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A CHARACTERIZATION OF THE BIHOLOMORPHISMS BETWEEN EQUIDIMENSIONAL CARTAN-HARTOGS DOMAINS

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Abstract: The holomorphic mappings F between equidimensional Cartan-Hartogs domains are considered. If a Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ is not the unit ball, then there is a function X on $\Omega^{B^m}(\mu)$ such that any holomorphic automorphism of $\Omega^{B^m}(\mu)$ leaves the function X on $\Omega^{B^m}(\mu)$ invariant. By direct calculations, we obtain that if a holomorphic mapping F between equidimensional Cartan-Hartogs domains leaves the functions X invariant, then F must be a biholomorphism. As a consequence of our result, if a Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ is not the unit ball, then, for any holomorphic self-mapping F on $\Omega^{B^m}(\mu)$, we have that F is a holomorphic automorphism of $\Omega^{B^m}(\mu)$ if and only if F leaves the function X on $\Omega^{B^m}(\mu)$ invariant.

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1 Introduction

The Cartan-Hartogs domains are defined as a class of Hartogs type domains over irreducible bounded symmetric domains. For an irreducible bounded symmetric domain $\Omega \subset \mathbb{C}^d$ in its Harish-Chandra realization, a positive integer number m and a positive real number μ, the Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ is defined by

$$\Omega^{B^m}(\mu) := \left\{(z,w) \in \Omega \times \mathbb{C}^m \subset \mathbb{C}^d \times \mathbb{C}^m : \|w\|^2 < N_\Omega(z,\overline{z})^\mu\right\}, \tag{1.1}$$

where $\|\cdot\|$ is standard Hermitian norm in $\mathbb{C}^m$. Note $\Omega \times \{0\} \subset \Omega^{B^m}(\mu)$ and $b\Omega \times \{0\} \subset b\Omega^{B^m}(\mu)$ (where bD denotes the boundary of the domain D). Obviously, each Cartan-Hartogs domain is a bounded complete circular domain.

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$. Let $\mathrm{Aut}(\Omega^{B^m}(\mu))$ be the holomorphic automorphism group of $\Omega^{B^m}(\mu)$. Let the family $G(\Omega^{B^m}(\mu))$ ($\subset \mathrm{Aut}(\Omega^{B^m}(\mu))$)

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be exactly the set of all mappings Φ (see [16]):

$$\Phi(z,w)=\left(\varphi(z),U(w)\frac{N_\Omega(z_0,\bar{z}_0)^{\mu/2}}{N_\Omega(z,\bar{z}_0)^{\mu}}\right) \tag{1.2}$$

for $(z,w)\in\Omega^{B^m}(\mu)$, where $\varphi\in\operatorname{Aut}(\Omega)$, U is a unitary transformation of $\mathbb{C}^m$, and $z_0=\varphi^{-1}(0)$. Then $G(\Omega^{B^m}(\mu))$ is a subgroup of $\operatorname{Aut}(\Omega^{B^m}(\mu))$ (see [1], Proposition 2.1). Obviously, as indicated in [16], every element of $G(\Omega^{B^m}(\mu))$ preserves the set $\Omega\times\{0\}$ $(\subset\Omega^{B^m}(\mu))$ and $G(\Omega^{B^m}(\mu))$ is transitive on $\Omega\times\{0\}$ $(\subset\Omega^{B^m}(\mu))$.

The Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ is homogeneous if and only if Ω is the unit ball in $\mathbb{C}^d$ and $\mu=1$ (see [1], Lemma 3.1), that is, $\Omega^{B^m}(\mu)$ must be the unit ball in this case. Therefore, with the exception of the unit ball which is obviously homogeneous, each Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ is a nonhomogeneous bounded domain. For the general reference of Cartan-Hartogs domains, see [1, 5–9, 11, 15–17], and references therein.

In 2012, by using the ball characterization theorem about noncompact automorphism groups (i.e., the Wong-Rosay theorem), Ahn-Byun-Park [1] proved the following theorem for irreducible bounded symmetric domains Ω of classical types.

Theorem 1.1 (see [1]) Let Ω be an irreducible bounded symmetric domain. If the Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ is not the unit ball, then $\operatorname{Aut}(\Omega^{B^m}(\mu))$ coincides with $G(\Omega^{B^m}(\mu))$ (see (1.2) for the definition).

Remark Let the Cartan-Hartogs domain $\Omega^{B^m}(\mu)$ be the unit ball. Since every element of $G(\Omega^{B^m}(\mu))$ preserves the set $\Omega\times\{0\}$ $(\subset\Omega^{B^m}(\mu))$ and the unit ball is homogeneous, we have $G(\Omega^{B^m}(\mu))\subsetneqq\operatorname{Aut}(\Omega^{B^m}(\mu))$ in this case.

In 2006, Wang-Yin-Zhang-Roos [16] proved the following result.

Theorem 1.2 (see [16]) Let $\Omega^{B^m}(\mu)$ be a Cartan-Hartogs domain. Let $X:\Omega^{B^m}(\mu)\to[0,1)$ be the function defined by

$$X=\frac{\|w\|^2}{N_\Omega(z,\overline{z})^{\mu}}. \tag{1.3}$$

If F is an automorphisms of $\Omega^{B^m}(\mu)$ with $X(F(z,w))\equiv X(z,w)$ on $\Omega^{B^m}(\mu)$, then $F\in G(\Omega^{B^m}(\mu))$.

In this paper, we prove the following conclusion.

Theorem 1.3 Let Ω_1,Ω_2 be two equidimensional irreducible bounded symmetric domains. Define

$$X_1(z,w):=\frac{\|w\|^2}{N_{\Omega_1}(z,\overline{z})^{\mu_1}}\ ((z,w)\in\Omega_1^{B^m}(\mu_1)),\ \ X_2(z,w):=\frac{\|w\|^2}{N_{\Omega_2}(z,\overline{z})^{\mu_2}}\ ((z,w)\in\Omega_2^{B^m}(\mu_2))$$

for Cartan-Hartogs domains $\Omega_1^{B^m}(\mu_1),\Omega_2^{B^m}(\mu_2)$, respectively. Let $\mathbf{U}\subset\Omega_1^{B^m}(\mu_1)$ be a neighborhood of the origin in $\Omega_1^{B^m}(\mu_1)$. Suppose that $F:\mathbf{U}\to\Omega_2^{B^m}(\mu_2)$ is a holomorphic mapping with $X_2(F(z,w))\equiv X_1(z,w)$ on $\mathbf{U}$. Then there exists a holomorphic automorphism $\Phi\in G(\Omega_2^{B^m}(\mu_2))$ such that $\Phi\circ F$ is the restriction on $\mathbf{U}$ of the standard linear isomorphism

$$\Phi_0(z,w)=(\mathcal{A}(z),U(w)), \tag{1.4}$$

where $\mathcal{A} : \mathbb{C}^d \mapsto \mathbb{C}^d$ is a complex linear isomorphism of $\mathbb{C}^d$ with $\mathcal{A}(\Omega_1) = \Omega_2$, and U is a unitary transformation of $\mathbb{C}^m$.

