

GLOBAL EXISTENCE AND BLOW-UP FOR A NONLINEAR POROUS MEDIUM EQUATION WITH NONLINEAR NONLOCAL BOUNDARY CONDITION

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Abstract: This paper studies a degenerate nonlinear porous medium equation $u_t = \Delta u^m + au^p \int_{\Omega} u^q dx$ with nonlinear and nonlocal boundary condition $u|_{\partial\Omega \times (0, \infty)} = \int_{\Omega} g(x, y) u^l(y, t) dy$. With the help of the comparison principle and super-, sub-solution methods, some criteria on this problem which determine whether the solutions blow up in a finite time or the solutions exist for all time are given. These results show that the global existence and blow-up results depend on the weight function $g(x, y)$ and the size of l . Finally, the blow-up rate of the blow-up solutions is given.

Keywords: porous medium equation; nonlinear nonlocal boundary condition; global existence; blow-up; blow-up rate

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1 Introduction

In the past decades, many physical phenomena have been formulated into nonlocal mathematical models and studied by many authors. In particular, there exist many articles dealing with properties of solutions to semilinear or degenerate parabolic equations with homogeneous Dirichlet boundary condition (see [1–4] and references therein). For example, Li et al. [5] studied the nonlinear nonlocal porous medium equation

$$u_t - \Delta u^m = au^p \int_{\Omega} u^q dx, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

which is subjected to homogeneous Dirichlet boundary condition. Under appropriate hypotheses for initial data, the following results were obtained:

(1) In case of $p + q < m$ or $p + q = m$ and a is sufficiently small, then there exists a global positive classical solution.

(2) In case of $p + q > m$, then the solution either blows up in finite time if initial value u_0 is large enough or exists globally provided that u_0 is sufficiently small.

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However, there are some important phenomena formulated into parabolic equations with nonlocal boundary conditions in mathematical modeling such as thermoelasticity theory (see [6]). In 1986, Friedman [7] studied the following parabolic equation with nonlocal boundary condition:

$$\begin{cases} u_t = \Delta u + c(x)u, & x \in \Omega, t > 0, \\ u(x, t) = \int_{\Omega} g(x, y)u(y, t)dy, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (1.2)$$

He gave the global existence of solutions provided that $\int_{\Omega} |g(x, y)|dy < 1$ for $x \in \partial\Omega$ and $c(x) \leq 0$. Recently, Gladkov and Kim [8] extended the type of (1.2), they considered the problem of the form:

$$\begin{cases} u_t = \Delta u + c(x, t)u^p, & x \in \Omega, t > 0, \\ u(x, t) = \int_{\Omega} g(x, y)u^l(y, t)dy, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega \end{cases} \quad (1.3)$$

with $p, l > 0$. And some criteria for the existence of global solution as well as for the solution to blow-up in finite time were obtained. For other works on parabolic equations with nonlocal nonlinear boundary conditions, we refer readers to [9–11] and references therein.

Motivated by those of works above, we will extend the works in [5] and investigate the positive solution of the following degenerate nonlinear nonlocal porous medium equation:

$$\begin{cases} u_t = \Delta u^m + au^p(x, t) \int_{\Omega} u^q(x, t)dx, & x \in \Omega, t > 0, \\ u(x, t) = \int_{\Omega} g(x, y)u^l(x, t)dy, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $a > 0$, $m > 1$, $p, q \geq 0$, $l > 0$. The aim of this paper is to understand how the weight function $g(x, y)$ and the nonlinear term $u^l(y, t)$ in the boundary condition play substantial roles in determining blow-up or not of solutions and give some criteria for the existence of global solutions as well as for a blow-up of solutions in a finite time.

In this paper, we give the following hypotheses (H1)–(H4):

(H1) $u_0(x) \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$, $\alpha \in (0, 1)$, $u_0(x) > 0$ in Ω , $u_0(x) = \int_{\Omega} g(x, y)u_0^l(y)dy$ on $\partial\Omega$.

