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# ESTIMATES FOR LOWER ORDER EIGENVALUES OF A CLASS OF OPERATORS ON RIEMANNIAN MANIFOLD

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**Abstract:** In this paper, we investigate the boundary value problem of the operator. By the Rayleigh-Ritz inequality, we obtain universal inequalities for lower order eigenvalues of these operator.

**Keywords:** lower order eigenvalues; Rayleigh-Ritz inequality

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## 1 Introduction

In recent years, there are some researches on eigenvalue estimates for quadratic polynomial operator of Laplacian:

$$\begin{cases} \Delta^2 u - p\Delta u + qu = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial v}|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in an $n$-dimensional complete Riemannian manifold $M, v$ denotes the outwards unit normal vector field of $\partial\Omega$, the constants $p, q \geq 0$, $\Delta(= -\mathrm{div}\nabla)$ is the Laplacian and $\Delta^2$ is the biharmonic operator on $M$. In 2011, Sun and Qi [3] obtained universal eigenvalue inequalities of problem (1.1),

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{n^2}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left((nH_0)^2 + (2n+4)E_i + np\right)\left((nH_0)^2 + 4E_i\right),$$

where $H_0$ is a constant which only depends on the mean curvature of $M$ and

$$E_i = \frac{1}{2}(-p + \sqrt{p^2 + 4(\lambda_i - q)}).$$

For other related results, one can see [2, 4, 5, 7, 8].

Let $(M^n, \langle,\rangle)$ be an $n(\geq 2)$-dimensional complete Riemannian manifold isometrically immersed in the Euclidean space $\mathbb{R}^N$ by $\Psi$, $\Omega$ is a bounded domain in $M^n$. In this paper, we

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are interested in extrinsic upper bounds for the lower order eigenvalues of following operator defined on $\Omega$:

$$\begin{cases} \Delta^2 u - \operatorname{div} A\nabla u + au = \lambda u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial v}|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $A$ is a positive definite symmetric (1,1)-tensor on $M^n$, the eigenvalues of $A$ on $\Omega$ are bounded below by a positive constant $b$, $\operatorname{tr}(A)|_\Omega \leq c$ (i.e., $A$ can also be viewed as a smooth symmetric and positive defined section of the bundle of all endomorphisms of $T(M^n)$, $\Delta(= \operatorname{div}\nabla)$ is the Laplacian, $\nabla$ is the gradient operator on $M$ and $a$ is a nonnegative constant.

It is well known that problem (1.2) has a real and discrete spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty,$$

where each eigenvalue is repeated according to its multiplicity. Our results are stated as follows.

**Theorem 1.1** Let $(M^n, \langle , \rangle)$ be an $n(\geq 2)$-dimensional complete Riemannian manifold isometrically immersed in the Euclidean space $\mathbb{R}^N$ by $\Psi$, $\Omega$ is a bounded domain in $M^n$. $A$ is a positive definite symmetric (1,1)-tensor on $M^n$. Assume that the eigenvalues of $A$ on $\Omega$ are bounded below by a positive constant $b$ (that is, $\bar{\lambda}(A)|_\Omega \geq b$) and that $\operatorname{tr(A)}|_\Omega \leq c$. Then for the eigenvalues of problem (1.2),

$$\sum_{\alpha=1}^{n} (\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \leq \sqrt{\left(n^2 \sup_\Omega |H|^2 + (n+2)B + c\right)\left(n^2 \sup_\Omega |H|^2 + 2B\right)}, \tag{1.3}$$

where $B = -b + \sqrt{b^2 + 4(\lambda_1 - a)}$ and $H$ is the mean curvature vector of the immersion $\Psi$.

## 2 Proof of Theorem 1.1

First, we give the following lemma.

