \varepsilon x - (1-\varepsilon) r x^t r$ is an automorphism of $T_n(R)$. We call w_ε a graph automorphism.

Lemma 2.1 (the main theorem of [11] and [12]) Let ψ be an automorphism of $T_n(R)$. Then there exist an inner automorphism θ_a, a graph automorphism w_ε and a central automorphism η_f of $T_n(R)$ such that $\psi = \theta_a w_\varepsilon \eta_f$ for $n \geq 3$; $\psi = \theta_a \eta_f$ when $n = 2$; $\psi = \eta_f$ for $n = 1$.

The following lemma is obvious.

Lemma 2.2 Let θ be an inner automorphism of $T_n(R)$. Then $\theta(E) = \theta(E)^2$ for every idempotent E in $T_n(R)$.

We will prove our main result in this section via the following lemmas.

Lemma 2.3 Let φ be a local automorphism of $T_n(R)$ $(n \geq 3)$. If $\varphi(e_{11}) = e_{11}$, then we may find an inner automorphism $\theta = \prod_{j=2}^{n} \theta_{b_j}$ and a central automorphism η_f such that $\eta_f^{-1}\theta^{-1}\varphi(e_{ii}) = e_{ii}$ for $i = 1, 2, \cdots, n$.

Proof For $e_{ii} \in T_n(R)$, since φ is a local automorphism, there exists an automorphism $\varphi_{e_{ii}}$, depending on e_{ii}, such that $\varphi(e_{ii}) = \varphi_{e_{ii}}(e_{ii})$. By Lemma 2.1, we know there exist $\varepsilon_i \in \Upsilon$, $f_i \in F$ and $a_i \in T_n^*(R)$ such that

$$\varphi(e_{ii}) = \varphi_{e_{ii}}(e_{ii}) = \theta_{a_i} w_{\varepsilon_i} \eta_{f_i}(e_{ii}). \tag{2.1}$$

In the following, we first prove that $\varepsilon_i = 1$ for $i = 2, 3, \cdots, n$ in (2.1).

From (2.1) we get

$$\begin{aligned} & \varphi(e_{11} + e_{ii}) = e_{11} + \varphi(e_{ii}) \\ \equiv \ & e_{11} + \varepsilon_i e_{ii} + (2\varepsilon_i - 1) f_i(e_{ii}) e - (1 - \varepsilon_i) e_{n+1-i,n+1-i} \bmod \mathbf{n}. \end{aligned} \tag{2.2}$$

On the other hand, we have $\varphi(e_{11} + e_{ii}) = \varphi_{e_{11}+e_{ii}}(e_{11} + e_{ii})$, where $\varphi_{e_{11}+e_{ii}}$ is an automorphism depending on $e_{11} + e_{ii}$. By Lemma 2.1, we have $\varphi_{e_{11}+e_{ii}} = \theta_{a_i'} w_{\varepsilon_i'} \eta_{f_i'}$ for some $a_i' \in T_n^*(R)$, $\varepsilon_i' \in \Upsilon$ and $f_i' \in F$, so

$$\begin{aligned} & \varphi(e_{11} + e_{ii}) = \theta_{a_i'} w_{\varepsilon_i'} \eta_{f_i'}(e_{11} + e_{ii}) \\ \equiv \ & \varepsilon_i'(e_{11} + e_{ii}) + (2\varepsilon_i' - 1) f_i'(e_{11} + e_{ii}) e - (1 - \varepsilon_i')(e_{nn} + e_{n+1-i,n+1-i}) \bmod \mathbf{n}. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we have $\varepsilon_i = \varepsilon_i' = 1$ for $i = 2, 3, \cdots, n$. That is to say

$$\varphi(e_{ii}) = \theta_{a_i} \eta_{f_i}(e_{ii}), i = 2, 3, \cdots, n. \tag{2.4}$$

Next we use induction to prove that there exists an inner automorphism θ such that $\theta^{-1}\varphi(e_{ii}) = e_{ii} + f_i(e_{ii})e$ for $i = 2, \cdots, n$, and $\theta^{-1}\varphi(e_{11}) = e_{11}$. Let $i = 2$ in (2.2) and (2.3), we have

$$f_2(e_{22}) = f_2^{'}(e_{11} + e_{22}).$$

So

$$\varphi(e_{11} + e_{22}) = \theta_{a_2'}\eta_{f_2'}(e_{11} + e_{22}) = \theta_{a_2'}(e_{11} + e_{22}) + f_2(e_{22})e. \tag{2.5}$$