Theorem 1.3 obviously implies the following corollaries.

Corollary 1.4 Let $\Omega^{B^m}(\mu)$ be a Cartan-Hartogs domain. Let $X : \Omega^{B^m}(\mu) \to [0, 1)$ be the function defined by (1.3). If F is a holomorphic self-mapping of $\Omega^{B^m}(\mu)$, then $F \in G(\Omega^{B^m}(\mu))$ if and only if $X(F(z, w)) \equiv X(z, w)$ on $\Omega^{B^m}(\mu)$.

Combining Theorem 1.1 and Corollary 1.4, we immediately have the following result.

Corollary 1.5 Let $\Omega^{B^m}(\mu)$ be a Cartan-Hartogs domain. Let $X : \Omega^{B^m}(\mu) \to [0, 1)$ be the function defined by (1.3). Assume that $\Omega^{B^m}(\mu)$ is not the unit ball. If F is a holomorphic self-mapping of $\Omega^{B^m}(\mu)$, then $F \in \mathrm{Aut}(\Omega^{B^m}(\mu))$ if and only if $X(F(z, w)) \equiv X(z, w)$ on $\Omega^{B^m}(\mu)$.

The paper is organized as follows. In next section, we collect basic material about classical domains, and we will prove two lemmas which are necessary for the proof of Theorem 1.3. Finally Section 3 is dedicated to the proof of Theorem 1.3.

2 Preliminaries

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$ with the rank r in its Harish-Chandra realization. The space of holomorphic polynomials on $\mathbb{C}^d$ can be decomposed into irreducible subspaces under the action of the isotropy group of the bounded symmetric domain Ω in $\mathbb{C}^d$. Let $\mathcal{K}$ be the connected component of the identity in the Lie group of the (complex linear) automorphisms of Ω leaving 0 fixed. Under the action $f \mapsto f \circ k$ $(k \in \mathcal{K})$ of $\mathcal{K}$, the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^d$ admits the Peter-Weyl decomposition (see Th. 2.1 in [3] for references)

$$\mathcal{P} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}, \tag{2.1}$$

where the summation is taken over all partitions $\mathbf{m} := (m_1, m_2, \cdots, m_r)$ of nonnegative integers such that $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, and the spaces $\mathcal{P}_{\mathbf{m}}$ are $\mathcal{K}$-invariant and irreducible. For each $\mathbf{m}$, we have $\mathcal{P}_{\mathbf{m}} \subset \mathcal{P}_{|\mathbf{m}|}$, where $\mathcal{P}_{|\mathbf{m}|}$ is the space of homogeneous holomorphic polynomials on $\mathbb{C}^d$ of degree $|\mathbf{m}|(:= \sum_{j=1}^{r} m_j)$ (Obviously, $\mathcal{P}_{|\mathbf{m}|}$ is a $\mathcal{K}$-invariant subspace of $\mathcal{P}$). Let

$$\langle f, g\rangle := \int_{\mathbb{C}^d} f(z)\overline{g(z)} e^{-m(z,\overline{z})} \frac{(\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} m(z, \overline{z}))^d}{d!} \tag{2.2}$$

be an inner product on the space $\mathcal{P}$, where $m(z, \overline{z}) := - \left.\frac{\partial N_\Omega(tz,\overline{z})}{\partial t}\right|_{t=0} = zC_\Omega \overline{z}^t$ (where C_Ω is a positive definite Hermite matrix, see (2.10) for the details). For every partition $\mathbf{m}$, let $K_{\mathbf{m}}(z, \overline{z})$ be the reproducing kernel of $\mathcal{P}_{\mathbf{m}}$ with respect to (2.2). Since $\mathcal{P}_{\mathbf{m}}(\subset \mathcal{P}_{|\mathbf{m}|})$ is of finite demension, by definition $K_{\mathbf{m}}(\cdot, \cdot)$ is homogeneous holomorphic polynomials on $\mathbb{C}^d \times \mathbb{C}^d$ of bidegrees $(|\mathbf{m}|, |\mathbf{m}|)$.

Let Ω be an irreducible bounded symmetric domainin $\mathbb{C}^d$ with rank r in its Harish-Chandra realization. Then there exists the Jordan triple product on $\mathbb{C}^d$ associated with the

Bergman kernel of Ω and the space $\mathbb{C}^d$ endowed with the triple product is a simple Hermitian positive Jordan triple system (e.g., see Appendix A in [16]). Let $e_1, e_2, \cdots, e_r \in \mathbb{C}^d$ be a frame for $\mathbb{C}^d$. Then each $z \in \mathbb{C}^d$ has the spectral decomposition (see Th. VI. 2.3 and Def. VI. 2.2 in [4], p. 512–513, and Def. VI. 2.3 and Prop. VI. 2.6 in [4], p. 515–516)

$$z = k(z) \cdot (\lambda_1(z)e_1 + \lambda_2(z)e_2 + \cdots + \lambda_r(z)e_r), \tag{2.3}$$

where $k(z) \in \mathcal{K},\ \lambda_1(z) \geq \lambda_2(z) \geq \cdots \geq \lambda_r(z) \geq 0$. The spectral norm of z is defined by

$$|||z||| := \lambda_1(z). \tag{2.4}$$

Then we have (see Def. VI.4.1 and Prop. VI.4.2 of [4], p. 524)

$$\Omega = \{z \in \mathbf{C}^d : |||z||| < 1\}.$$

Therefore, we have $1 > \lambda_1(z) \geq \lambda_2(z) \geq \cdots \geq \lambda_r(z) \geq 0$ for any $z \in \Omega$.