**Lemma 2.1** Let $\lambda_i$ be the $i$-th eigenvalue of problem (1.2) and $u_i$ the orthonormal eigenfunction corresponding to $\lambda_i$ (i.e. $\int_\Omega u_i u_j = \delta_{ij}$). For any function $m_\alpha \in C^2(\bar{\Omega})$ satisfying

$$\int_\Omega m_\alpha u_1 u_\beta = 0 \quad \text{for } \beta = 2, \cdots, \alpha, \tag{2.1}$$

we have for any positive constant $\delta$,

$$\begin{aligned} &(\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_\Omega u_1^2 |\nabla m_\alpha|^2 \\ \leq &\frac{\delta}{2}\left(||u_1 \Delta m_\alpha + 2\langle \nabla m_\alpha, \nabla u_1\rangle||^2 - 2\int_\Omega |\nabla m_\alpha|^2 u_1 \Delta u_1 + \int_\Omega u_1^2 \langle \nabla m_\alpha, A\nabla m_\alpha\rangle\right) \\ &+ \frac{1}{2\delta}||u_1 \Delta m_\alpha + 2\langle \nabla m_\alpha, \nabla u_1\rangle||^2, \end{aligned} \tag{2.2}$$

where $||f||^2 = \int_\Omega f^2$.

**Proof** We consider the function $T_\alpha = m_\alpha u_1 - u_1\int_\Omega m_\alpha u_1^2$. Then, it is easy to check that $T_\alpha|_{\partial\Omega} = 0$ and

$$\int_\Omega T_\alpha u_1 = 0. \tag{2.3}$$

Hence

$$\int_\Omega T_\alpha u_l = 0 \quad \text{for } l = 1, \cdots, \alpha. \tag{2.4}$$

It follows from the Rayleigh-Ritz inequality(or min-max principle) that

$$\lambda_{\alpha+1}||T_\alpha||^2 \le \int_\Omega T_\alpha[\Delta^2 - \mathrm{div}A\nabla + a]T_\alpha. \tag{2.5}$$

By virtue of (2.3), a direct calculation yields

$$\begin{aligned}\lambda_{\alpha+1}||T_\alpha||^2 &\le \int_\Omega T_\alpha[\Delta^2 - \mathrm{div}A\nabla + a]T_\alpha = \int_\Omega T_\alpha[\Delta^2 - \mathrm{div}A\nabla + a](m_\alpha u_1)\\ &= \int_\Omega T_\alpha[u_1\Delta^2 m_\alpha + 2\Delta m_\alpha \Delta u_1 + 2\Delta\langle\nabla m_\alpha, \nabla u_1\rangle\\ &\quad + 2\langle\nabla m_\alpha, \nabla\Delta u_1\rangle + 2\langle\nabla u_1, \nabla\Delta m_\alpha\rangle\\ &\quad + m_\alpha\Delta^2 u_1 - m_\alpha \mathrm{div}A\nabla u_1 + a m_\alpha u_1 - u_1\mathrm{div}A\nabla m_\alpha - 2\langle\nabla m_\alpha, A\nabla u_1\rangle]\\ &= \int_\Omega T_\alpha[u_1\Delta^2 m_\alpha + 2\Delta m_\alpha \Delta u_1 + 2\Delta\langle\nabla m_\alpha, \nabla u_1\rangle\\ &\quad + 2\langle\nabla m_\alpha, \nabla\Delta u_1\rangle + 2\langle\nabla u_1, \nabla\Delta m_\alpha\rangle\\ &\quad - u_1\mathrm{div}A\nabla m_\alpha - 2\langle\nabla m_\alpha, A\nabla u_1\rangle] + \lambda_1||T_\alpha||^2.\end{aligned} \tag{2.6}$$

Substituting (2.6) into inequality (2.5) and utilizing the Green's formula, we have