On the other hand, by (2.4) we have

$$\varphi(e_{11} + e_{22}) \;=\; e_{11} + \varphi(e_{22}) = e_{11} + \theta_{a_2}(e_{22}) + f_2(e_{22})e. \tag{2.6}$$

(2.5) and (2.6) imply that $\theta_{a_2'}(e_{11}+e_{22}) = e_{11}+\theta_{a_2}(e_{22})$. The idempotence of $e_{11}+e_{22}$ shows that the image of it under φ is also idempotent. So $\theta_{a_2}(e_{22}) \in \mathcal{S}_1$. Suppose $a_2 = (a_{ij}^{(2)})_{n\times n}$. Let $b_2 = (b_{ij}^{(2)})_{n\times n}$, where $b_{11}^{(2)} = a_{11}^{(2)}, b_{ij}^{(2)} = a_{ij}^{(2)}$ for $2 \le i \le j \le n$, and $b_{1j}^{(2)} = 0$ for $2 \le j \le n$. Then $\theta_{b_2}(e_{22}) = \theta_{a_2}(e_{22})$. So

$$\theta_{b_2}^{-1}\varphi(e_{22}) = e_{22} + f_2(e_{22})e \quad \text{and} \quad \theta_{b_2}^{-1}\varphi(e_{11}) = e_{11}.$$

Denote $\theta_{b_2}^{-1}\varphi$ by φ_1.

By induction we assume that there are $\theta_{b_j}, j = 3, 4, \cdots, k-1$ such that

$$(\prod_{j=3}^{k-1}\theta_{b_j})^{-1}\varphi_1(e_{11}) = e_{11}$$

and $(\prod_{j=3}^{k-1}\theta_{b_j})^{-1}\varphi_1(e_{ii}) = e_{ii}+f_i(e_{ii})e$, where $f_i \in F, i = 2, 3, \cdots, k-1$. Denote $(\prod_{j=3}^{k-1}\theta_{b_j})^{-1}\varphi_1$ by φ_{k-2}. By (2.4), we know there exist some $u \in T_n^*(R)$ and $f_k \in F$ such that

$$\varphi_{k-2}(e_{kk}) = \theta_u(e_{kk} + f_k(e_{kk})e) = \theta_u(e_{kk}) + f_k(e_{kk})e.$$

So

$$\begin{aligned}
&\varphi_{k-2}(e_{11} + \cdots + e_{k-1,k-1} + e_{kk})\\
=\;& e_{11} + \cdots + e_{k-1,k-1} + (f_2(e_{22}) + \cdots + f_k(e_{kk}))e + \theta_u(e_{kk}) &(2.7)\\
\equiv\;& e_{11} + \cdots + e_{kk} + (f_2(e_{22}) + \cdots + f_k(e_{kk}))e \bmod \mathbf{n}. &(2.8)
\end{aligned}$$

On the other hand, since φ_{k-2} is a local automorphism, there exists an automorphism $\psi = \theta_t w_\alpha \eta_\sigma$, where $t \in T_n^*(R),\ \alpha \in \Upsilon$ and $\sigma \in F$, depending on $e_{11} + \cdots + e_{kk}$, such that

$$\begin{aligned}
\varphi_{k-2}(e_{11} + \cdots + e_{kk}) \;=\;& \psi(e_{11} + \cdots + e_{kk})\\
\equiv\;& \alpha(e_{11} + \cdots + e_{kk}) + (2\alpha - 1)\sigma(e_{11} + \cdots + e_{kk})e\\
-\;& (1-\alpha)(e_{nn} + \cdots + e_{n+1-k,n+1-k}) \bmod \mathbf{n}. &(2.9)
\end{aligned}$$

By (2.8) and (2.9) we get

$$\varphi_{k-2}(e_{11}+\cdots+e_{kk}) \quad = \quad \theta_t(e_{11}+\cdots+e_{kk})+(f_2(e_{22})+\cdots+f_k(e_{kk}))e. \qquad (2.10)$$