The generic minimal polynomial of $\mathbb{C}^d$ (see Prop. VI. 2.6 and its proof in [4], p. 515–517)

$$m(t, z_1, \overline{z_2}) = t^r - m_1(z_1, \overline{z_2})t^{r-1} + \cdots + (-1)^r m_r(z_1, \overline{z_2})$$

satisfies

$$m(t, z, \overline{z}) = \prod_{j=1}^{r}(t - \lambda_j^2(z)),$$

where $m_1(\cdot,\cdot), \cdots, m_r(\cdot,\cdot)$ on $\mathbb{C}^d \times \mathbb{C}^d$ are homogeneous holomorphic polynomials of, respectively, bidegrees $(1,1), \cdots, (r,r)$, $z = k(z) \cdot (\lambda_1(z)e_1 + \lambda_2(z)e_2 + \cdots + \lambda_r(z)e_r)$ is the spectral decomposition of z. The generic norm N_Ω is defined by

$$N_\Omega(z_1, \overline{z_2}) = m(1, z_1, \overline{z_2}).$$

Then

$$N_\Omega(z, \overline{z}) = \prod_{j=1}^{r}(1 - \lambda_j^2(z)). \tag{2.5}$$

Note that, by definition, $N_\Omega(\cdot,\cdot)$ is a holomorphic polynomials on $\mathbb{C}^d \times \mathbb{C}^d$. The generic norm N_Ω is related to the kernels $K_{\mathbf{m}}$ by the formula (see Th. 3.8 in [3])

$$N_\Omega(z_1, \overline{z_2})^{-s} = \sum_{\mathbf{m}}(s)_{\mathbf{m}} K_{\mathbf{m}}(z_1, \overline{z_2}) \ \ (s \in \mathbb{C},\ z_1, z_2 \in \Omega), \tag{2.6}$$

where the series converges uniformly and absolutely on compact subsets of $\Omega \times \Omega$, and $(s)_{\mathbf{m}}$ denotes the generalized Pochhammer symbol

$$(s)_{\mathbf{m}} := \prod_{j=1}^{r}\left(s - \frac{j-1}{2}a\right)_{m_j},\ (x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\cdot\cdots\cdot(x+k-1). \tag{2.7}$$

By using logarithmic expansion, (2.5) implies the formula

$$\ln N_{\Omega}(z,\overline{z}) = -\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{r}\lambda_j^{2k}(z) \ \ (z\in\Omega). \tag{2.8}$$

Using the spectral decomposition of z in (2.3), $K_{\mathbf{m}}$ can be rewritten as (see Lemma 3.2 in [3], p. 235 for references)

$$K_{\mathbf{m}}(z,\overline{z}) = K_{\mathbf{m}}(\sum_{j=1}^{r}\lambda_j^2(z)e_j,\overline{e}) \ \ (e=\sum_{j=1}^{r}e_j).$$

Let

$$e_k(t_1,\cdots,t_r) = \sum_{1\le i_1<i_2<\cdots<i_k\le r} t_{i_1}t_{i_2}\cdots t_{i_r} \ (1\le k\le r)$$

be elementary symmetric polynomials. For a partition $\mathbf{m}=(1^{k_1}2^{k_2}\cdots r^{k_r})$, where k_i is the number of parts of $\mathbf{m}$ that are equal to i for $i\ge 1$. We define

$$e_{\mathbf{m}} = \prod_{j=1}^{r} e_1^{k_1}e_2^{k_2}\cdots e_r^{k_2}.$$

Then the set $\{e_{\mathbf{m}} : |\mathbf{m}|=n\}$ is a basis of a space consist of all symmetric polynomials of degree n in variables $t_1,t_2,\cdots,t_r$, $n\in\mathbb{N}$ (see [13] page 21). Thus, there exist constants $c_{\mathbf{m}}$ such that

$$\sum_{j=1}^{r}t_j^k = \sum_{|\mathbf{m}|=k} c_{\mathbf{m}}e_{\mathbf{m}}(t_1,\cdots,t_r)$$

for $k\ge 1, k\in\mathbb{N}$.

For $z=k\cdot(\lambda_1(z)e_1+\lambda_2(z)e_2+\cdots+\lambda_r(z)e_r)$, coefficients $m_k(z,\overline{z})$ of the generic minimal polynomial $m(t,z,\overline{z})$ may be written as

$$m_k(z,\overline{z}) = \sum_{1\le i_1<i_2<\cdots<i_k\le r}\lambda_{i_1}^2(z)\lambda_{i_2}^2(z)\cdots\lambda_{i_r}^2(z) = e_k(\lambda_1^2(z),\lambda^2(z),\cdots,\lambda_r^2(z)) \ (1\le k\le r).$$

Therefore

$$\sum_{j=1}^{r}\lambda_j^{2k}(z) = \sum_{\substack{\mathbf{m}=(1^{k_1}2^{k_2}\cdots r^{k_r}) \\ |\mathbf{m}|=\sum_{j=1}^{r}jk_j=k}} c_{\mathbf{m}}m_1^{k_1}(z,\overline{z})m_2^{k_2}(z,\overline{z})\cdots m_r^{k_r}(z,\overline{z}).$$

This means that there exist constants $c_{\alpha\beta}$ such that

$$\sum_{j=1}^{r}\lambda_j^{2k}(z) = \sum_{|\alpha|=|\beta|=k} c_{\alpha\beta}z^{\alpha}\overline{z}^{\beta}, \tag{2.9}$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_d)$, $\alpha_i\in\mathbb{N}, 1\le i\le d$, $|\alpha|=\sum\limits_{i=1}^{d}\alpha_i$ and $z^{\alpha}=\prod\limits_{i=1}^{d}z_i^{\alpha_i}$.

Let $z = k \cdot (\lambda_1(z)e_1 + \lambda_2(z)e_2 + \cdots + \lambda_r(z)e_r)$ be the spectral decomposition of z. For give $t \geq 0$, the spectral decomposition of tz is $tz = k \cdot (t\lambda_1(z)e_1 + t\lambda_2(z)e_2 + \cdots + t\lambda_r(z)e_r)$. From (2.5) and (2.6), we obtain

$$N_\Omega(tz, \overline{tz}) = \prod_{j=1}^{r}(1 - t^2\lambda_j^2(z)) = 1 - t^2\sum_{j=1}^{r}\lambda_j^2(z) + \cdots$$

and

$$N_\Omega(tz, \overline{tz}) = \sum_{\mathbf{m}}(-1)_{\mathbf{m}}K_{\mathbf{m}}(tz, \overline{tz}) = \sum_{\mathbf{m}}(-1)_{\mathbf{m}}t^{2|\mathbf{m}|}K_{\mathbf{m}}(z, \overline{z}) = 1 - t^2K_{(1,0,\cdots,0)}(z, \overline{z}) + \cdots$$

(Note the r-tuples $\mathbf{m}$ in (2.1) with $|\mathbf{m}| = 1$ if and only if $\mathbf{m} = (1, 0, \cdots, 0)$). Since $K_{(1,0,\cdots,0)}(\cdot, \cdot)$ is homogeneous holomorphic polynomials on $\mathbb{C}^d \times \mathbb{C}^d$ of bidegrees $(1, 1)$, then

$$\sum_{j=1}^{r}\lambda_j^2(z) = K_{(1,0,\cdots,0)}(z, \overline{z}) \equiv zC_\Omega\overline{z}^t, \tag{2.10}$$

where C_Ω is a Hermite matrix. Since $K_{(1,0,\cdots,0)}(z, \overline{z}) \geq 0$ and $K_{(1,0,\cdots,0)}(z, \overline{z}) = 0$ iff $z = 0$ by the definition, we get that C_Ω is a positive definite Hermite matrix.

