

# MULTIPLE SOLUTIONS FOR SYMMETRIC AND NON-SYMMETRIC QUASILINEAR ELLIPTIC EQUATIONS: AN ORLICZ-SOBOLEV SPACE SETTING

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**Abstract:** In this paper, we study multiplicity of solutions for the quasilinear elliptic problem in a bounded domain with smooth boundary. By using variational and perturbed methods in Orlicz-Sobolev space, we prove the existence of multiple solutions both in symmetric and non-symmetric case.

**Keywords:** Orlicz-Sobolev spaces; quasilinear elliptic equations; perturbed methods; symmetry

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## 1 Introduction

In this paper, we discuss multiplicity of solutions for the following boundary value problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

and

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x, u) + \epsilon g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.2)$$

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where  $\Omega \subset R^N$  is a bounded domain with smooth boundary  $\partial\Omega$ . The function  $a$  is such that  $p : R \rightarrow R$  defined by

$$p(t) = \begin{cases} a(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

which is an increasing homeomorphism from  $R$  onto itself and the continuous function  $f(x, t) \in C(\bar{\Omega} \times R, R)$  satisfies  $f(x, 0) = 0$ ,  $x \in \bar{\Omega}$  and odd in  $t$ .  $g \in C(\bar{\Omega} \times R, R)$  and  $\epsilon$  is a parameter.

As we all know, when  $a(t) = |t|^{p-2}$ , problem (1.1) is the well known  $p$ -Laplacian equation. There is a large number of papers on the existence of solutions for  $p$ -Laplacian equation. But problem (1.1) possesses more complicated nonlinearities, for example, it is inhomogeneous, and has important physical background, e.g.,

(a) nonlinear elasticity:  $P(t) = (1 + t^2)^\gamma - 1$ ,  $\gamma > \frac{1}{2}$ ;

(b) plasticity:  $P(t) = t^\alpha (\log(1 + t))^\beta$ ,  $\alpha \geq 1$ ,  $\beta > 0$ ;

(c) generalized Newtonian fluids:  $P(t) = \int_0^t s^{1-\alpha} (\sin h^{-1} s)^\beta ds$ ,  $0 \leq \alpha \leq 1$ ,  $\beta > 0$ .

So in the discussions, some special techniques will be needed, and problem (1.1) is studied in an Orlicz-Sobolev space and received considerable attention in recent years, see, for instance, papers [1–9]. Motivated by their results, the aim of this paper is to state some multiplicity results under more general assumptions for the nonlinearity  $f(x, t)$  in the symmetric case and the case that the breaking of symmetry  $g(x, t)$ .

The paper is organized as follows: in Section 2, we present some preliminary knowledge on the Orlicz-Sobolev spaces and give the main results. In Section 3, we will give the proof for symmetric case (1.1) by symmetric mountain pass theorem, and in Section 4, with the method used in [13], we give the proof for perturbed problem (1.2).

## 2 Preliminaries

Obviously, our problems allow a nonhomogeneous function  $p$  in the differential operator. To deal with this situation, we introduce an Orlicz-Sobolev space setting as follows.

Let

$$P(t) = \int_0^t p(s)ds, \quad \tilde{P}(t) = \int_0^t p^{-1}(s)ds, \quad t \in R,$$

then  $P$  and  $\tilde{P}$  are complementary  $N$ -functions (see [10]), which define the Orlicz spaces  $L^P := L^P(\Omega)$  and  $L^{\tilde{P}} := L^{\tilde{P}}(\Omega)$ , respectively.

Throughout this paper, we assume the following condition on  $P$ :

$$(p) 1 < p^- := \inf_{t>0} \frac{tp(t)}{P(t)} \leq p^+ := \sup_{t>0} \frac{tp(t)}{P(t)} < +\infty.$$