$$\begin{aligned}(\lambda_{\alpha+1} - \lambda_1)||T_\alpha||^2 &\le \int_\Omega T_\alpha[u_1\Delta^2 m_\alpha + 2\Delta m_\alpha \Delta u_1 + 2\Delta\langle\nabla m_\alpha, \nabla u_1\rangle\\ &\quad + 2\langle\nabla m_\alpha, \nabla\Delta u_1\rangle + 2\langle\nabla u_1, \nabla\Delta m_\alpha\rangle - u_1\mathrm{div}A\nabla m_\alpha - 2\langle\nabla m_\alpha, A\nabla u_1\rangle]\\ &\le \int_\Omega (m_\alpha u_1)[u_1\Delta^2 m_\alpha + 2\Delta m_\alpha \Delta u_1 + 2\Delta\langle\nabla m_\alpha, \nabla u_1\rangle\\ &\quad + 2\langle\nabla m_\alpha, \nabla\Delta u_1\rangle + 2\langle\nabla u_1, \nabla\Delta m_\alpha\rangle\\ &\quad - u_1\mathrm{div}A\nabla m_\alpha - 2\langle\nabla m_\alpha, A\nabla u_1\rangle]\\ &\quad - \int_\Omega m_\alpha u_1^2\int_\Omega u_1[u_1\Delta^2 m_\alpha + 2\Delta m_\alpha \Delta u_1 + 2\Delta\langle\nabla m_\alpha, \nabla u_1\rangle\\ &\quad + 2\langle\nabla m_\alpha, \nabla\Delta u_1\rangle + 2\langle\nabla u_1, \nabla\Delta m_\alpha\rangle\\ &\quad - u_1\mathrm{div}A\nabla m_\alpha - 2\langle\nabla m_\alpha, A\nabla u_1\rangle]\\ &= \int_\Omega (m_\alpha u_1)[u_1\Delta^2 m_\alpha + 2\Delta m_\alpha \Delta u_1 + 2\Delta\langle\nabla m_\alpha, \nabla u_1\rangle\\ &\quad + 2\langle\nabla m_\alpha, \nabla\Delta u_1\rangle + 2\langle\nabla u_1, \nabla\Delta m_\alpha\rangle\\ &\quad - u_1\mathrm{div}A\nabla m_\alpha - 2\langle\nabla m_\alpha, A\nabla u_1\rangle].\end{aligned} \tag{2.7}$$

From the divergence theorem, we get

$$\begin{aligned}
&\int_\Omega 2m_\alpha u_1 \Delta\langle\nabla m_\alpha, \nabla u_1\rangle \\
={}& 2\int_\Omega (u_1\langle\nabla m_\alpha, \nabla u_1\rangle\Delta m_\alpha + m_\alpha\langle\nabla m_\alpha, \nabla u_1\rangle\Delta u_1 + 2\langle\nabla m_\alpha, \nabla u_1\rangle^2), \qquad (2.8)\\
&\int_\Omega 2m_\alpha u_1 \langle\nabla m_\alpha, \nabla\Delta u_1\rangle \\
={}& -2\int_\Omega (m_\alpha\langle\nabla m_\alpha, \nabla u_1\rangle\Delta u_1 + \langle\nabla m_\alpha, \nabla m_\alpha\rangle u_1\Delta u_1 + m_\alpha u_1\Delta u_1\Delta m_\alpha), \qquad (2.9)\\
&\int_\Omega 2m_\alpha u_1 \langle\nabla u_1, \Delta\nabla m_\alpha\rangle \\
={}& \int_\Omega (\Delta m_\alpha)^2 u_1^2 + \langle\nabla u_1^2, \nabla m_\alpha\rangle\Delta m_\alpha - m_\alpha u_1^2\Delta^2 m_\alpha. \qquad (2.10)
\end{aligned}$$

Then, putting (2.8)–(2.10) into (2.7), one get

$$\begin{aligned}
&(\lambda_{\alpha+1} - \lambda_1)||T_\alpha||^2 \\
\le& \int_\Omega [u_1^2(\Delta m_\alpha)^2 + 4\langle\nabla m_\alpha, \nabla u_1\rangle^2 - 2\langle\nabla m_\alpha, \nabla m_\alpha\rangle u_1\Delta u_1 \\
&+ 4u_1\langle\nabla m_\alpha, \nabla u_1\rangle\Delta m_\alpha - m_\alpha u_1^2(\mathrm{div} A\nabla m_\alpha) - \frac{1}{2}\langle\nabla(m_\alpha^2), A\nabla(u_1^2)\rangle] \\
=& ||u_1\Delta m_\alpha + 2\langle\nabla m_\alpha, \nabla u_1\rangle||^2 - 2\int_\Omega |\nabla m_\alpha|^2 u_1\Delta u_1 + \int_\Omega u_1^2\langle\nabla m_\alpha, A\nabla m_\alpha\rangle.
\end{aligned} \qquad (2.11)$$