From (2.7) and (2.10), we have $\theta_t(e_{11}+\cdots+e_{kk}) = e_{11}+\cdots+e_{k-1,k-1}+\theta_u(e_{kk})$. Since $e_{11}+\cdots+e_{kk}$ is idempotent, then by Lemma 2.2, we get

$$[e_{11}+\cdots+e_{k-1,k-1}+\theta_u(e_{kk})]^2 = e_{11}+\cdots+e_{k-1,k-1}+\theta_u(e_{kk}),$$

which means that $\theta_u(e_{kk}) \in \mathcal{S}_{k-1}$. Suppose $u=(u_{ij})_{n\times n}$. Let $b_k=(b_{ij}^{(k)})_{n\times n}$, where $b_{ii}^{(k)} = u_{ii},\ i=1,2,\cdots,k-1, b_{ij}^{(k)}=u_{ij}$ for $k\le i\le j\le n$, and $b_{ts}^{(k)}=0$ for $t=1,2,\cdots,k-1, t< s\le n$. By calculating, we have $\theta_{b_k}(e_{kk})=\theta_u(e_{kk})$. So

$$\begin{aligned}\theta_{b_k}^{-1}\varphi_{k-2}(e_{11}) &= e_{11},\\ \theta_{b_k}^{-1}\varphi_{k-2}(e_{ii}) &= e_{ii}+f_i(e_{ii})e \text{ for } i=2,\cdots,k.\end{aligned}$$

When $k=n$, let $\theta=\prod_{j=2}^{n}\theta_{b_j}$. Then $\theta^{-1}\varphi(e_{11})=e_{11}$, and $\theta^{-1}\varphi(e_{ii})=e_{ii}+f_i(e_{ii})e$ for $i=2,\cdots,n$.

Let f be an $R-$linear map satisfying $f(e_{11})=0, f(e_{ii})=f_i(e_{ii})$ for $i=2,\cdots,n$ and $f(e_{ij})=0$ for $1\le i<j\le n$. Then $f\in\mathrm{Hom}_R(T_n(R),R)$. It is easy to check that $1+f(e)\in R^*$. Thus $f\in F$ and $\eta_f^{-1}\theta^{-1}\varphi(e_{ii})=e_{ii}$ for $i=1,2,\cdots,n$.

Lemma 2.4 Let φ be a local automorphism of $T_n(R)$ satisfying $\varphi(e_{ii})=e_{ii}$ for $i=1,2,\cdots,n$. Then $\varphi(e_{ij})=a_{ij}e_{ij}$, where $1\le i<j\le n$ and $a_{ij}\in R^*$.

Proof For $1\le i<j\le n$, there exists an automorphism $\varphi_{e_{ii}+e_{jj}}$ which agree with φ at $e_{ii}+e_{ij}$. By Lemma 2.1, we know there exist $\beta_{ij}\in\Upsilon$, $\tau_{ij}\in F$ and $u_{ij}\in T_n^*(R)$ such that $\varphi_{e_{ii}+e_{ij}}=\theta_{u_{ij}}w_{\beta_{ij}}\eta_{\tau_{ij}}$. So

$$\begin{aligned}&\varphi(e_{ii}+e_{ij})=\theta_{u_{ij}}w_{\beta_{ij}}\eta_{\tau_{ij}}(e_{ii}+e_{ij})\\ \equiv\ &\beta_{ij}e_{ii}+(2\beta_{ij}-1)\tau_{ij}(e_{ii}+e_{ij})e-(1-\beta_{ij})e_{n+1-i,n+1-i} \bmod \mathbf{n}.\end{aligned} \qquad (2.11)$$

On the other hand,

$$\varphi(e_{ii}+e_{ij})=e_{ii}+\varphi(e_{ij})\equiv e_{ii} \bmod \mathbf{n}. \qquad (2.12)$$

So $\beta_{ij}=1$ and $\tau_{ij}(e_{ii}+e_{ij})=0$ follow from (2.11) and (2.12). Thus

$$e_{ii}+\varphi(e_{ij})=\varphi(e_{ii}+e_{ij})=\theta_{u_{ij}}(e_{ii}+e_{ij}).$$

Similarly, there exists some $h_{ij}\in T_n^*(R)$ such that

$$e_{jj}+\varphi(e_{ij})=\varphi(e_{jj}+e_{ij})=\theta_{h_{ij}}(e_{jj}+e_{ij}).$$

Since $e_{ii}+e_{ij}$ and $e_{jj}+e_{ij}$ are idempotents, by Lemma 2.2, we know that the image of them under φ are also idempotent, which imply that $\varphi(e_{ij})=a_{ij}e_{ij}$ for some $a_{ij}\in R$. Clearly, $a_{ij}\in R^*$.