Under the condition (p), the Orlicz space  $L^P$  coincides with the set (equivalence classes) of measurable functions  $u : \Omega \rightarrow R$  such that

$$\int_{\Omega} P(|u|)dx < +\infty$$

and is equipped with the (luxemburg) norm, i.e.,

$$|u|_P := \inf\{k > 0 : \int_{\Omega} P(\frac{|u|}{k})dx < 1\}.$$

We will denote by  $W^{1,P}(\Omega)$  the corresponding Orlicz-Sobolev space with the norm

$$\|u\|_{W^{1,P}(\Omega)} := |u|_P + |\nabla u|_P$$

and define  $W_0^{1,P}(\Omega)$  as the closure of  $C_0^\infty$  in  $W^{1,P}(\Omega)$ . In this paper, we will use the following equivalent norm on  $W_0^{1,P}(\Omega)$ :

$$\|u\| := \inf\{k > 0 : \int_{\Omega} P(\frac{|\nabla u|}{k})dx < 1\}.$$

Now we introduce the Orlicz-Sobolev conjugate  $P_*$  of  $P$ , which is given by

$$P_*^{-1}(t) := \int_0^t \frac{p^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}d\tau,$$

where we suppose that

$$\lim_{t \rightarrow 0} \int_t^1 \frac{p^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}d\tau < +\infty, \lim_{t \rightarrow \infty} \int_1^t \frac{p^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}d\tau = +\infty.$$

Let  $p_*^- := \inf_{t>0} \frac{tP_*'(t)}{P_*(t)}, p_*^+ := \sup_{t>0} \frac{tP_*'(t)}{P_*(t)}$ . Throughout this paper, we assume that  $p^+ < p_*^-$ . Now we will make the following assumptions on  $f(x, t)$ .

( $f_*$ ) There exists an odd increasing homeomorphism  $h$  from  $R$  to  $R$ , and nonnegative constants  $c_1, c_2$  such that

$$\begin{aligned} |f(x, t)| &\leq c_1 + c_2h(|t|), \forall t \in R, \forall x \in \bar{\Omega}, \\ \lim_{t \rightarrow +\infty} \frac{H(t)}{P_*(kt)} &= 0, \forall k > 0, \end{aligned} \tag{2.1}$$

where

$$H(t) := \int_0^t h(s)ds.$$

Let

$$\tilde{H}(t) := \int_0^t h^{-1}(s)ds,$$

then we can obtain complementary N-functions which define corresponding Orlicz spaces  $L^H$  and  $L^{H_*}$ .

Similar to condition ( $p$ ), we also assume the following condition on  $H$ :

$$(h)1 < h^- := \inf_{t>0} \frac{th(t)}{H(t)} \leq h^+ := \sup_{t>0} \frac{th(t)}{H(t)} < +\infty.$$

**Lemma 2.1** [10] Under condition  $(p)$ , the spaces  $L^P(\Omega)$ ,  $W_0^{1,P}(\Omega)$  and  $W^{1,P}(\Omega)$  are separable and reflexive Banach spaces.

**Lemma 2.2** [10] Under the condition  $(f_*)$  and (2.1), the imbedding  $W_0^{1,P}(\Omega) \hookrightarrow L^H(\Omega)$  is compact.

Next we assume

$$N < p^- < \liminf_{t \rightarrow \infty} \frac{\log(P(t))}{\log(t)}. \quad (2.2)$$

**Lemma 2.3** Under condition (2.2), the imbedding  $W_0^{1,P}(\Omega) \hookrightarrow C(\overline{\Omega})$  is compact.

**Proof** Using Lemma D.2 in [14], it follows that  $W_0^{1,P}(\Omega)$  is continuously embedded in  $W^{1,p^-}(\Omega)$ . On the other hand, since we assume  $p^- > N$ , we deduce that  $W_0^{1,p^-}(\Omega)$  is compactly embedded in  $C(\overline{\Omega})$ . Thus, we obtain that  $W_0^{1,P}(\Omega)$  is compactly embedded in  $C(\overline{\Omega})$ .

**Lemma 2.4** [2] Let  $\rho(u) = \int_{\Omega} P(u)dx$ , we have

- (1) if  $|u|_P < 1$ , then  $|u|_P^{p^+} \leq \rho(u) \leq |u|_P^{p^-}$ ;
- (2) if  $|u|_P > 1$ , then  $|u|_P^{p^-} \leq \rho(u) \leq |u|_P^{p^+}$ ;
- (3) if  $0 < t < 1$ , then  $t^{p^+} P(u) \leq P(tu) \leq t^{p^-} P(u)$ ;
- (4) if  $t > 1$ , then  $t^{p^-} P(u) \leq P(tu) \leq t^{p^+} P(u)$ .