(H2) $g(x, y)$ is a nonnegative and continuous function defined for $x \in \partial\Omega, y \in \Omega$.

(H3) $\Delta u_0^m + au_0^p \int_{\Omega} u_0^q dx > 0, x \in \Omega$.

(H4) There exists a constant $\rho \geq \rho_0$ such that

$$\Delta u_0^m + au_0^p \int_{\Omega} u_0^q dx - \rho u_0^{p+q} \geq 0,$$

where

$$\rho_0 = (a|\Omega|(p + q - 1)/(m - 1))((q + m - 1)/(p + 2q - 1))^{(p+2q-1)/(p+q-1)}.$$

The plan of this paper is as follows: In Section 2, we establish the comparison principle and the local existence. Some criteria regarding to global existence and finite time blow-up for problem (1.4) are given in Section 3. In Section 4, we give the blow-up rate estimate for the special case of $l = 1$.

2 Comparison Principle and Local Existence

At first, we give the definition of sub-solution and super-solution of (1.4) and comparison principle. Let $Q_T = \Omega \times (0, T), S_T = \partial\Omega \times (0, T), \bar{Q}_T = \bar{\Omega} \times [0, T)$.

Definition 1 A function $\tilde{u}(x, t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is called a sub-solution of (1.4) on Q_T if

$$\begin{cases} \tilde{u}_t \leq \Delta \tilde{u}^m + a\tilde{u}^p \int_{\Omega} \tilde{u}^q dx, & x \in \Omega, t > 0, \\ \tilde{u}(x, t) \leq \int_{\Omega} g(x, y)\tilde{u}^l(y, t)dy, & x \in \partial\Omega, t > 0, \\ \tilde{u}(x, 0) \leq u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Similarly, a super-solution $\bar{u}(x, t)$ of (1.4) is defined by the opposite inequalities. We say that u is a solution of (1.4) in Q_T if both a sub-solution and a super-solution of (1.4) in Q_T . Further, we say that u is a global solution of (1.4) if it is a solution of (1.4) in Q_T for any $T > 0$. By similar arguments in [12] we have

Lemma 1 Suppose that $w(x, t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and satisfies

$$\begin{cases} w_t - d(x, t)\Delta w \geq c_1(x, t)w + c_2(x, t) \int_{\Omega} c_3(x, t)w(x, t)dx, & (x, t) \in Q_T, \\ w(x, t) \geq c_4(x, t) \int_{\Omega} c_5(x, y)w^l(y, t)dy, & (x, t) \in S_T, \end{cases} \tag{2.2}$$

where $d(x, t), c_i(x, t)$ ($i = 1, 2, 3, 4$) are bounded functions and $d(x, t), c_i(x, t) \geq 0$ ($i = 1, 2, 3, 4$) in Q_T , $c_5(x, y) \geq 0$ for $x \in \partial\Omega, y \in \Omega$ and is not identically zero. Then, $w(x, 0) > 0$ for $x \in \bar{\Omega}$ implies that $w(x, t) > 0$ in Q_T . Moreover, $c_5(x, t) \equiv 0$ or $c_4(x, t) \int_{\Omega} c_5(x, y)dy \leq 1$ on S_T , then $w(x, 0) \geq 0$ for $x \in \bar{\Omega}$ implies that $w(x, t) \geq 0$ in Q_T .

Lemma 2 Let \tilde{u} and \bar{u} be nonnegative sub-solution and super-solution of (1.4), respectively, with $\tilde{u}(x, 0) \leq \bar{u}(x, 0)$ in $\bar{\Omega}$. Then, $\tilde{u}(x, t) \leq \bar{u}(x, t)$ in \bar{Q}_T if $\tilde{u} > \eta$ or $\bar{u} > \eta$ for some small positive constant η holds.

Local in time existence of positive classical solutions of (1.4) can be obtained by using fixed point theorem [13], the representation formula and the contraction mapping principle as in [12], the proof is more or less standard, so is omitted here.