Lemma 2.1 Let Ω_1 and Ω_2 be two irreducible bounded symmetric domains in $\mathbb{C}^d$ in their Harish-Chandra realization. Suppose that $\phi: \Omega_1 \to \Omega_2$ is a holomorphic mapping such that

$$N_{\Omega_1}(z, \overline{z}) \equiv N_{\Omega_2}(\phi(z), \overline{\phi(z)})^\mu \quad (z \in \Omega_1), \tag{2.11}$$

where $\mu > 0$. Then ϕ is a biholomorphic mapping of Ω_1 onto Ω_2 with $\phi(0) = 0$ and $\mu = 1$. Therefore, ϕ must be a complex linear automorphism of $\mathbb{C}^d$ with $\phi(\Omega_1) = \Omega_2$.

Proof Let $z = (z_1, z_2, \cdots, z_d)$, $w = \phi(z) = (w_1(z), w_2(z), \cdots, w_d(z))$. Assume that K_i, γ_i and V_i are the Bergman kernel, the genus and the volume of Ω_i $(1 \leq i \leq 2)$.

Combining $N(z, \overline{z}) = 0$ iff $z = 0$ and (2.11), we obtain $\phi(0) = 0$.

Using (2.11), we have

$$\left(\frac{\partial^2 \ln N_{\Omega_1}}{\partial z_i \partial \overline{z_j}}(z, \overline{z})\right)_{1\leq i,j\leq d} = \mu\phi'(z)\left(\frac{\partial^2 \ln N_{\Omega_2}}{\partial w_i \partial \overline{w_j}}(\phi(z), \overline{\phi(z)})\right)_{1\leq i,j\leq d}\overline{\phi'(z)}^t,$$

where

$$\phi'(z) = \begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_2}{\partial z_1} & \cdots & \frac{\partial w_d}{\partial z_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial w_1}{\partial z_d} & \frac{\partial w_2}{\partial z_d} & \cdots & \frac{\partial w_d}{\partial z_d} \end{pmatrix}.$$

Since $K_i = N_{\Omega_i}^{-\gamma_i}/V_i$, we get

$$\frac{\partial^2 \ln K_1}{\partial z_i \partial \overline{z_j}} = -\gamma_1\frac{\partial^2 \ln N_{\Omega_1}}{\partial z_i \partial \overline{z_j}}, \quad \frac{\partial^2 \ln K_2}{\partial w_i \partial \overline{w_j}} = -\gamma_2\frac{\partial^2 \ln N_{\Omega_2}}{\partial w_i \partial \overline{w_j}}.$$

Since $\left(\frac{\partial^2 \ln K_1}{\partial z_i \partial \overline{z_j}}\right)_{1\le i,j\le d}$ and $\left(\frac{\partial^2 \ln K_2}{\partial w_i \partial \overline{w_j}}\right)_{1\le i,j\le d}$ are positive definite Hermite matrices, we obtain that $\left(\frac{\partial^2 \ln N_{\Omega_1}}{\partial z_i \partial \overline{z_j}}\right)_{1\le i,j\le d}$ and $\left(\frac{\partial^2 \ln N_{\Omega_2}}{\partial w_i \partial \overline{w_j}}\right)_{1\le i,j\le d}$ are negative definite Hermite matrices. Therefore $\phi'(z)$ is an invertible matrix for any $z \in \Omega_1$.

Now we show that ϕ is a proper holomorphic mapping between Ω_1 and Ω_2. In fact, if there exists a sequence $\{p_j\}$ in Ω_1 such that $p_j \to p_0 \in b\Omega_1$ ($b\Omega$ stand for the boundary of Ω in $\mathbb{C}^d$) and $q_j = \phi(p_j) \to q_0 \in \Omega_2$. Then (2.11) implies

$$N_{\Omega_1}(p_0, \overline{p_0}) = N_{\Omega_2}(q_0, \overline{q_0})^{\mu}.$$

This is a contradiction with $N_{\Omega_1}(p_0, \overline{p_0}) = 0,\ 0 < N_{\Omega_2}(q_0, \overline{q_0}) \le 1$. Therefore, ϕ must be a proper holomorphic mapping between Ω_1 and Ω_2.

Since the irreducible bounded symmetric domains Ω_1 is simply connected, we have that ϕ is a biholomorphic mapping between Ω_1 and Ω_2. Since $\phi(0) = 0$, from the Cartan's theorem, ϕ is a complex linear automorphism of $\mathbb{C}^d$ with $\phi(\Omega_1) = \Omega_2$.

Finally, we show that $\mu = 1$. Since ϕ is a holomorphically isomorphism of Ω_1 onto Ω_2, we have Ω_1 and Ω_2 have the same genus $\gamma (:= \gamma_1 = \gamma_2)$. Let $\phi(z) = zA$ $(z \in \mathbb{C}^d)$ is the complex linear automorphism of $\mathbb{C}^d$ with $\phi(\Omega_1) = \Omega_2$ (where A is an invertible $d \times d$ matrix). Then we have

$$V_2 = |\det A|^2 V_1, \quad K_1(z, \bar{z}) = |\det A|^2 K_2(zA, \overline{zA}).$$

Thus, from $K_i(z, \bar{z}) = N_{\Omega_i}(z, \bar{z})^{-\gamma}/V_i$ $(1 \le i \le 2)$, we get

$$N_{\Omega_1}(z, \bar{z}) \equiv N_{\Omega_2}(zA, \overline{zA}) \ \ (z \in \Omega_1).$$

Since $N_{\Omega_1}(z, \bar{z})$ $(z \in \Omega_1)$ takes any number in (0, 1], we have $\mu = 1$ by (2.11). This proves Lemma 2.1.