With all the lemmas mentioned above, we now state our results and make the following assumptions:

- ( $f_1$ ) there exist  $\eta > p^+$ ,  $1 < \sigma < p^-$ , and  $a_1, a_2 > 0$ , such that

$$\frac{1}{\eta} f(x, t)t - F(x, t) \geq -a_1 - a_2 |t|^\sigma$$

for every  $t \in R$ , a.e. in  $\Omega$ .

- ( $f_2$ )  $\liminf_{|t| \rightarrow \infty} F(x, t)/|t|^{p^+} = \infty$  uniformly a.e. in  $\Omega$ .

**Theorem 2.1** (Symmetric case) Assume that  $f(x, t)$  is odd in  $t$ , satisfies  $(f_*)$  ( $f_1$ ) ( $f_2$ ) and (2.1) with  $p^- < h^+ \leq p^+$ . Then problem (1.1) possesses infinitely many nontrivial solutions.

**Theorem 2.2** (Non-symmetric case) Assume that  $f(x, t)$  is odd in  $t$ , satisfies  $(f_*)$ , ( $f_1$ ), ( $f_2$ ), (2.1), (2.2) with  $p^- < h^+ \leq p^+$ ,  $g \in C(\overline{\Omega} \times R)$ . Then for any  $m \in N$ , there exists  $\epsilon_m > 0$  such that if  $|\epsilon| \leq \epsilon_m$ , problem (1.2) possesses at least  $m$  distinct solutions.

### 3 Proof for Symmetric Case

In this section, we assume that  $N \geq 1$  and  $E = W_0^{1,P}(\Omega)$ ,  $u \in E$  is called a weak solution of problem (1.1) if

$$\int_{\Omega} a(|\nabla u|) \nabla u \nabla \phi dx = \int_{\Omega} f(x, u) \phi dx, \forall \phi \in E.$$

Set

$$I(u) = \int_{\Omega} P(|\nabla u|) dx - \int_{\Omega} F(x, u) dx, \forall u \in E$$

and we know that the critical points of  $I$  are just the weak solutions of problem (1.1).

Now we state the symmetric mountain pass theorem used in this section.

**Lemma 3.1** [11] Let  $E$  be an infinite dimensional Banach space and let  $I \in C^1(E, R)$  be even, satisfy (PS), and  $I(0) = 0$ , if  $E = V \oplus X$ , where  $V$  is finite dimensional, and  $I$  satisfies

(I<sub>1</sub>) there are constants  $\rho, \alpha > 0$  such that  $I|_{\partial B_\rho \cap X} \geq \alpha$ ;

(I<sub>2</sub>) for each finite dimensional subspace  $\tilde{E} \subset E$ , there is an  $R = R(\tilde{E})$  such that  $I \leq 0$  on  $\tilde{E} \setminus B_{R(\tilde{E})}$ , then  $I$  possesses an unbounded sequence of critical values.

For  $E$  a separable and reflexive Banach space, there exist (see [9])  $\{e_n\}_{n=1}^\infty \subset E$  and  $\{e_n^*\}_{n=1}^\infty \subset E^*$  such that

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m \end{cases} \quad \text{and } e_n^*(v) = \alpha_n \text{ for } v = \sum_{i=1}^\infty \alpha_i e_i \in E.$$

Now we set

$$V_j = \{u \in W_0^{1,P}(\Omega) : e_i^*(u) = 0, i > j\},$$

$$X_j = \{u \in W_0^{1,P}(\Omega) : e_i^*(u) = 0, i \leq j\},$$

so

$$W_0^{1,P}(\Omega) = V_j \oplus X_j. \tag{3.1}$$

**Lemma 3.2** Given  $\delta > 0$ , there is  $j \in N$  such that for all  $u \in X_j$ ,  $|u|_H \leq \delta \|u\|$ .

**Proof** We prove the lemma by contradiction. Suppose that there exist  $\delta > 0$  and  $u_j \in X_j$  for every  $j \in N$  such that  $|u_j|_H \geq \delta \|u_j\|$ . Taking  $v_j = \frac{u_j}{|u_j|_H}$ , we have  $|v_j|_H = 1$ , for every  $j \in N$  and  $\|v_j\| \leq \frac{1}{\delta}$ . Hence  $\{v_j\} \subset W_0^{1,P}(\Omega)$  is a bounded sequence, and we may suppose, without loss of generality, that  $v_j \rightharpoonup v$  in  $W_0^{1,P}(\Omega)$ . Furthermore,  $e_n^*(v) = 0$  for every  $n \in N$  since  $e_n^*(v_j) = 0$  for all  $j \geq n$ . This shows that  $v = 0$ . On the other hand, by the compactness of embedding  $W_0^{1,P}(\Omega) \hookrightarrow L^H(\Omega)$ , we conclude that  $|v|_H = 1$ . This proves the lemma.