3 Global Existence and Blow-Up in Finite Time

In this section, we shall get some results about the existence and nonexistence of global solution for problem (1.4).

Theorem 1 Assume that $0 < l \leq 1$ and $\int_{\Omega} g(x, y) dy < 1$ for all $x \in \partial\Omega$.

- (1) If $p + q < m$, then the solution of (1.4) exists globally.
- (2) If $p + q = m$, and a is sufficiently small, then the solution of (1.4) exists globally.
- (3) If $p + q > m$, and a is sufficiently small, then the solution of (1.4) exists globally provided that $u_0(x)$ satisfies

$$u_0(x) \leq \left(\frac{\varepsilon_0}{a|\Omega|} \right)^{\frac{1}{p+q-m}} \psi^{\frac{1}{m}}(x),$$

where $\psi(x), \varepsilon_0$ are defined by (3.1).

Proof Since $\int_{\Omega} g(x, y) dy < 1$, there exists a sufficiently small constant $\varepsilon_0 > 0$ such that $\psi(x)$ is the unique solution of the linear elliptic problem

$$-\Delta\psi(x) = \varepsilon_0, \quad x \in \Omega; \quad \psi(x) = \left(\int_{\Omega} g(x, y) dy \right)^m, \quad x \in \partial\Omega. \quad (3.1)$$

Moreover, $0 < \psi(x) < 1$ for $x \in \bar{\Omega}$. Denote $\bar{u}(x, t) = K\psi^{\frac{1}{m}}(x)$, $x \in \Omega, t > 0$, where $K > 1$ is a constant to be determined later. A series of computations yields

$$\begin{cases} \bar{u}_t - \Delta\bar{u}^m = \varepsilon_0 K^m, \\ a\bar{u}^p \int_{\Omega} \bar{u}^q dx = aK^{p+q} \psi^{\frac{p}{m}}(x) \int_{\Omega} \psi^{\frac{q}{m}}(x) dx \leq a|\Omega|K^{p+q}. \end{cases} \quad (3.2)$$

- (1) If $p + q < m$, we choose that K large enough such that

$$K > \max\{(\varepsilon_0^{-1}a|\Omega|)^{1/(m-p-q)}, 1\},$$

then we can get

$$\begin{cases} \bar{u}_t - \Delta\bar{u}^m \geq a\bar{u}^p \int_{\Omega} \bar{u}^q dx, & x \in \Omega, t > 0, \\ \bar{u}(x, 0) = K\psi^{\frac{1}{m}}(x) \geq u_0(x), & x \in \Omega. \end{cases} \quad (3.3)$$

On the other hand, we have for $x \in \partial\Omega, t > 0$,

$$\bar{u}(x, t) = K \int_{\Omega} g(x, y) dy \geq K \int_{\Omega} g(x, y) \psi^{\frac{1}{m}}(y) dy \geq \int_{\Omega} g(x, y) \bar{u}^l(y, t) dy. \quad (3.4)$$

Then that $\bar{u}(x, t)$ is a super-solution of (1.4). Therefore, the solution of (1.4) exists globally.

(2) If $p + q = m$, set $a_0 = \varepsilon_0|\Omega|^{-1}$, then (3.2)–(3.4) hold provided that $a \leq a_0$ and $K(> 1)$ is large enough. Thus, the solution $u(x, t)$ of (1.4) exists globally.

(3) If $p + q > m$, we can choose a is sufficiently small such that

$$K = K_0 = \left(\frac{\varepsilon_0}{a|\Omega|}\right)^{\frac{1}{p+q-m}} \geq 1.$$

Then it is easy to verify that (3.2)–(3.4) hold provided $u_0(x) \leq \left(\frac{\varepsilon_0}{a|\Omega|}\right)^{\frac{1}{p+q-m}} \psi^{\frac{1}{m}}(x)$. Therefore, $\bar{u}(x, t) = K_0\psi^{\frac{1}{m}}(x)$ is a super-solution of (1.4), thus, the solution $u(x, t)$ of (1.4) exists globally.