Lemma 2.2 Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $\beta = (\beta_1, \cdots, \beta_m)$ be tuples of non-negative integers. For $z \in \mathbb{C}^d$, $w \in \mathbb{C}^m$, set

$$P_N(z) = \sum_{|\alpha|=N} d_\alpha z^\alpha, \tag{2.12}$$

$$Q_N(z, w) = \sum_{\substack{|\alpha|+|\beta|=N \\ |\alpha|\ge 1, |\beta|\ge 1}} e_{\alpha\beta} z^\alpha w^\beta, \tag{2.13}$$

$$R_N(z, w) = \sum_{\substack{|\alpha|+|\beta|=N \\ |\alpha|\ge 1, |\beta|\ge 1}} f_{\alpha\beta} z^\alpha w^\beta, \tag{2.14}$$

$$S_{2k}(z, \bar{z}) = \sum_{\substack{|\alpha|=k \\ |\beta|=k}} g_{\alpha\beta} z^\alpha \bar{z}^\beta, \tag{2.15}$$

where $N \ge 1$, $k \ge 1$, d_α and $e_{\alpha\beta}$ are d-dimensional row vectors, $f_{\alpha\beta}$ are m-dimensional row vectors, and $g_{\alpha\beta}$ are complex numbers. Assume that A is an invertible matrix of order d, B is an invertible matrix of order m, and $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^k$ ($k = m$ or d).

(i) If

$$2\mathrm{Re}\langle wB, R_2(z,w)\rangle \equiv 0 \tag{2.16}$$

and

$$2\mathrm{Re}\langle wB, R_3(z,w)\rangle + \|R_2(z,w)\|^2 + \|w\|^2 S_2(z,\overline{z}) \equiv 0, \tag{2.17}$$

then

$$R_2(z,w) \equiv 0,\ R_3(z,w) \equiv 0,\ S_2(z,\overline{z}) \equiv 0. \tag{2.18}$$

(ii) Suppos that $N = 2k+1$, $k \geq 2$ and

$$2\mathrm{Re}\langle wB, R_N(z,w)\rangle + 2\mu\|w\|^2\mathrm{Re}\langle zA, P_{N-2}(z) + Q_{N-2}(z,w)\rangle + \frac{1}{k}\|w\|^2 S_{2k}(z,\overline{z}) \equiv 0, \tag{2.19}$$

where $\|w\|^2 \equiv \langle w, w\rangle := \sum\limits_{j=1}^{m} w_j\overline{w_j}$. Then

$$P_{N-2}(z) \equiv 0,\ Q_{N-2}(z,w) \equiv 0,\ R_N(z,w) \equiv 0,\ S_{2k}(z,\overline{z}) \equiv 0. \tag{2.20}$$

(iii) Let $N = 2k$ $(k \geq 2)$. If

$$2\mathrm{Re}\langle wB, R_N(z,w)\rangle + 2\mu\|w\|^2\mathrm{Re}\langle zA, P_{N-2}(z) + Q_{N-2}(z,w)\rangle \equiv 0, \tag{2.21}$$

then

$$P_{N-2}(z) \equiv 0,\ Q_{N-2}(z,w) \equiv 0,\ R_N(z,w) \equiv 0. \tag{2.22}$$

Proof We only prove (2.20) here (the proof of (2.18) and (2.22) are the same as that of (2.20)).

Let $wB = \sum\limits_{j=1}^{m} \varepsilon_j w_j$, $zA = \sum\limits_{j=1}^{d} \eta_j z_j$, where $\{\varepsilon_j : 1 \leq j \leq m\}$ and $\{\eta_j : 1 \leq j \leq d\}$ are bases of $\mathbb{C}^m$ and $\mathbb{C}^d$, respectively. Since

$$\begin{aligned}
2\mathrm{Re}\langle wB, R_N(z,w)\rangle &= \sum_{1\leq j\leq m}\ \sum_{\substack{|\alpha|+|\beta|=N\\ |\alpha|\geq 1, |\beta|\geq 1}} \left\{\langle \varepsilon_j, f_{\alpha\beta}\rangle w_j \overline{z}^\alpha \overline{w}^\beta + \langle f_{\alpha\beta}, \varepsilon_j\rangle \overline{w_j} z^\alpha w^\beta\right\},\\
&= \|w\|^2\mathrm{Re}\langle zA, P_{N-2}(z) + Q_{N-2}(z,w)\rangle\\
&= \|w\|^2 \sum_{1\leq j\leq d}\left\{\sum_{|\alpha|=N-2}(\langle \eta_j, d_\alpha\rangle z_j\overline{z}^\alpha + \langle d_\alpha, \eta_j\rangle z^\alpha \overline{z_j})\right.\\
&\qquad \left. + \sum_{\substack{|\alpha|+|\beta|=N-2\\ |\alpha|\geq 1, |\beta|\geq 1}}\left(\langle \eta_j, e_{\alpha\beta}\rangle z_j \overline{z}^\alpha \overline{w}^\beta + \langle e_{\alpha\beta}, \eta_j\rangle z^\alpha w^\beta \overline{z_j}\right)\right\},\\
\|w\|^2 S_{2k}(z,\overline{z}) &= \|w\|^2 \sum_{\substack{|\alpha|=k\\ |\beta|=k}} g_{\alpha\beta} z^\alpha \overline{z}^\beta,
\end{aligned}$$

and sets $\{z^\alpha, \overline{z}^\alpha : 1 \le |\alpha| \le N-1\}$, $\{z_j\overline{z}^\alpha, \overline{z_j}z^\alpha : 1 \le j \le d, |\alpha| = N-2\}$, $\{z_j\overline{z}^\alpha, \overline{z_j}z^\alpha : 1 \le j \le d, 1 \le |\alpha| \le N-3\}$ and $\{z^\alpha\overline{z}^\beta : |\alpha| = |\beta| = k(>1)\}$ are pairwise disjoint, (2.19) implies

$$\sum_{1\le j\le m}\ \sum_{1\le|\beta|=N-|\alpha|} < \varepsilon_j, f_{\alpha\beta} > w_j\overline{w}^\beta \equiv 0 \quad (1 \le |\alpha| \le N-1), \tag{2.23}$$

$$< \eta_j, d_\alpha >= 0 \quad (1 \le j \le d, |\alpha| = N-2), \tag{2.24}$$

$$\sum_{1\le|\beta|=N-2-|\alpha|} < \eta_j, e_{\alpha\beta} > \overline{w}^\beta \equiv 0 \quad (1 \le j \le d, 1 \le |\alpha| \le N-3), \tag{2.25}$$

$$g_{\alpha\beta} = 0 \quad (|\alpha| = k, |\beta| = k). \tag{2.26}$$

Therefore, (2.23) implies

$$< \varepsilon_j, f_{\alpha\beta} >= 0 \quad (1 \le j \le m, |\alpha| + |\beta| = N, |\alpha| \ge 1, |\beta| \ge 1).$$