**Lemma 3.3** Suppose  $f$  satisfy  $(f_*)$ , then there exist  $j \in N$  and  $\rho, \alpha > 0$ , such that  $I|_{\partial B_\rho \cap X_j} \geq \alpha$ .

**Proof** Now suppose that  $\|u\| > 1$ , from  $(f_*)$ , we know that

$$\begin{aligned} I(u) &= \int_\Omega P(|\nabla u|) dx - \int_\Omega F(x, u) dx \\ &\geq \|u\|^{p^-} - C_1 |u|_H^{h^+} - C_2. \end{aligned}$$

Consequently, considering  $\delta > 0$  to be chosen posteriorly by Lemma 3.2, we have for all  $u \in X_j$  and  $j$  sufficiently large,

$$I(u) \geq \|u\|^{p^-} (1 - C_1 \delta^{h^+} \|u\|^{h^+ - p^-}) - C_2.$$

Now, taking  $\|u\| = \rho(\delta) = (\frac{1}{2C\delta^{h^+}})^{\frac{1}{h^+ - p^-}}$ , and noting that  $\rho(\delta) \rightarrow +\infty$ , if  $\delta \rightarrow 0$ . We can choose  $\delta > 0$  such that  $\frac{1}{2}\rho^{p^-} > C_2$  with  $\rho > 1$ , then  $I(u) > 0$  for every  $u \in X_j$ ,  $\|u\| = \rho$ , the proof is complete.

**Lemma 3.4** Suppose  $f$  satisfy  $(f_2)$ , then for each finite dimensional subspace  $\tilde{E} \subset E$ , there is an  $R = R(\tilde{E})$  such that  $I \leq 0$  on  $\tilde{E} \setminus B_{R(\tilde{E})}$ .

**Proof** From condition  $(f_2)$ , given  $L > 0$ , there is a  $C > 0$  such that for every  $t \in R$ , a.e.  $x \in \Omega$ ,

$$F(x, t) \geq L|t|^{p^+} - C.$$

Now let  $\tilde{E}$  be a finite dimensional subspace, suppose that  $u \in \tilde{E}$  with  $\|u\| > 1$ , then

$$\begin{aligned} I(u) &= \int_{\Omega} P(\nabla u) dx - \int_{\Omega} F(x, u) dx \\ &\leq \|u\|^{p^+} - L \int_{\Omega} |u|^{p^+} dx + C \\ &\leq (1 - CL)\|u\|^{p^+} + C. \end{aligned}$$

Choose  $L$  large enough such that  $1 - LC < 0$  then there exists  $R(\tilde{E}) > 1$  such that  $I(u) \leq 0$  for all  $\|u\| \geq R(\tilde{E})$  and the proof is complete.

**Lemma 3.5** Suppose  $f$  satisfies  $(f_1)$ , then  $I$  satisfies (PS) condition.

**Proof** We suppose that  $\|u_n\| > 1$ ,

$$\begin{aligned} M + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{\eta} I'(u_n) u_n \\ &= \int_{\Omega} P(|\nabla u_n|) dx - \frac{1}{\eta} \int_{\Omega} p(|\nabla u_n|) \nabla u_n dx + \int_{\Omega} (\frac{1}{\eta} f(x, u_n) u_n - F(x, u_n)) dx \\ &\geq (1 - \frac{p^+}{\eta}) \|u_n\|^{p^-} - a_1 |\Omega| - C \|u_n\|^\sigma. \end{aligned}$$

Noting that  $1 < \sigma < p^-$ ,  $\eta > p^+$ ,  $\{u_n\}$  is bounded. By [9] Lemma 3.1, we know that  $I$  satisfies (PS) condition.

**Proof of Theorem 1.1** First, we recall that  $W_0^{1,P}(\Omega) = V_j \oplus X_j$ , where  $V_j$  and  $X_j$  are defined in (3.1). Invoking Lemma 3.3, we find  $j \in N$ , and  $I$  satisfies  $(I_1)$  with  $X = X_j$ . Now by Lemma 3.4  $I$  satisfies  $(I_2)$ . Since  $I(0) = 0$  and  $I$  is even, we may apply Lemma 3.1 to conclude that  $I$  possesses infinitely many nontrivial critical points. The proof is completed.