Theorem 2 Assume that $l > 1$ and $\int_{\Omega} g(x, y)dy < 1$ for all $x \in \partial\Omega$.

(1) If $p + q = m$, a and $u_0(x)$ are sufficiently small, then the solution of (1.4) exists globally.

(2) If $p + q > m$, and $u_0(x)$ is sufficiently small, then the solution of (1.4) exists globally. The proof is similar to Theorem 1, here we omit them.

Theorem 3 Assume that $l \geq 1$ and $\int_{\Omega} g(x, y)dy < 1$ for all $x \in \partial\Omega$. If $p + q > m$, then the solution of (1.4) blows up in finite time provided that $u_0(x)$ is large enough.

Proof To prove the blow-up result, denote by $\varphi(x)$ the first eigenfunction of the problem

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega; \quad \varphi(x) = \int_{\Omega} g(x, y)\varphi^l(y)dy, \quad x \in \partial\Omega \tag{3.5}$$

with the first eigenvalue $\lambda_1 > 0$. Let $K_1 = \min_{x \in \Omega} \varphi(x)$, $K_2 = \max_{x \in \Omega} \varphi(x)$, and $s(t)$ be the solution of the following ODE

$$s'(t) = bs^\mu(t), \quad t > 0; \quad s(0) = \delta > 1, \tag{3.6}$$

where $b > 0$ and $\delta, \mu > 1$ to be fixed later. Clearly, $s(t) \geq \delta > 1$ becomes unbounded in finite time $T(\delta)$. Set $\tilde{u}(x, t) = s^l(t)\varphi^l(x)$, a series of computations yields

$$\begin{aligned} & \Delta\tilde{u}^m + a\tilde{u}^p \int_{\Omega} \tilde{u}^q dx \\ &= ml(ml - 1)s^{ml}\varphi^{ml-2}|\nabla\varphi|^2 - \lambda_1 mls^{ml}\varphi^{ml} + as^{(p+q)l}\varphi^{pl}(x) \int_{\Omega} \varphi^{ql}(x)dx \\ &\geq ls^{l-1}\varphi^l \frac{a|\Omega|K_1^{(p+q)l}}{lK_2^l} s^{ml-l+1} \left(s^{(p+q-m)l} - \frac{\lambda_1 mlK_2^{ml}}{a|\Omega|K_1^{(p+q)l}} \right), \end{aligned} \tag{3.7}$$

$$\tilde{u}_t = ls^{l-1}\varphi^l s'(t) = ls^{l-1}\varphi^l bs^\mu(t). \tag{3.8}$$

Since $p + q > m$, thus, if we choose

$$b \leq \frac{a|\Omega|K_1^{(p+q)l}}{lK_2^l}, \quad \mu = ml - l + 1, \quad \delta \geq \left(1 + \frac{\lambda_1 mlK_2^{ml}}{a|\Omega|K_1^{(p+q)l}}\right)^{\frac{1}{(p+q-m)l}}.$$

Then, from (3.7) and (3.8) we have

$$\tilde{u}_t \leq \Delta\tilde{u}^m + a\tilde{u}^p \int_{\Omega} \tilde{u}^q dx, \quad (x, t) \in \Omega \times (0, T(\delta)). \tag{3.9}$$

On the other hand, if $l = 1$, then we have

$$\begin{aligned}\tilde{u}(x, t) &= s(t)\varphi(x) = s(t) \int_{\Omega} g(x, y)\varphi(y)dy \\ &= \int_{\Omega} g(x, y)s(t)\varphi(y)dy = \int_{\Omega} g(x, y)\tilde{u}(y, t)dy \quad \text{for } (x, t) \in \partial\Omega \times (0, T(\delta)).\end{aligned}$$