Since $\{\varepsilon_j : 1 \le j \le m\}$ is a basis of $\mathbb{C}^m$, we have

$$f_{\alpha\beta} = 0 \quad (|\alpha| + |\beta| = N, |\alpha| \ge 1, |\beta| \ge 1),$$

by (2.14), we get

$$R_N(z, w) \equiv 0.$$

Similarly, from (2.24), (2.25) and (2.26), we have $P_{N-2}(z) \equiv 0$, $Q_{N-2}(z, w) \equiv 0$, $S_{2k}(z, \overline{z}) \equiv 0$. This proves Lemma 2.2.

3 Proof of Theorem 1.3

Proof We divide our proof into four steps.

(i) Let $F(z, w) = (F_1(z, w), F_2(z, w))$. Then, from $X_2 \circ F \equiv X_1$ on $\mathbf{U}$, we have

$$\frac{\|F_2(z, w)\|^2}{N_{\Omega_2}(F_1(z, w), \overline{F_1(z, w)})^{\mu_2}} \equiv \frac{\|w\|^2}{N_{\Omega_1}(z, \overline{z})^{\mu_1}} \quad ((z, w) \in \mathbf{U}).$$

Thus we have $F_2(z, 0) = 0, (z, 0) \in \mathbf{U}$, and so $F(0, 0) = (\widetilde{u}_0, 0)(\in \Omega_2 \times \{0\})$. Therefore, there exists $\Phi \in G(\Omega_2^{B^m}(\mu_2))$ with $\Phi \circ F(0, 0) = (0, 0)$ (Note Φ leaves the function X_2 on $\Omega_2^{B^m}(\mu_2)$ invariant). Let $H := \Phi \circ F$. Then

$$H : \mathbf{U} \to \Omega_2^{B^m}(\mu_2)$$

is a holomorphic mapping with $X_2(H(z, w)) \equiv X_1(z, w)$ on $\mathbf{U}$ and $H(0, 0) = (0, 0)$.

Write $H(0, w)$ in the following form

$$H(0, w) \equiv (h_1(w), h_2(w)) = (wV + \sum_{j\ge2} f_j(w), wU + \sum_{j\ge2} g_j(w))(\in \Omega_2^{B^m}(\mu_2)), (0, w) \in \mathbf{U},$$

where all components of $f_j(w)$ and $g_j(w)$ are homogeneous polynomials of degree j ($j \ge 2$). For $(0, w) \in \mathbf{U}$ (i.e., $w \in B^m$), there exists a positive number δ_w such that $(0, tw) \in \mathbf{U}$, $\forall t \in [0, \delta_w]$. By $X_2 \circ H(0, tw) \equiv X_1(0, tw)$, we have

$$\left\| wU + \sum_{j\ge2} t^{j-1} g_j(w) \right\|^2 \equiv \|w\|^2 \left(N_{\Omega_2} \left(twV + \sum_{j\ge2} t^j f_j(w), \overline{twV + \sum_{j\ge2} t^j f_j(w)} \right) \right)^{\mu_2}.$$

Take $t \to 0^+$, we get

$$\|wU\|^2 \equiv \|w\|^2,$$

that is, U is a unitary matrix of order m.

From

$$\|h_2(w)\|^2 \equiv \|w\|^2 \left(N_{\Omega_2}\left(h_1(w), \overline{h_1(w)}\right)\right)^{\mu_2}, (0, w) \in \mathbf{U}, \tag{3.1}$$

we get

$$\|h_2(w)\| \le \|w\|, (0, w) \in \mathbf{U}.$$

For $\zeta \in \mathbb{C}^m, \|\zeta\| = 1$, there exists a positive number η_ζ such that $(0, \lambda\zeta) \in \mathbf{U}$ for all $|\lambda| \le \eta_\zeta$. We define

$$g(\lambda) := < h_2(\lambda\zeta), \zeta U >, |\lambda| \le \eta_\zeta.$$

Then $g(0) = 0, g'(0) = 1$ and $|g(\lambda)| \le |\lambda|$ for $|\lambda| \le \eta_\zeta$.

Let

$$\widetilde{g}(\lambda) := \begin{cases} \frac{g(\lambda)}{\lambda}, & 0 < |\lambda| \le \eta_\zeta, \\ 1, & \lambda = 0. \end{cases}$$

Then $\widetilde{g}$ is a holomorphic map on $\{\lambda \in \mathbb{C} : |\lambda| < \eta_\zeta\}$, and by $\|h_2(w)\| \le \|w\|, (0, w) \in \mathbf{U}$, we have $|\widetilde{g}(\lambda)| \le 1$. Since $\widetilde{g}(0) = 1$, according to maximum modulus principle, it follows that $\widetilde{g} \equiv 1$, thus $g(\lambda) = \lambda, |\lambda| \le \eta_\zeta$, that is

$$< \lambda^{-1} h_2(\lambda\zeta), \zeta U >= 1, 0 < |\lambda| \le \eta_\zeta.$$

Using $\|h_2(\lambda\zeta)\| \le \lambda$ and $\|\zeta U\| = 1$, we get $h_2(\lambda\zeta) = \lambda\zeta U, |\lambda| \le \eta_\zeta$. Thus $h_2(w) = wU, (0, w) \in \mathbf{U}$.

Owing to (3.1), we get

$$N_{\Omega_2}\left(h_1(w), \overline{h_1(w)}\right) \equiv 1, (0, w) \in \mathbf{U},$$

that is, $h_1(w) \equiv 0, (0, w) \in \mathbf{U}$.

Let $H(z, w) = (H_1(z, w), H_2(z, w))$. Then we have

$$H(0, 0) = (0, 0),\ H_2(z, 0) \equiv 0,\ H_1(0, w) \equiv 0,\ H_2(0, w) = wU, (z, 0) \in \mathbf{U}, (0, w) \in \mathbf{U},$$

where U is a unitary matrix of order m. This means

$$H_1(z, w) = zA + \sum_{j \ge 2}(P_j(z) + Q_j(z, w)), \quad H_2(z, w) = wU + \sum_{j \ge 2} R_j(z, w), (z, w) \in \mathbf{U},$$

where P_j, Q_j and R_j are homogeneous polynomials of degree j, which are given by (2.12), (2.13) and (2.14) respectively.