Lemma 2.5 Let φ be a local automorphism of $T_n(R)$ satisfying $\varphi(e_{ii}) = e_{ii}$ for $i = 1, 2, \cdots, n$. Then there exists an inner automorphism θ_d such that $\theta_d\varphi(e_{i,i+1}) = e_{i,i+1}$ for $i = 1, 2, \cdots, n-1$, and $\theta_d\varphi(e_{ii}) = e_{ii}$ for $i = 1, 2, \cdots, n$.

Proof By Lemma 2.4, we have $\varphi(e_{i,i+1}) = a_{i,i+1}e_{i,i+1}$ with $a_{i,i+1} \in R^*$. Let

$$d = \mathrm{diag}(1, a_{12}^{-1}, (a_{12}a_{23})^{-1}, \cdots, (a_{12}a_{23}\cdots a_{n-1,n})^{-1}).$$

Then $\theta_d^{-1}\varphi(e_{i,i+1}) = e_{i,i+1}$ for $i = 1, 2, \cdots, n-1$, and $\theta_d^{-1}\varphi(e_{ii}) = e_{ii}$ for $i = 1, 2, \cdots, n$.

Lemma 2.6 Let φ be a local automorphism of $T_n(R)$. If $\varphi(e_{ii}) = e_{ii}$ for $i = 1, 2, \cdots, n$, and $\varphi(e_{i,i+1}) = e_{i,i+1}$ for $i = 1, 2, \cdots, n-1$, then for any $e_{i,i+k} \in T_n(R)$, we have

$$\varphi(e_{i,i+k}) = e_{i,i+k}.$$

Proof We will prove this lemma by induction on k ($k \geq 2$). When $k = 2$, since φ is a local automorphism, we have

$$\varphi(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) = \phi_i^{(2)}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}),$$

where $\phi_i^{(2)}$ is an automorphism corresponding to $e_{i,i+1}+e_{i+1,i+2}+e_{i,i+2}+e_{i+1,i+1}$. By Lemma 2.1, we know there exist $\gamma_i^{(2)} \in \Upsilon$, $\sigma_i^{(2)} \in F$ and $x_i^{(2)} \in T_n^*(R)$ such that $\phi_i^{(2)} = \theta_{x_i^{(2)}} w_{\gamma_i^{(2)}} \eta_{\sigma_i^{(2)}}$. So

$$\begin{aligned} & \varphi(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) \\ \equiv\ & \gamma_i^{(2)} e_{i+1,i+1} - (1-\gamma_i^{(2)})e_{n-i,n-i} + (2\gamma_i^{(2)} - 1)\sigma_i^{(2)}(e_{i+1,i+1})e \bmod \mathbf{n}. \end{aligned} \tag{2.13}$$

On the other hand, by Lemma 2.4, we have

$$\begin{aligned} & \varphi(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) \\ =\ & e_{i,i+1} + e_{i+1,i+2} + e_{i+1,i+1} + a_{i,i+2}e_{i,i+2} \equiv e_{i+1,i+1} \bmod \mathbf{n}. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we have $\gamma_i^{(2)} = 1$ and $\sigma_i^{(2)}(e_{i+1,i+1}) = 0$. So

$$\theta_{x_i^{(2)}}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) = e_{i,i+1} + e_{i+1,i+2} + e_{i+1,i+1} + a_{i,i+2}e_{i,i+2}.$$

The idempotence of $e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}$ and Lemma 2.2 impliy that $a_{i,i+2} = 1$. So $\varphi(e_{i.i+2}) = e_{i.i+2}$.

By induction we assume that $\varphi(e_{i,i+m}) = e_{i,i+m}$ for $m = 2, \cdots, k-1$. For $e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+1}$, similar to the case $k = 2$, we can get that there exists some $x_i^{(k)} \in T_n^*(R)$ such that

$$\theta_{x_i^{(k)}}(e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+1}) = e_{i,i+1} + e_{i+1,i+k} + e_{i+1,i+1} + a_{i,i+k}e_{i,i+k}.$$

Also by the idempotence of $e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+1}$ and Lemma 2.2, we can prove that $a_{i,i+k} = 1$. That is to say $\varphi(e_{i,i+k}) = e_{i,i+k}$ for any $e_{i,i+k} \in T_n(R)$.

