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THE CHARACTERIZATION OF n-REGULAR MONOIDS BY FLATNESS PROPERTY

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Abstract: In this paper, we investigate the problem of generalization of principal weak flatness. Using equal systems of tensor products and the method of homological classification, we obtain the characterizations of generally regular monoids, which extends the main results on the characterizations of regular monoids.

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1 Introduction

Throughout this paper, S always stands for a monoid. A nonempty set A is called a right S-act, if there exists a mapping $A \times S \to A, (a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. Let S be a monoid and A be a right S-act. A is called principally weakly flat if the functor $A \otimes -$ preserves embeddings of principal left ideals of S into S. In this article, we generalize regular monoids, and define a new class of monoids, which is called n-regular monoids. Using a generalization of principal weak flatness, we give some characterizations of these monoids, and some known results are easily obtained.

Definition 1.1 Let S be a monoid and n a positive integer. A right S-act A is called principally weakly n-flat if for every $s \in S$, $A \otimes Ss^n \to A \otimes S$ is monic.

Remark 1.2 It is clear that in the above definition, when $n = 1$, then a right S-act A is principally weakly n-flat if and only if A is principally weakly flat. When $n \geq 2$, we will show that there exists a right S-act A which is principally weakly n-flat but not principally weakly flat.

Suppose I is a proper right ideal of S, by [1] the amalgam of two copies of S with the core I is defined as follows. If x, y and z denote elements not belonging to S, define

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$A(I) = (\{x, y\} \times (S - I)) \cup (\{z\} \times I)$, and define a right S-action on $A(I)$ by

$$\begin{aligned}(x,u)s &= \begin{cases}(x,us), & \text{if } us \notin I,\\ (z,us), & \text{if } us \in I,\end{cases}\\ (y,u)s &= \begin{cases}(y,us), & \text{if } us \notin I,\\ (z,us), & \text{if } us \in I,\end{cases}\\ (z,u)s &= (z,us).\end{aligned}$$

Then $A(I)$ is a right S-act.

Lemma 1.3 [3] Let S be a monoid, $a, a' \in A_S, b, b' \in {}_SB$. Then $a\otimes b = a'\otimes b'$ in $A_S \otimes {}_SB$ if and only if there exist a positive integer m and elements $a_1, \cdots, a_m \in A_S, b_2, \cdots, b_m \in {}_SB$, $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{aligned}&a = a_1s_1,\\ &a_1t_1 = a_2s_2, \qquad s_1b = t_1b_2,\\ &a_2t_2 = a_3s_3, \qquad s_2b_2 = t_2b_3,\\ &\qquad\vdots \qquad\qquad\qquad \vdots\\ &a_mt_m = a', \qquad s_mb_m = t_mb'.\end{aligned}$$

Lemma 1.4 Let S be a monoid, A an S-act and n a positive integer. The following conditions are equivalent:

(1) A is principally weakly n-flat.

(2) For every $s \in S$, any $a, a' \in A$, $a \otimes s^n = a' \otimes s^n$ in $A \otimes S$ implies that there exist $m \in N$, elements $a_1, \cdots, a_m \in A_S$, and $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{aligned}&a = a_1s_1,\\ &a_1t_1 = a_2s_2, \qquad s_1s^n = t_1s^n,\\ &a_2t_2 = a_3s_3, \qquad s_2s^n = t_2s^n,\\ &\qquad\vdots \qquad\qquad\qquad \vdots\\ &a_mt_m = a', \qquad s_ms^n = t_ms^n.\end{aligned}$$