## 4 Proof for Non-Symmetric Case

First of all, let us recall some notions and facts from Degiovanni and Lancelotti [12]. Let  $E$  be a Banach space and  $I \in C^1(E, R)$ . For  $b \in \tilde{R} := R \cup \{-\infty, +\infty\}$ , set  $I^b = \{u \in E | I(u) \leq b\}$ .

**Definition 4.1** [12] Let  $a, b \in \tilde{R}$  with  $a \leq b$ . The pair  $(I^b, I^a)$  is said to be trivial, if for every neighborhood  $[\alpha', \alpha'']$  of  $a$  and  $[\beta', \beta'']$  of  $b$  ( $\alpha', \alpha'', \beta', \beta'' \in R$ ), there exist two closed subsets  $A$  and  $B$  such that  $I^{\alpha'} \subset A \subset I^{\alpha''}$ ,  $I^{\beta'} \subset B \subset I^{\beta''}$  and  $A$  is a strong deformation retract of  $B$ .

**Definition 4.2** [12] A real number  $c$  is said to be an essential value of  $I$ , if for every  $\epsilon > 0$ , there exist  $a, b \in (c - \epsilon, c + \epsilon)$  with  $a < b$  such that the pair  $(I^b, I^a)$  is not trivial.

**Lemma 4.1** [12] Let  $a, b \in \widetilde{R}$  with  $a < b$ . Let us assume that  $I$  has no essential value in  $(a, b)$ . Then the pair  $(I^b, I^a)$  is trivial.

**Lemma 4.2** [12] Let  $c$  be an essential value of  $I$ . Then for every  $\epsilon > 0$  there exists  $\delta > 0$  such that every  $J \in C^1(E, R)$  with  $\sup\{|J(u) - I(u)| | u \in E\} < \delta$  admits an essential value in  $(c - \epsilon, c + \epsilon)$ .

**Lemma 4.3** [12] Let  $c$  be an essential value of  $I$ . If  $(PS)_c$  holds for  $I$ , then  $c$  is a critical value of  $I$ .

**Proof of Theorem 1.2** Fix a number  $m \in N$ . For  $k \in N$ , choose a continuous function  $\beta_k(t) = 1$  if  $|t| \leq k$ ,  $\beta_k(t) = 0$  if  $|t| \geq k + 1$  and  $0 < \beta_k(t) < 1$ , if  $k < |t| < k + 1$ . Let  $g_k(x, t) = \beta_k(t)g(x, t)$  and  $G_k(x, t) = \int_0^t g_k(x, s)ds$ . For any  $k \in N$ , choose  $\epsilon_1(k) > 0$  such that for all  $x \in \Omega$  and  $t \in R$ ,

$$\epsilon_1(k)|g_k(x, t)| < 1, \epsilon_1(k)|G_k(x, t)| < 1, \epsilon_1(k)tg_k(x, t) < 1.$$

For  $k \in N$  and  $|\epsilon| \leq \epsilon_1(k)$ , set

$$I(u) = \int_{\Omega} P(|\nabla u|)dx - \int_{\Omega} F(x, u)dx, u \in W_0^{1,P}(\Omega)$$

and

$$I_{\epsilon k}(u) = \int_{\Omega} P(|\nabla u|)dx - \int_{\Omega} (F(x, u) + \epsilon G_k(x, u))dx, u \in W_0^{1,P}(\Omega).$$

Then  $I$  and  $I_{\epsilon k}$  satisfy  $(PS)_c$  for every real number  $c$ . Denote  $E_k = \text{span}\{e_1, e_2, \dots, e_k\}$ . By  $(f_2)$ , there exists an increasing sequence of positive numbers  $\{R_k\}$  such that

$$I(u) \leq 0, \forall u \in E_k, \|u\| \geq R_k.$$

Let  $D_k = \{u | u \in E_k, \|u\| \leq R_k\}$ , and  $\partial D_k$  be the boundary of  $D_k$  in  $E_k$ .