Theorem 2.1 Let R be a commutative ring with identity 1 and unit 2, $T_n(R)$ the Lie algebra consisting of all upper triangular $n \times n$ matrices over R. Then every local automorphism φ of $T_n(R)$ is an automorphism.

Proof Let φ be a local automorphism of $T_n(R)$. When $n \geq 3$, for $e_{11} \in T_n(R)$, by the definition of φ, there exists an automorphism $\varphi_{e_{11}}$, depending on e_{11}, such that $\varphi(e_{11}) = \varphi_{e_{11}}(e_{11})$. So $\varphi_{e_{11}}^{-1}\varphi(e_{11}) = e_{11}$. Obviously, $\varphi_{e_{11}}^{-1}\varphi$ is also a local automorphism of $T_n(R)$. By Lemmas 2.3, 2.5 and 2.6, there are η_f^{-1}, θ^{-1} and θ_d^{-1} such that

$$\theta_d^{-1}\eta_f^{-1}\theta^{-1}\varphi_{e_{11}}^{-1}\varphi(e_{ij}) = e_{ij} \text{ for } 1 \leq i \leq j \leq n,$$

which mean that $\varphi = \varphi_{e_{11}}\theta\eta_f\theta_d$. So φ is an automorphism.

When $n = 1$, suppose that $\varphi(1) = a$, then for any $x \in T_1(R) = R$ we have $\varphi(x) = x\varphi(1) = xa = \eta_f(x)$, where $f : R \to R, x \mapsto (a-1)x$ is an $R-$linear map from R to R. So φ is an automorphism.

When $n = 2$, similar to the case $n \geq 3$, there is an automorphism $\varphi_{e_{11}}$, depending on e_{11}, such that $\varphi_{e_{11}}^{-1}\varphi(e_{11}) = e_{11}$. Denote $\varphi_{e_{11}}^{-1}\varphi$ by φ_1. Clearly, φ_1 is also a local automorphism of $T_2(R)$, by Lemma 2.1, there exist an inner automorphism θ_{a_2} and a central automorphism η_{f_2}, corresponding to e_{22}, such that $\varphi_1(e_{22}) = \theta_{a_2}\eta_{f_2}(e_{22}) = \theta_{a_2}(e_{22}) + f_2(e_{22})e$. So

$$\varphi_1(e_{11} + e_{22}) = e_{11} + \theta_{a_2}(e_{22}) + f_2(e_{22})e \equiv e + f_2(e_{22})e \bmod \mathbf{n}. \tag{2.15}$$

On the other hand, there exist an inner automorphism θ_{b_2} and a central automorphism η_{g_2}, depending on $e_{11} + e_{22}$, such that

$$\varphi_1(e_{11} + e_{22}) = \theta_{b_2}\eta_{g_2}(e_{11} + e_{22}) = e + g_2(e)e. \tag{2.16}$$

From (2.15) and (2.16), we get $\theta_{a_2}(e_{22}) = e_{22}$. Now we have $\varphi_1(e_{11}) = e_{11}$ and $\varphi_1(e_{22}) = e_{22} + f_2(e_{22})$. Let f be an $R-$linear map satisfying $f(e_{11}) = 0, f(e_{22}) = f_2(e_{22})$, it is easy to check that $f \in F$ and $\eta_f^{-1}\varphi_1(e_{ii}) = e_{ii}$ for $i = 1, 2$.

Denote $\eta_f^{-1}\varphi_1$ by φ_2. Since φ_2 is a local automorphism, by Lemma 2.1, we have $\varphi_2(e_{12}) = \theta_x(e_{12}) = ae_{12}$, where θ_x is an inner automorphism depending on e_{12} and $a \in R^*$. Let z =diag$(1, a^{-1})$, then $\theta_z^{-1}\varphi_2(e_{ij}) = e_{ij}, 1 \leq i \leq j \leq 2$, which mean $\theta_z^{-1}\eta_f^{-1}\varphi_{e_{11}}^{-1}\varphi = 1$, that is $\varphi = \varphi_{e_{11}}\eta_f\theta_z$. So φ is an automorphism.