Proof (1)$\Rightarrow$(2) Since A is principally weakly n-flat, for every $s \in S$, any $a, a' \in A$, $a \otimes s^n = a' \otimes s^n$ in $A \otimes S$ implies that $a \otimes s^n = a' \otimes s^n$ in $A \otimes Ss^n$. By Lemma 1 there exists a positive integer m and elements $a_1, \cdots, a_m \in A_S, q_2, \cdots, q_m \in Ss^n$, $u_1, v_1, \cdots, u_m, v_m \in S$

such that

$$\begin{array}{ll} a = a_1u_1, & \\ a_1v_1 = a_2u_2, & u_1s^n = v_1q_2, \\ a_2v_2 = a_3u_3, & u_2q_2 = v_2q_3, \\ \vdots & \vdots \\ a_mv_m = a', & u_mq_m = v_ms^n. \end{array}$$

Since $q_2, q_3, \cdots q_m \in Ss^n$, there exist $c_2, c_3, \cdots, c_m \in S$ such that $q_i = c_is^n$ $(i = 2, \cdots, m)$. Hence we have

$$\begin{array}{ll} a = a_1u_1, & \\ a_1v_1c_2 = a_2u_2c_2, & u_1s^n = v_1c_2s^n, \\ a_2v_2c_3 = a_3u_3c_3, & u_2c_2s^n = v_2c_3s^n, \\ \vdots & \vdots \\ a_{m-1}v_{m-1}c_m = a_mu_mc_m, & u_{m-1}c_{m-1}s^n = v_{m-1}c_ms^n, \\ a_mv_m = a', & u_mc_ms^n = v_ms^n. \end{array}$$

Denote $u_1 = s_1, v_m = t_m, u_ic_i = s_i, v_{i-1}c_i = t_{i-1}, i = 2, \cdots, m$. Then the result follows.

(2)⇒(1) For every $s \in S$, any $a, a' \in A$, $a \otimes s^n = a' \otimes s^n$ in $A \otimes S$. By (2) there exist $m \in N$, elements $a_1, \cdots, a_m \in A_S$, and $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{array}{ll} a = a_1s_1, & \\ a_1t_1 = a_2s_2, & s_1s^n = t_1s^n, \\ a_2t_2 = a_3s_3, & s_2s^n = t_2s^n, \\ \vdots & \vdots \\ a_mt_m = a', & s_ms^n = t_ms^n. \end{array}$$

Now

$$\begin{aligned} a \otimes s^n &= a_1s_1 \otimes s^n = a_1 \otimes s_1s^n = a_1 \otimes t_1s^n = a_1t_1 \otimes s^n \\ &= a_2s_2 \otimes s^n = a_2 \otimes s_2s^n = a_2 \otimes t_2s^n = a_2t_2 \otimes s^n = \cdots = a' \otimes s^n \end{aligned}$$

in $A \otimes Ss^n$.

2 Main Results

A monoid S is called regular, if for every $s \in S$, there exists $x \in S$ such that $s = sxs$. Now we have the following:

Definition 2.1 Let n be a positive integer. A monoid S is called a n-regular monoid, if for every $s \in S$, there exists $x \in S$ such that $s^n = s^n x s^n$.

Lemma 2.2 Let S be a monoid, I a proper right ideal of S and n a positive integer. Then the following conditions are equivalent:

(1) $A(I)$ is principally weakly n-flat.

(2) For every $s \in S$, if there exists $r \in S - I$ such that $rs^n \in I$, then there exists $j \in I$ such that $rs^n = js^n$.

Proof (1)⇒(2) Since $A(I)$ is principally weakly n-flat, for every $s \in S$, if there exists $r \in S - I$ such that $rs^n \in I$, then $(x, r) \otimes s^n = (y, r) \otimes s^n$ in $A(I) \otimes S$. By Lemma 1.4 there exist $m \in N$, $p_1, \cdots, p_m \in S$, $s_1, t_1, \cdots, s_m, t_m \in S$, and $w_1, \cdots, w_m \in \{x, y, z\}$ such that

$$\begin{aligned} (x, r) &= (w_1, p_1)s_1, & & \\ (w_1, p_1)t_1 &= (w_2, p_2)s_2, & s_1 s^n &= t_1 s^n, \\ &\vdots & &\vdots \\ (w_{m-1}, p_{m-1})t_{m-1} &= (w_m, p_m)s_m, & s_{m-1}s^n &= t_{m-1}s^n, \\ (w_m, p_m)t_m &= (y, r), & s_m s^n &= t_m s^n. \end{aligned}$$