Define a sequence  $\{\Phi_k\}$  of sets of functions inductively as

$$\Phi_1 = \{h | h \in C(D_1, E), h \text{ is odd, and } h|_{\partial D_1} = id\},$$

and for  $k = 1, 2, \dots$ ,

$$\Phi_{k+1} = \{h | h \in C(D_{k+1}, E), h \text{ is odd, } h|_{\partial D_{k+1}} = id, \text{ and } h|_{D_k} \in \Phi_k\}.$$

Define for  $k = 1, 2 \dots$

$$b_k = \inf_{h \in \Phi_k} \max_{u \in D_k} I(h(u)).$$

It is obvious that  $b_1 \leq b_2 \leq b_3 \leq \dots$ . Define for  $k = 1, 2 \dots$ ,

$$G_k = \{h | h \in C(D_k, E), h \text{ is odd, and } h|_{\partial D_k} = id\},$$

$$\Gamma_k = \{h(\overline{D_j} \setminus Y) | h \in G_j, j \geq k, Y \in \Sigma, \text{ and } \gamma(Y) \leq j - k\}$$

and

$$c_k = \inf_{B \in \Gamma_k} \max_{u \in B} I(u).$$

where  $\Sigma$  is the class of closed symmetric subsets of  $E$  and  $\gamma(Y)$  is the genus of  $Y$  for  $Y \in \Sigma$ . By Rabinowitz [11, Proposition 9.33],  $c_k \rightarrow \infty$ , as  $k \rightarrow \infty$ . From the definition of  $b_k$  and  $c_k$ , it is clear that  $b_k \geq c_k$  for all  $k \in N$ . So  $b_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Let  $\Lambda = \{c \in R, c \text{ is an essential value of } I\}$ . Now we prove that  $\Lambda \neq \emptyset$ , and  $\sup \Lambda = +\infty$ . If this statement was false, then there would exist  $k \in N$  such that  $0 < b_k < b_{k+1}$  and  $[b_k, +\infty) \cap \Lambda = \emptyset$ . Choose real numbers  $\alpha', a, \alpha''$  such that

$$b_k < \alpha' < a < \alpha'' < b_{k+1}. \quad (4.1)$$

Let  $h \in \Phi_k$  be such that  $\max_{u \in D_k} I(h(u)) < \alpha'$ . for  $k \in N$ , define

$$D_{k+1}^+ = \{u | u = v + te_{k+1}, v \in E_k, t \geq 0, \|u\| \leq R_{k+1}\},$$

and let  $\partial D_{k+1}^+$  be the boundary of  $D_{k+1}^+$  in  $E_{k+1}$ . Extend  $h$  to be a function  $h_1 \in C(\partial D_{k+1}^+, E)$  as

$$h_1(u) = \begin{cases} h(u), & u \in D_k; \\ u, & u \in \partial D_{k+1}^+ \setminus D_k. \end{cases}$$

Clearly,  $h_1$  is well defined, continuous and  $h_1(\partial D_{k+1}^+) \subset I^{\alpha'}$ . Extend  $h_1$  to a function  $h_2 \in C(D_{k+1}^+, E)$  and let  $\beta = \max\{I(h_2(u)) | u \in D_{k+1}^+\}$ . By lemma 4.1, the pair  $(I^{+\infty}, I^a)$  is trivial. So there exist closed subsets  $A$  and  $B$  of  $E$  such that  $I^{\alpha'} \subset A \subset I^{\alpha''}$ ,  $I^\beta \subset B$ , and there exists a strong deformation interaction  $\eta : B \times [0, 1] \rightarrow B$  of  $B$  to  $A$ . Define  $h_3 \in C(D_{k+1}^+, E)$  as  $h_3(u) = \eta(h_2(u), 1)$  then  $h_3(u)$  satisfies

$$I(h_3(D_{k+1}^+)) \subset I^{\alpha''}, \quad (4.2)$$

$$h_3 \text{ is odd on } D_{k+1}^+ \cap E_k, \quad (4.3)$$

$$h_3 = id \text{ on } \partial D_{k+1}^+ \cap \partial D_{k+1}, \quad (4.4)$$

$$h_3|_{D_k} = h. \quad (4.5)$$

Define  $h_4 \in C(D_{k+1}, E)$  as

$$h_4(u) = \begin{cases} h_3(u), & u \in D_{k+1}^+; \\ -h_3(-u), & u \in D_{k+1} \setminus D_{k+1}^+. \end{cases}$$