3 Local Derivation

In [14], Wang and Yu characterized the derivations of $T_n(R)$ by the following lemma. Before giving this lemma, we first introduce two standard derivations of $T_n(R)$.

(A) Inner derivations

Let $t \in T_n(R)$, then ad $t : x \mapsto [t, x], x \in T_n(R)$ is a derivation of $T_n(R)$, which is called an inner derivation of $T_n(R)$ induced by t.

(B) Central derivations

We denote by Hom$(D_n(R), R)$ the set of all R-module homomorphisms from $D_n(R)$ to R. For any $\sigma \in$Hom$(D_n(R), R)$, σ may be extended to a derivation η_σ of $T_n(R)$ by:

$\eta_\sigma(d + x) = \sigma(d)e$ for all $d \in D_n(R), x \in \mathbf{n}$. η_σ is called a central derivation of $T_n(R)$ induced by σ.

Lemma 3.1 (the theorem of [14]) Let R be a commutative ring with identity. Then

(1) every derivation of $T_n(R)$ can be uniquely written as the sum of an inner derivation and a central derivation when $n \geq 2$.

(2) every derivation of $T_n(R)$ is a central derivation when $n = 1$.

In order to achieve our goal, we also need other lemmas.

Lemma 3.2 Let δ be a local derivation of $T_n(R), n \geq 2$. If $\delta(e_{11}) = 0$, then there exist an inner derivation ad $m = \sum_{j=2}^{n}$ad m_j and a central derivation η_σ such that $(\delta - \text{ad } m - \eta_\sigma)(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$.

Proof By the definition of δ and Lemma 3.1, there exists a derivation $\delta_{e_{22}} = \text{ad } t_2 + \eta_{\sigma_2}$, corresponding to e_{22}, such that

$$\delta(e_{11} + e_{22}) = \delta(e_{11}) + \delta(e_{22}) = 0 + \delta_{e_{22}}(e_{22}) = \text{ad } t_2(e_{22}) + \sigma_2(e_{22})e. \tag{3.1}$$

On the other hand, there is a derivation $\delta_{e_{11}+e_{22}}$=ad $s_2 + \eta_{\alpha_2}$, depending on $e_{11} + e_{22}$, such that

$$\delta(e_{11} + e_{22}) = \delta_{e_{11}+e_{22}}(e_{11} + e_{22}) = \text{ad } s_2(e_{11} + e_{22}) + \alpha_2(e_{11} + e_{22})e. \tag{3.2}$$

Suppose $t_2 = (t_{ij}^{(2)})_{n\times n}$, from (3.1) and (3.2), we have $t_{12}^{(2)} = 0$. Let $m_2 = (m_{ij}^{(2)})_{n\times n}$, where $m_{ij}^{(2)} = t_{ij}^{(2)}$ for $2 \leq i \leq j \leq n$, and $m_{1j}^{(2)} = 0$ for $1 \leq j \leq n$. Then $(\delta - \text{ad } m_2)(e_{11}) = 0$ and $(\delta - \text{ad } m_2)(e_{22}) = \sigma_2(e_{22})e$. Denote $\delta - \text{ad } m_2$ by δ_1.

By induction we assume that there are ad $m_j, j = 3, 4, \cdots, k-1$ such that

$$(\delta_1 - \sum_{j=3}^{k-1} \text{ad } m_j)(e_{11}) = 0 \text{ and } (\delta_1 - \sum_{j=3}^{k-1} \text{ad } m_j)(e_{ii}) = \sigma_i(e_{ii})e,$$

where $\sigma_i \in$Hom $(D_n(R), R), i = 2, \cdots, k-1$. Denote $\delta_1 - \sum_{j=3}^{k-1}$ ad m_j by δ_{k-2}. It is obvious that δ_{k-2} is also a local derivation. By Lemma 3.1, there exist an inner derivation ad t_k and a central derivation η_{σ_k}, depending on e_{kk}, such that

$$\begin{aligned} & \delta_{k-2}(e_{11} + e_{22} + \cdots + e_{kk}) \\ = \ & \sigma_2(e_{22})e + \cdots + \sigma_{k-1}(e_{k-1,k-1})e + \text{ad } t_k(e_{kk}) + +\sigma_k(e_{kk})e. \end{aligned} \tag{3.3}$$