There exists $k \in \{1, \cdots, m-1\}$ such that $w_k \neq w_{k+1}$, and so, there exists $j \in I$ such that $p_k t_k = p_{k+1}s_{k+1} = j$. So we have

$$rs^n = p_1 s_1 s^n = p_1 t_1 s^n = p_2 s_2 s^n = \cdots = p_k t_k s^n = js^n.$$

(2)⇒(1) Let for every $s \in S$, any $a, a' \in A(I)$, $a \otimes s^n = a' \otimes s^n$ in $A(I) \otimes S$. Since $(x, 1)S \cong S \cong (y, 1)S$(a free hence principally weakly n-flat S-act), without loss of generality we may restrict ourselves to the case in which $a = (x, r_1), a' = (y, r_2), r_1, r_2 \in S - I$. Since $(x, r_1) \otimes s^n = (y, r_2) \otimes s^n$, then $r_1 s^n = r_2 s^n \in I$. By (2) there exists $j \in I$ such that $r_1 s^n = js^n = r_2 s^n$. Hence

$$(x, r_1) \otimes s^n = (x, 1) \otimes r_1 s^n = (x, 1) \otimes js^n = (y, 1) \otimes js^n = (y, r_2) \otimes s^n$$

in $A(I) \otimes Ss^n$.

Corollary 2.3 [3] Let S be a monoid and I be a proper right ideal of S. Then the following conditions are equivalent:

(1) $A(I)$ is principally weakly flat.

(2) For every $i \in I$, there exists $j \in I$ such that $i = ji$.

Proposition 2.4 Let S be a monoid, I a proper right ideal of S and n a positive integer. Then the following conditions are equivalent:

(1) S/I is principally weakly n-flat.

(2) For every $s \in S$, if there exists $r \in S - I$ such that $rs^n \in I$, then there exist $j \in I$ such that $rs^n = js^n$.

Proof (1)⇒(2) Since S/I is principally weakly n-flat, for every $s \in S$, if there exists $r \in S - I$ such that $rs^n \in I$, then for every $j' \in I$, we have $[r] \otimes s^n = [j'] \otimes s^n$ in $S/I \otimes S$. By Lemma 1.4 there exist $m \in N$, $u_1, \cdots, u_m \in S$, and $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{aligned} [r] &= [u_1]s_1, & & \\ [u_1]t_1 &= [u_2]s_2, & s_1 s^n &= t_1 s^n, \\ &\vdots & &\vdots \\ [u_{m-1}]t_{m-1} &= [u_m]s_m, & s_{m-1}s^n &= t_{m-1}s^n, \\ [u_m]t_m &= [j'], & s_m s^n &= t_m s^n. \end{aligned} \qquad (*)$$

Since $j' \in I$, we have $u_m t_m \in I$. Let k be the least number such that $k \in \{1, 2, \cdots, m\}$ and $u_k t_k \in I$. If $j = u_k t_k$, then $u_{k-1}t_{k-1} \in S - I$. Since $[u_{k-1}]t_{k-1} = [u_k]s_k$, we have $u_{k-1}t_{k-1} = u_k s_k$ and so

$$rs^n = u_1 s_1 s^n = u_1 t_1 s^n = u_2 s_2 s^n = u_2 t_2 s^n = \cdots = u_{k-1}t_{k-1}s^n = u_k s_k s^n = u_k t_k s^n = js^n.$$

(2)⇒(1) For every $s \in S$, any $u, u' \in S$, if $[u] \otimes s^n = [u'] \otimes s^n$ in $S/I \otimes S$, we have the following four cases to consider:

Case 1 $u, u' \in I$. Then it is clear that $[u] \otimes s^n = [u'] \otimes s^n$ in $S/I \otimes Ss^n$.