Then (4.3) implies that  $h_4$  is odd, (4.4) implies  $h_4|_{\partial D_{k+1}} = id$ , and (4.5) implies  $h_4|_{D_k} \in \Phi_k$ . So  $h_4 \in \Phi_{k+1}$  which is a contradiction, since by (4.1) and (4.2) we have

$$b_{k+1} \leq \max_{u \in D_{k+1}} I(h_4(u)) = \max_{u \in D_{k+1}^+} I(h_3(u)) \leq \alpha'' < b_{k+1}.$$

Therefore  $\Lambda \neq \emptyset$  and  $\sup \Lambda = +\infty$ . Choose a strictly increasing sequence of positive number  $\{d_k\} \subset \Lambda$ , such that  $d_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ . By Lemma 4.2 and the definition of  $I_{\epsilon_k}$ , there

exists  $\epsilon_2(k) \in (0, \epsilon_1(k))$  such that if  $|\epsilon| \leq \epsilon_2(k)$ , then  $I_{\epsilon k}$  has at least  $m$  essential values  $d_{\epsilon k 1}, d_{\epsilon k 2}, \dots, d_{\epsilon k m}$  such that

$$0 < d_{\epsilon k 1} < d_{\epsilon k 2} < \dots < d_{\epsilon k m} < d_{m+1}.$$

According to Lemma 4.3, there are  $m$  distinct critical points  $u_{\epsilon k i}$  ( $i = 1, 2, \dots, m$ ) of  $I_{\epsilon k}$  such that

$$\int_{\Omega} P(|\nabla u_{\epsilon k i}|) dx - \int_{\Omega} (F(x, u_{\epsilon k i}) + \epsilon G_k(x, u_{\epsilon k i})) dx = d_{\epsilon k i},$$

and

$$\int_{\Omega} p(|\nabla u_{\epsilon k i}|) \nabla u_{\epsilon k i} dx = \int_{\Omega} (f(x, u_{\epsilon k i}) u_{\epsilon k i} + \epsilon g_k(x, u_{\epsilon k i}) u_{\epsilon k i}) dx.$$

Then there exists a constant  $C_m > 0$  independent of  $\epsilon$  and  $k$  such that for every  $k \in N$ ,  $|\epsilon| \leq \epsilon_2(k)$  and  $i = 1, 2, \dots, m$ ,

$$\begin{aligned} \int_{\Omega} P(|\nabla u_{\epsilon k i}|) dx - \int_{\Omega} F(x, u_{\epsilon k i}) dx &\leq C_m, \\ \int_{\Omega} p(|\nabla u_{\epsilon k i}|) \nabla u_{\epsilon k i} dx - \int_{\Omega} f(x, u_{\epsilon k i}) u_{\epsilon k i} &\geq -C_m. \end{aligned}$$

Then we have for every  $k \in N$ ,  $|\epsilon| \leq \epsilon_2(k)$ , and  $i = 1, 2, \dots, m$ ,

$$\|u_{\epsilon k i}\| \leq C'_m, \quad (4.6)$$

where  $C'_m$  is a constant independent of  $\epsilon$  and  $k$  and so there exists a constant  $C''_m$  independent of  $\epsilon$  and  $k$  such that for every  $k \in N$ ,  $|\epsilon| \leq \epsilon_2(k)$  and  $i = 1, 2, \dots, m$ ,

$$\|u_{\epsilon k i}\|_{C(\bar{\Omega})} \leq C''_m. \quad (4.7)$$

So if  $k > C''_m$  then for any  $\epsilon$  with  $|\epsilon| \leq \epsilon_2(k)$  the problem possesses  $m$  distinct solutions  $u_{\epsilon k 1}, u_{\epsilon k 2}, \dots, u_{\epsilon k m}$ . The proof is complete.

**Remark 4.1** In this theorem we assume (2.2) just because we can get (4.7) from (4.6) by the embedding result such that  $C''_m$  is independent of  $\epsilon$  and  $k$ .

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## Orlicz-Sobolev 空间上对称及非对称拟线性椭圆方程的多解性

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**摘要:** 本文研究了具光滑边界的有界域上拟线性椭圆问题的多解性. 在Orlicz-Sobolev空间中利用变分及扰动的方法, 得到了方程在对称及非对称情况下解的存在性和多解性.

**关键词:** Orlicz-Sobolev 空间; 拟线性椭圆方程; 扰动方法; 对称性

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