On the other hand, since δ_{k-2} is a local derivation, we have

$$\begin{aligned} & \delta_{k-2}(e_{11} + e_{22} + \cdots + e_{kk}) \\ = \ & \text{ad } s_k(e_{11} + e_{22} + \cdots + e_{kk}) + \alpha_k(e_{11} + e_{22} + \cdots + e_{kk})e, \end{aligned} \tag{3.4}$$

where $s_k \in T_n(R)$ and $\alpha_k \in$Hom $(D_n(R), R)$, depending on $e_{11} + e_{22} + \cdots + e_{kk}$. Suppose $t_k = (t_{ij}^{(k)})_{n\times n}$. By (3.3) and (3.4), we have $t_{jk}^{(k)} = 0$ for $1 \leq j \leq k-1$. Let $m_k = (m_{ij}^{(k)})_{n\times n}$, where $m_{ij}^{(k)} = t_{ij}^{(k)}$ for $k \le i \le j \le n$, and $m_{st}^{(k)} = 0$ for $1 \le s \le k-1, s \le t \le n$. Then $(\delta_{k-2} - \text{ad } m_k)(e_{11}) = 0$ and $(\delta_{k-2} - \text{ad } m_k)(e_{ii}) = \sigma_i(e_{ii})e$ for $2 \le i \le k$.

When $k = n$, let $m = \sum_{j=2}^{n} m_j$. Then

$$(\delta - \text{ad } m)(e_{11}) = 0, \text{ and } (\delta - \text{ad } m)(e_{ii}) = \sigma_i(e_{ii})e \text{ for } i = 2, 3, \cdots, n.$$

Let σ be an $R-$linear map from $D_n(R)$ to R, and define $\sigma(e_{11}) = 0, \sigma(e_{ii}) = \sigma_i(e_{ii})$ for $i = 2, 3, \cdots, n$. Then $\sigma \in$Hom $(D_n(R), R)$ and $(\delta - \text{ad } m - \eta_\sigma)(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$.

Lemma 3.3 Let δ be a local derivation of $T_n(R)$ satisfying $\delta(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$. Then $\delta(e_{ij}) = a_{ij}e_{ij}$ for some $a_{ij} \in R$ and $1 \le i < j \le n$.

Proof For $e_{ii} + e_{ij}, j \ne i$, since δ is a local derivation, from Lemma 3.1 we know there exist an inner derivation ad x_{ij} and a central derivation $\eta_{\gamma_{ij}}$, depending on $e_{ii} + e_{ij}$, such that

$$\delta(e_{ii} + e_{ij}) = (\text{ad } x_{ij} + \eta_{\gamma_{ij}})(e_{ii} + e_{ij}) = \text{ad } x_{ij}(e_{ii} + e_{ij}) + \gamma_{ij}(e_{ii} + e_{ij})e. \tag{3.5}$$

On the other hand, by the definition of δ and Lemma 3.1, we have

$$\delta(e_{ii} + e_{ij}) = \delta(e_{ii}) + \delta(e_{ij}) = \delta(e_{ij}) = \text{ad } p_{ij}(e_{ij}) \in \mathbf{n}, \tag{3.6}$$

where $p_{ij} \in T_n(R)$ depending on e_{ij}. By (3.5) and (3.6), we have

$$\text{ad } x_{ij}(e_{ii} + e_{ij}) = \text{ad } p_{ij}(e_{ij}). \tag{3.7}$$

Similarly, there exists some $y_{ij} \in T_n(R)$ such that

$$\text{ad } y_{ij}(e_{jj} + e_{ij}) = \text{ad } p_{ij}(e_{ij}). \tag{3.8}$$

(3.7) and (3.8) imply that $\delta(e_{ij}) = a_{ij}e_{ij}$ for some $a_{ij} \in R$ and $1 \le i < j \le n$.

Lemma 3.4 Let δ be a local derivation of $T_n(R)$. If $\delta(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$, then there exists some $h \in T_n(R)$ such that $(\delta - \text{ad } h)(e_{i,i+1}) = 0$ for $i = 1, 2, \cdots, n-1$, and $(\delta - \text{ad } h)(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$.