(ii) For $(z,w)\in\mathbf{U}$ with $w\neq 0$, there exists a positive number $\delta_{z,w}$ such that $(tw,tw)\in\mathbf{U}$ for all $t\in[0,\delta_{z,w}]$. Since $X_2\circ H(tz,tw)=X_1(tz,tw)$ $(\forall t\in[0,\delta_{z,w}])$, it follows

$$\begin{aligned}&\left\|\tfrac{1}{\|w\|}wU+\tfrac{1}{\|w\|}\textstyle\sum_{j\geq 2}t^{j-1}R_j(z,w)\right\|^2\\ &=\frac{N_{\Omega_2}\left(tzA+\sum_{j\geq 2}t^j(P_j(z)+Q_j(z,w)),\overline{tzA+\sum_{j\geq 2}t^j(P_j(z)+Q_j(z,w))}\right)^{\mu_2}}{N_{\Omega_1}(tz,t\overline{z})^{\mu_1}}.\end{aligned}\tag{3.2}$$

By (2.8) we obtain

$$\begin{aligned}&\ln N_{\Omega_2}\left(tzA+\sum_{j\geq 2}t^j(P_j(z)+Q_j(z,w)),\overline{tzA+\sum_{j\geq 2}t^j(P_j(z)+Q_j(z,w))}\right)\\ =&\ -t^2\left(\sum_{j=1}^{r_2}\Lambda_j^2(zA+\sum_{j\geq 2}t^{j-1}(P_j(z)+Q_j(z,w)))+o(1)\right),\end{aligned}\tag{3.3}$$

$$\ln N_{\Omega_1}(tz,t\overline{z})=-t^2\left(\sum_{j=1}^{r_1}\lambda_j^2(z)+o(1)\right),\tag{3.4}$$

where r_1 and r_2 are the ranks of Ω_1 and Ω_2, $z\in\Omega_1$ has the spectral decomposition $z=k(z)\cdot(\lambda_1(z)e_1+\lambda_2(z)e_2+\cdots+\lambda_r(z)e_r)$ and $u\in\Omega_2$ has the spectral decomposition $u=\tilde{k}(u)\cdot(\Lambda_1(u)\tilde{e}_1+\Lambda_2(u)\tilde{e}_2+\cdots+\Lambda_r(u)\tilde{e}_r)$ (Note $\lambda_j(tz)=t\lambda_j(z)$ for $t\geq 0$ here). By substituting (3.3) and (3.4) into (3.2), for $t\in[0,\delta_{z,w}]$, we have

$$\begin{aligned}&\ln\Bigg(1+2\mathrm{Re}<wU,R_2(z,w)>\frac{t}{\|w\|^2}+2\mathrm{Re}<wU,R_3(z,w)>\frac{t^2}{\|w\|^2}\\ &+\|R_2(z,w)\|^2\frac{t^2}{\|w\|^2}+o(t^2)\Bigg)\\ =&\ -t^2\left(\mu_2\sum_{j=1}^{r_2}\Lambda_j^2(zA+\sum_{j\geq 2}t^{j-1}(P_j(z)+Q_j(z,w)))-\mu_1\sum_{j=1}^{r_1}\lambda_j^2(z)+o(1)\right).\end{aligned}\tag{3.5}$$

Dividing the two sides of the equation (3.5) by t^2 and taking $t\to 0^+$, we get

$$\begin{aligned}&2\mathrm{Re}<wU,R_2(z,w)>\equiv 0,\\ &2\mathrm{Re}<wU,R_3(z,w)>+\|R_2(z,w)\|^2+\|w\|^2S_2(z,\overline{z})\equiv 0,\end{aligned}$$

where

$$S_{2k}(z,\overline{z}):=\mu_2\sum_{j=1}^{r_2}\Lambda_j^{2k}(zA)-\mu_1\sum_{j=1}^{r_1}\lambda_j^{2k}(z),\tag{3.6}$$

in view of (2.9), there exist constants $g_{\alpha\beta}$ such that

$$S_{2k}(z,\overline{z})=\sum_{|\alpha|=|\beta|=k}g_{\alpha\beta}z^\alpha\overline{z}^\beta.$$

By Lemma 2.2 and (2.10), we have

$$R_2(z,w)\equiv 0,\ R_3(z,w)\equiv 0,\ S_2(z,\overline{z})\equiv 0,\ \mu_2 AC_{\Omega_2}\overline{A}^t=\mu_1 C_{\Omega_1}. \tag{3.7}$$

Since C_{Ω_1} and C_{Ω_2} are positive definite Hermite matrices and $\mu_2 AC_{\Omega_2}\overline{A}^t=\mu_1 C_{\Omega_1}$, we get that A is an invertible matrix of order d.

(iii) Now we show that for all $j>3$, $P_{j-2}(z)\equiv 0, Q_{j-2}(z,w)\equiv 0, R_j(z,w)\equiv 0, S_{2[\frac{j-1}{2}]}\equiv 0$ by the reduction to absurdity. Let

$$N:=\min\left\{j: P_{j-2}(z)\not\equiv 0,\ Q_{j-2}(z,w)\not\equiv 0,\ R_j(z,w)\not\equiv 0 \text{ or } S_{2[\frac{j-1}{2}]}\not\equiv 0\right\}. \tag{3.8}$$

From (3.7), we know $N\geq 4$. Now assume $N<+\infty$ here.

Using (3.2), we have

$$\begin{aligned}&\left\|\frac{1}{\|w\|}wU+\frac{1}{\|w\|}\sum_{j\geq N}t^{j-1}R_j(z,w)\right\|^2\\ =\ &\frac{N_2\left(tzA+\sum_{j\geq N-2}t^j(P_j(z)+Q_j(z,w)),\overline{tzA+\sum_{j\geq N-2}t^j(P_j(z)+Q_j(z,w))}\right)^{\mu_2}}{N_1(tz,t\overline{z})^{\mu_1}}.\end{aligned} \tag{3.9}$$