Utilizing the divergence theorem again, it holds

$$\begin{aligned}
2\int_\Omega u_1 m_\alpha\langle\nabla m_\alpha, \nabla u_1\rangle &= \frac{1}{2}\int_\Omega\langle\nabla(m_\alpha^2), \nabla(u_1^2)\rangle = -\frac{1}{2}\int_\Omega u_1^2\Delta(m_\alpha^2) \\
&= -\int_\Omega u_1^2 m_\alpha\Delta m_\alpha - \int_\Omega u_1^2|\nabla m_\alpha|^2.
\end{aligned} \qquad (2.12)$$

Hence

$$\begin{aligned}
&\int_\Omega T_\alpha(u_1\Delta m_\alpha + 2\langle\nabla m_\alpha, \nabla u_1\rangle) \\
=& \int_\Omega u_1^2 m_\alpha\Delta m_\alpha + 2\int_\Omega u_1 m_\alpha\langle\nabla m_\alpha, \nabla u_1\rangle \\
&- 2\int_\Omega m_\alpha u_1^2(\frac{1}{2}\int_\Omega u_1^2\Delta m_\alpha + \int_\Omega u_1\langle\nabla m_\alpha, \nabla u_1\rangle) \\
=& -\int_\Omega u_1^2\,|\nabla m_\alpha|^2\,.
\end{aligned} \qquad (2.13)$$

By virtue of (2.11), (2.13) and the Cauchy's inequality, it is not difficult to get

$$\begin{aligned}
&(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2|\nabla m_\alpha|^2\\
=&-(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega T_\alpha(u_1\Delta m_\alpha+2\langle\nabla m_\alpha,\nabla u_1\rangle)\\
\le&\frac{\delta}{2}(\lambda_{\alpha+1}-\lambda_1)\|T_\alpha\|^2+\frac{1}{2\delta}\|u_1\Delta m_\alpha+2\langle\nabla m_\alpha,\nabla u_1\rangle\|^2\\
=&\frac{\delta}{2}(\|u_1\Delta m_\alpha+2\langle\nabla m_\alpha,\nabla u_1\rangle\|^2-2\int_\Omega|\nabla m_\alpha|^2u_1\Delta u_1+\int_\Omega u_1^2\langle\nabla m_\alpha,A\nabla m_\alpha\rangle)\\
&+\frac{1}{2\delta}\|u_1\Delta m_\alpha+2\langle\nabla m_\alpha,\nabla u_1\rangle\|^2.
\end{aligned}\tag{2.14}$$

Now we give the proof of Theorem 1.1:

**Proof of Theorem 1.1** Let $x^1,x^2,\cdots,x^N$ be the standard coordinate functions on $\mathbb{R}^N$. We consider the $N\times N$ matrix

$$D\triangleq(d_{\alpha\beta}),\ \ d_{\alpha\beta}=\int_\Omega x^\alpha u_1u_{\beta+1}\ \ (\alpha,\beta=1,\cdots N).$$

From the orthogonalization of Gram-Schmidt (QR-factorization theorem), we know that $D$ can be written by $R=QD$, where $Q=(q_{\alpha\beta})$ is an orthogonal $N\times N$ matrix and $R=(r_{\alpha\beta})$ is an upper triangular matrix. Hence, we have

$$r_{\alpha\beta}=\sum_{\gamma=1}^N q_{\alpha\gamma}d_{\gamma\beta}=\int_\Omega\sum_{\gamma=1}^N q_{\alpha\gamma}x^\gamma u_1u_{\beta+1}=0\ \text{ for } \beta=1,\cdots,\alpha-1.$$