Case 2 $u \in I$. $u' \in S - I$. By assumption there exists $j \in I$ such that $u's^n = js^n$. Then

$$[u] \otimes s^n = [j] \otimes s^n = [1] \otimes js^n = [1] \otimes u's^n = [u'] \otimes s^n$$

in $S/I \otimes Ss^n$.

Case 3 $u \in S - I$. $u' \in I$. It is similar to Case 2.

Case 4 $u, u' \in S - I$. By $[u] \otimes s^n = [u'] \otimes s^n$ in $S/I \otimes S$, we have $us^n = u's^n$ or $us^n, u's^n \in I$. If $us^n = u's^n$, the result follows. Otherwise by assumption there exists $j_1, j_2 \in I$ such that $us^n = j_1 s^n$ and $u's^n = j_2 s^n$. So

$$[u] \otimes s^n = [1] \otimes us^n = [1] \otimes j_1 s^n = [j_1] \otimes s^n = [j_2] \otimes s^n = [1] \otimes j_2 s^n = [1] \otimes u's^n = [u'] \otimes s^n$$

in $S/I \otimes Ss^n$.

Corollary 2.5 [4] Let S be a monoid and I be a proper right ideal of S. Then the following conditions are equivalent:

(1) S/I is principally weakly flat.

(2) For every $i \in I$, there exists $j \in I$ such that $i = ji$.

Theorem 2.6 Let n be a positive integer. The following conditions on a monoid S are equivalent:

(1) All right S-acts are principally weakly n-flat.

(2) All finitely generated right S-acts are principally weakly n-flat.

(3) S is a n-regular monoid.

Proof (1)⇒(2) is clear.

(2)⇒(3) Let $s \in S$. If $s^nS = S$, it is clear that there exists $x \in S$ such that $s^n = s^nxs^n$. Otherwise s^nS is a proper right ideal of S. Denote $I = s^nS$, then $A(I)$ is a finitely generated S-act, by assumption $A(I)$ is principally weakly n-flat. Since $s^n \in I$ and by Lemma 2.2 there exists $j \in I$ such that $s^n = js^n$. Hence there exists $x \in S$ such that $j = s^nx$, that is $s^n = s^nxs^n$.

(3)⇒(1) Suppose A is a right S-act. Since S is n-regular, for every $s \in S$, there exists $x \in S$ such that $s^n = s^nxs^n$. For any $a, a' \in A$, if $a \otimes s^n = a' \otimes s^n$ in $A \otimes S$. Then

$$a \otimes s^n = a \otimes s^nxs^n = as^n \otimes xs^n = a's^n \otimes xs^n = a' \otimes s^nxs^n = a' \otimes s^n$$

in $A \otimes Ss^n$.

Corollary 2.7 [5] The following conditions on a monoid S are equivalent:

(1) All right S-acts are principally weakly flat.

(2) All finitely generated right S-acts are principally weakly flat.

(3) S is a regular monoid.

Theorem 2.8 Let n be a positive integer. The following conditions on a monoid S are equivalent:

(1) All cyclic right S-acts are principally weakly n-flat.

(2) All Rees factor right S-acts are principally weakly n-flat.

(3) S is a n-regular monoid.

Proof (1)⇒(2) is clear.

(2)⇒(3) By Proposition 2.4, the proof is similar to Theorem 2.6.

(3)⇒(1) By Theorem 2.6, it is clear.

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n-正则幺半群的平坦性刻画

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摘要: 本文研究了主弱平坦性质的推广问题. 利用张量积相等的等式组, 以及同调分类方法, 获得了对广义正则的幺半群的刻画结果, 推广了关于正则幺半群刻画的主要的结果.

关键词: 正则幺半群; n-正则幺半群; 主弱平坦性; 同调分类

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