By (2.8) and (2.10), we get

$$\begin{aligned}&\ln N_2\left(tzA+\sum_{j\geq N-2}t^j(P_j(z)+Q_j(z,w)),\overline{tzA+\sum_{j\geq N-2}t^j(P_j(z)+Q_j(z,w))}\right)\\ =\ &-t^2\sum_{j=1}^{r_2}\Lambda_j^2(zA+\sum_{l\geq N-2}t^{l-1}(P_l(z)+Q_l(z,w))-\sum_{k=2}^{[\frac{N-1}{2}]}\frac{t^{2k}}{k}\sum_{j=1}^{r_2}\Lambda_j^{2k}(zA)+o(t^{N-1})\\ =\ &-2t^{N-1}Re<zAC_{\Omega_2},P_{N-2}(z)+Q_{N-2}(z,w)>-\sum_{k=1}^{[\frac{N-1}{2}]}\frac{t^{2k}}{k}\sum_{j=1}^{r_2}\Lambda_j^{2k}(zA)+o(t^{N-1})\end{aligned} \tag{3.10}$$

and

$$\ln N_1(tz,t\overline{z})=-\sum_{k=1}^{[\frac{N-1}{2}]}\frac{t^{2k}}{k}\sum_{j=1}^{r_1}\lambda_j^{2k}(z)+o(t^{N-1}). \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we obtain for all $t\in[0,\delta_{z,w}]$,

$$\begin{aligned}&\ln\left(1+2\mathrm{Re}<wU,R_N(z,w)>\frac{t^{N-1}}{\|w\|^2}+o(t^{N-1})\right)\\ =\ &-\left\{2\mu_2t^{N-1}\mathrm{Re}<zAC_{\Omega_2},P_{N-2}(z)+Q_{N-2}(z,w)>+\sum_{j=1}^{[\frac{N-1}{2}]}\frac{t^{2j}}{j}S_{2j}(z,\overline{z})+o(t^{N-1})\right\}\\ =\ &-\left\{2\mu_2t^{N-1}\mathrm{Re}<zAC_{\Omega_2},P_{N-2}(z)+Q_{N-2}(z,w)>\right.\\ &\left.+\frac{t^{2[\frac{N-1}{2}]}}{[\frac{N-1}{2}]}S_{2[\frac{N-1}{2}]}(z,\overline{z})+o(t^{N-1})\right\},\end{aligned} \tag{3.12}$$

where $S_{2k}(z,\overline{z})$ is the same as (3.6).

When $N=2k+1$, by (3.12), we have

$$\begin{aligned}&2\mathrm{Re} < wU, R_N(z,w) > +2\mu_2\|w\|^2\mathrm{Re} < zAC_{\Omega_2}, P_{N-2}(z)\\ &+Q_{N-2}(z,w) > +\tfrac{1}{k}\|w\|^2 S_{2k}(z,\overline{z}) \equiv 0.\end{aligned}$$

By Lemma 2.2, we obtain

$$P_{N-2}(z)\equiv 0,\ Q_{N-2}(z,w)\equiv 0,\ R_N(z,w)\equiv 0,\ S_{2[\frac{N-1}{2}]}(z,\overline{z})\equiv S_{2k}(z,\overline{z})\equiv 0.$$

This is the contradiction with (3.8).

When $N=2k$, by (3.12), we get

$$\begin{aligned}&2\mathrm{Re} < wU, R_N(z,w) > +2\mu_2\|w\|^2\mathrm{Re} < zAC_{\Omega_2}, P_{N-2}(z)+Q_{N-2}(z,w) >\equiv 0,\\ &S_{2[\frac{N-1}{2}]}(z,\overline{z})\equiv 0.\end{aligned}$$

From Lemma 2.2 we have

$$P_{N-2}(z)\equiv 0, Q_{N-2}(z,w)\equiv 0, R_N(z,w)\equiv 0.$$

This is also the contradiction with (3.8).

(iv) From (i),(ii) and (iii), we get

$$\Phi\circ F(z,w)\equiv H(z,w)=(zA,wU), N_1(z,\overline{z})^{\mu_1}=N_2(zA,\overline{zA})^{\mu_2}, (z,w)\in \mathbf{U}$$

and

$$F(z,w)=\left(\psi(zA), \frac{N_2(u_0,\overline{u_0})^{\frac{\mu_2}{2}}}{N_2(zA,\overline{u_0})^{\mu_2}}wU\right), (z,w)\in\mathbf{U}.$$

Let $D:=\{z\in\Omega_1 : (z,w)\in\mathbf{U}\}$. Then D is a neighborhood of the origin in Ω_1. By $T(z,w):=N_1(z,w)^{\frac{\mu_1}{\mu_2}}-N_2(zA,w\overline{A})$ is a holomorphic function on $\Omega_1\times\Omega_1$ and $T(z,w)\equiv 0$ on $D\times D$, we obtain $T(z,w)\equiv 0$ on $\Omega_1\times\Omega_1$. Thus

$$N_1(z,\overline{z})^{\mu_1}=N_2(zA,\overline{zA})^{\mu_2}, z\in\Omega_1.$$

By Lemma 2.1, we have that mapping $\mathcal{A}: z\in\Omega_1\mapsto zA\in\Omega_2$ is a biholomorphic mapping of Ω_1 onto Ω_2 with $\mu_1=\mu_2$. So $\Phi_0(z,w):=(zA,wU)$ is a the standard linear isomorphism of $\Omega_1^{B^m}(\mu_1)$ into $\Omega_1^{B^m}(\mu_2)$. Therefore $\Phi\circ F$ is the restriction on $\mathbf{U}$ of a biholomorphic mapping Φ_0.

Let $\phi(z)=\psi(zA)$, $z_0=u_0A^{-1}$, then ϕ is a biholomorphic mapping of Ω_1 onto Ω_2 with $\phi(z_0)=0$, thus

$$F(z,w)=\left(\phi(z), \frac{N_1(z_0,\overline{z_0})^{\frac{\mu_1}{2}}}{N_1(z,\overline{z_0})^{\mu_1}}wU\right), (z,w)\in\mathbf{U}.$$

This proves Theorem 1.3.

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等维Cartan-Hartogs域双全纯映射的特征

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摘 要: 本文考虑了等维Cartan-Hartogs域之间的全纯映射. 如果Cartan-Hartogs域$\Omega^{B^m}(\mu)$不是球, 则它上面存在一函数X使得它在$\Omega^{B^m}(\mu)$的任一全纯自同构作用下不变. 通过直接计算得到: 如果等维Cartan-Hartogs 域间的全纯映射F保持函数X不变, 则F必是双全纯映射. 由此可得如果Cartan-Hartogs域$\Omega^{B^m}(\mu)$不是球, $\Omega^{B^m}(\mu)$ 的全纯自映射是自同构的充要条件是F保持函数X不变.

关键词: 双全纯映射; 有界对称域; Cartan-Hartogs 域

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