Taking $m_\alpha=\sum\limits_{\gamma=1}^N q_{\alpha\gamma}x^\gamma$, thus $\int_\Omega m_\alpha u_1u_l=0$ for $l=2,\cdots,\alpha$. Applying Lemma 2.1 to $m_\alpha=\sum\limits_{\gamma=1}^N q_{\alpha\gamma}x^\gamma$ and summing on $\alpha$ from 1 to $N$, we get

$$\begin{aligned}
&\sum_{\alpha=1}^N(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2|\nabla m_\alpha|^2\\
\le\ &\frac{\delta}{2}\sum_{\alpha=1}^N\Big(||u_1\Delta m_\alpha+2\langle\nabla m_\alpha,\nabla u_1\rangle||^2-2\int_\Omega|\nabla m_\alpha|^2u_1\Delta u_1\\
&+\int_\Omega u_1^2\langle\nabla m_\alpha,A\nabla m_\alpha\rangle\Big)+\frac{1}{2\delta}\sum_{\alpha=1}^N||u_1\Delta m_\alpha+2\langle\nabla m_\alpha,\nabla u_1\rangle||^2\\
\le\ &\frac{\delta}{2}\sum_{\gamma=1}^N\Big(||u_1\Delta x^\gamma+2\langle\nabla x^\gamma,\nabla u_1\rangle||^2-2\int_\Omega|\nabla x^\gamma|^2u_1\Delta u_1\\
&+\int_\Omega u_1^2\langle\nabla x^\gamma,A\nabla x^\gamma\rangle\Big)+\frac{1}{2\delta}\sum_{\gamma=1}^N||u_1\Delta x^\gamma+2\langle\nabla x^\gamma,\nabla u_1\rangle||^2
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{\delta}{2}\left(n^2\int_\Omega u_1^2|H|^2+(2n+4)\int_\Omega|\nabla u_1|^2+\int_\Omega u_1^2(\mathrm{tr}A)\right)\\
&+\frac{1}{2\delta}\int_\Omega\left(n^2u_1^2|H|^2+4|\nabla u_1|^2\right)\\
\leq\ & \frac{\delta}{2}\left(n^2\int_\Omega u_1^2|H|^2+(2n+4)\int_\Omega|\nabla u_1|^2+c\right)\\
&+\frac{1}{2\delta}\int_\Omega\left(n^2u_1^2|H|^2+4|\nabla u_1|^2\right),
\end{aligned}
\tag{2.15}
$$

where

$$
\begin{aligned}
&\sum_{\gamma=1}^N|\nabla x^\gamma|^2=n,\ \sum_{\gamma=1}^N\langle\nabla x^\gamma,\ \nabla u_1\rangle^2=|\nabla u_1|^2,\\
&\sum_{\gamma=1}^N(\Delta x^\gamma)^2=n^2|H|^2\ ,\ \sum_{\gamma=1}^N(\langle\nabla x^\gamma,\nabla u_1\rangle\Delta x^\gamma)=0,
\end{aligned}
$$

see [1, Lemma 2.1]. Since

$$
\begin{aligned}
\int_\Omega|\nabla u_1|^2&=\int_\Omega-u_1\Delta u_1\leq(\int_\Omega u_1^2)^{\frac12}(\int_\Omega(\Delta u_1)^2)^{\frac12}=(\int_\Omega u_1\Delta^2u_1)^{\frac12}\\
&=(\int_\Omega u_1(\mathrm{div}A\nabla u_1-au_1+\lambda_1u_1))^{\frac12}\\
&=(-a+\lambda_1-\int_\Omega\langle\nabla u_1,A\nabla u_1\rangle)^{\frac12}\\
&\leq(-a+\lambda_1-b\int_\Omega|\nabla u_1|^2)^{\frac12}.
\end{aligned}
\tag{2.16}
$$

Hence

$$
\int_\Omega|\nabla u_1|^2\leq\frac{-b+\sqrt{b^2+4(\lambda_1-a)}}{2}.
$$

Then, we obtain

$$
\begin{aligned}
&\sum_{\alpha=1}^N(\lambda_{\alpha+1}-\lambda_1)^{\frac12}\int_\Omega u_1^2|\nabla m_\alpha|^2\\
\leq&\frac{\delta}{2}(n^2\sup_\Omega|H|^2+(n+2)B+c)+\frac{1}{2\delta}(n^2\sup_\Omega|H|^2+2B),
\end{aligned}
\tag{2.17}
$$

where $B=-b+\sqrt{b^2+4(\lambda_1-a)}$ and $H$ is the mean curvature vector of its immersion $\Psi$.