Proof By Lemma 3.3, we have $\delta(e_{i,i+1}) = a_{i,i+1}e_{i,i+1}$ for some $a_{i,i+1} \in R$. Let

$$h = \text{diag}(0, -a_{12}, -(a_{12} + a_{23}), \cdots, -(a_{12} + a_{23} + \cdots + a_{n-1,n})).$$

Then $(\delta - \text{ad } h)(e_{i,i+1}) = 0$ for $i = 1, 2, \cdots, n-1$, and $(\delta - \text{ad } h)(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$.

Lemma 3.5 Let δ be a local derivation of $T_n(R)$ satisfying $\delta(e_{ii}) = 0$ for $i = 1, 2, \cdots, n$, and $\delta(e_{i,i+1}) = 0$ for $i = 1, 2, \cdots, n-1$. Then we have $\delta(e_{i,i+k}) = 0$ for any $e_{i,i+k} \in T_n(R)$.

Proof We will prove this lemma by induction on $k, k \ge 2$.

When $k = 2$, for $e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}$, since δ is a local derivation, by Lemma 3.1, there exist an inner derivation ad $q_i^{(2)}$ and a central derivation $\eta_{\chi_i^{(2)}}$, depending on $e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}$, such that

$$\begin{aligned} & \delta(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) \\ = \ & \text{ad } q_i^{(2)}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) \\ & + \chi_i^{(2)}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1})e. \end{aligned} \tag{3.9}$$

On the other hand, By Lemma 3.3, we have

$$\delta(e_{i,i+1}+e_{i+1,i+2}+e_{i,i+2}+e_{i+1,i+1})=a_{i,i+2}e_{i,i+2}. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\text{ad } q_i^{(2)}(e_{i,i+1}+e_{i+1,i+2}+e_{i,i+2}+e_{i+1,i+1})=a_{i,i+2}e_{i,i+2},$$

this forces that $a_{i,i+2}=0$, that is to say $\delta(e_{i,i+2})=0$.

By induction we assume that $\delta(e_{i,i+m})=0$ for $m=2,3,\cdots,k-1$. For

$$e_{i,i+1}+e_{i+1,i+k}+e_{i,i+k}+e_{i+1,i+1},$$

similar to the case $k=2$, we can get there exists some $q_i^{(k)}\in T_n(R)$ such that

$$\text{ad } q_i^{(k)}(e_{i,i+1}+e_{i+1,i+k}+e_{i,i+k}+e_{i+1,i+1})=a_{i,i+k}e_{i,i+k},$$

which means that $a_{i,i+k}=0$. So $\delta(e_{i,i+k})=0$ for any $e_{i,i+k}\in T_n(R)$.

By those lemmas, we can prove the following theorem.

Theorem 3.1 Let R be a commutative ring with identity, $T_n(R)$ the Lie algebra consisting of all upper triangular $n\times n$ matrices over R. Then every local derivation δ of $T_n(R)$ is a derivation.

Proof Let δ be a local derivation of $T_n(R)$. When $n\geq 2$, for $e_{11}\in T_n(R)$, there exists a derivation $\delta_{e_{11}}$, depending on e_{11}, such that $\delta(e_{11})=\delta_{e_{11}}(e_{11})$. So $(\delta-\delta_{e_{11}})(e_{11})=0$. Clearly, $\delta-\delta_{e_{11}}$ is also a local derivation of $T_n(R)$. By Lemmas 3.2–3.5, we know there exist η_σ, ad m and ad h such that

$$(\delta-\text{ad } m-\eta_\sigma-\text{ad } h)(e_{ij})=0 \text{ for } 1\leq i\leq j\leq n,$$

which imply that $\delta=\text{ad } m+\eta_\sigma+\text{ad } h$, so δ is a derivation.

When $n=1$, suppose that $\delta(1)=b$, then for any $x\in T_1(R)=R$, we have

$$\delta(x)=x\delta(1)=xb=\eta_\sigma(x),$$

where $\sigma: R\to R, x\mapsto bx$ is an R-linear from R to R. So δ is a derivation.

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可换环上上三角矩阵李代数的局部自同构和局部导子

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摘要: 本文刻画了$T_n(R)$上的局部自同构和局部导子. 利用关于$T_n(R)$的自同构和导子的主要结果和矩阵计算技巧, 本文证明了$T_n(R)$上的每一个局部自同构是自同构, 每一个局部导子是导子, 这推广了文献关于$T_n(R)$的自同构和导子的主要结果.

关键词: 局部自同构; 局部导子; 上三角矩阵李代数；可换环

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