Minimizing the right hand side of (2.17), one get

$$
\begin{aligned}
&\sum_{\alpha=1}^N(\lambda_{\alpha+1}-\lambda_1)^{\frac12}\int_\Omega u_1^2|\nabla m_\alpha|^2\\
\leq&\sqrt{(n^2\sup_\Omega|H|^2+(n+2)B+c)(n^2\sup_\Omega|H|^2+2B)}.
\end{aligned}
\tag{2.18}
$$

It is easy to get $\sum\limits_{\alpha=1}^{N}|\nabla m_\alpha|^2=n$ and $|\nabla m_\alpha|^2\le 1$ for $\alpha=1,\cdots,n$, see [6]. Therefore,

$$\begin{aligned}&\sum_{\alpha=1}^{N}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}|\nabla m_\alpha|^2\\ \ge&\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}|\nabla m_\alpha|^2+(\lambda_{n+1}-\lambda_1)^{\frac{1}{2}}\sum_{t=n+1}^{N}|\nabla m_t|^2\\ =&\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}|\nabla m_\alpha|^2+(\lambda_{n+1}-\lambda_1)^{\frac{1}{2}}(n-\sum_{\beta=1}^{n}|\nabla m_\beta|^2)\\ =&\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}|\nabla m_\alpha|^2+(\lambda_{n+1}-\lambda_1)^{\frac{1}{2}}\sum_{\beta=1}^{n}(1-|\nabla m_\beta|^2)\\ \ge&\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}.\end{aligned}\tag{2.19}$$

From (2.18) and (2.19), we have

$$\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}\le\sqrt{(n^2\sup_{\Omega}|H|^2+(n+2)B+c)(n^2\sup_{\Omega}|H|^2+2B)}.$$

This completes the proof of Theorem 1.1.

**Corollary 2.2** Let $(M^n,\langle,\rangle)$ be an $n(\ge 2)$-dimensional complete Riemannian manifold isometrically immersed in the Euclidean space $\mathbb{R}^N$ by $\Psi$, $\Omega$ is a bounded domain in $M^n$. For the lower order eigenvalues of problem (1.1), where $p>0$, it holds that

$$\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}\le\sqrt{(n^2\sup_{\Omega}|H|^2+(n+2)C-2p)(n^2\sup_{\Omega}|H|^2+2C-2p)}\tag{2.20}$$

where $C=\sqrt{p^2+4(\lambda_1-q)}$ and $H$ is the mean curvature vector of its immersion $\Psi$.

**Corollary 2.3** Let $\Omega$ be a bounded domain in an $n(\ge 2)$-dimensional unit sphere $\mathbb{S}^n(1)$. For the lower order eigenvalues of problem (1.1), where $p>0$, It holds that

$$\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}-\lambda_1)^{\frac{1}{2}}\le\sqrt{(n^2+(n+2)C-2p)\,(n^2+2C-2p)},\tag{2.21}$$

where $C=\sqrt{p^2+4(\lambda_1-q)}$.

**Remark** For the unit sphere $\mathbb{S}^n(1)$, by taking $\Omega=\mathbb{S}^n(1)$, $\partial\Omega=\emptyset$, we know that eigenvalues of problem (1.1) satisfy $\lambda_1=q$ and $\lambda_2=\cdots=\lambda_{n+1}=n^2+pn+q$. Then inequality in (2.21) become equality. So inequality (2.21) is sharp.

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# 黎曼流形上一类算子的低阶特征值估计

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**摘要:** 本文研究$n(\geq 2)$维完备黎曼流形$M$的有界区域$\Omega$上算子的低阶特征值估计问题. 利用Rayleigh-Ritz不等式, 获得了该算子低阶特征值的万有不等式.

**关键词:** 低阶特征值; Rayleigh-Ritz 不等式

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