Second, if $l > 1$, by Hölder inequality,

$$\begin{aligned}\int_{\Omega} g(x, y)\varphi^l(y)dy &= \int_{\Omega} g(x, y)^{\frac{l-1}{l}}g(x, y)^{\frac{1}{l}}\varphi^l(y)dy \\ &\leq \left(\int_{\Omega} g(x, y)dy\right)^{\frac{l-1}{l}} \left(\int_{\Omega} g(x, y)\varphi^{l^2}(y)dy\right)^{\frac{1}{l}}.\end{aligned}$$

Thus, for $(x, t) \in \partial\Omega \times (0, T(\delta))$ and together with $\int_{\Omega} g(x, y)dy < 1$, $s(t) > 1$ and Jensen's inequality, we obtain

$$\begin{aligned}\tilde{u}(x, t) &= s^l(t) \left(\int_{\Omega} g(x, y)\varphi^l(y)dy\right)^l \\ &\leq s^l(t) \left(\int_{\Omega} g(x, y)dy\right)^{l-1} \int_{\Omega} g(x, y)\varphi^{l^2}(y)dy \\ &\leq s^l(t) \int_{\Omega} g(x, y)\varphi^{l^2}(y)dy = \int_{\Omega} g(x, y)s^l(t)\varphi^{l^2}(y)dy \\ &\leq \int_{\Omega} g(x, y)s^{l^2}(t)\varphi^{l^2}(y)dy = \int_{\Omega} g(x, y)\tilde{u}^l(y, t)dy.\end{aligned}\quad (3.10)$$

Also for $x \in \bar{\Omega}$, we have

$$\tilde{u}(x, 0) = s^l(0)\varphi^l(x) = \delta^l\varphi^l(x) \leq u_0(x).\quad (3.11)$$

Inequalities (3.9)–(3.11) imply that solution $u(x, t)$ of (1.4) blows up in finite time.

Theorem 4 Assume that $l \geq 1$ and $\int_{\Omega} g(x, y)dy \geq 1$ for all $x \in \partial\Omega$. If $p + q > 1$, then the solution of (1.4) blows up in finite time provided that $u_0(x)$ is large enough.

Proof Since $p + q > 1$, there exists a constant μ_0 such that $p + q > \mu_0 > 1$. Let $\tilde{u}(x, t) = s(t)$, which $s(t)$ is a solution of (3.6) with $b = 1$, $\mu = \mu_0$, $\delta = \max\{1, (a|\Omega|)^{\frac{1}{\mu_0 - p - q}}\}$, moreover, $s(t) \geq \delta \geq 1$. For $x \in \Omega, t > 0$, computing directly shows

$$\tilde{u}_t - \Delta\tilde{u}^m - a\tilde{u}^p \int_{\Omega} \tilde{u}^q dx = s^{p+q}(t)(s^{\mu_0 - p - q}(t) - a|\Omega|) \leq 0.\quad (3.12)$$

For $x \in \partial\Omega, t > 0$, we obtain

$$\tilde{u}(x, t) = s(t) \leq \int_{\Omega} g(x, y)s(t)dy \leq \int_{\Omega} g(x, y)s^l(t)dy = \int_{\Omega} g(x, y)\tilde{u}^l(y, t)dy.\quad (3.13)$$

Therefore, $\tilde{u}(x, t)$ is a sub-solution of (1.4) provided that $\tilde{u}(x, 0) = s(0) = \delta \leq u_0(x)$, which imply that the solution of (1.4) blows up in finite time.

4 Blow-Up Rate Estimate

In this section, we consider the problem (1.4) with $l = 1, p + q > m$ and $\int_{\Omega} g(x, y)dy < 1$ for $x \in \partial\Omega$. By introducing some transformations $v = u^m, t = m\tau$, then (1.4) becomes

$$\begin{cases} v_{\tau} = v^r (\Delta v + av^{p_1} \int_{\Omega} v^{q_1} dx), & x \in \Omega, \tau > 0, \\ v(x, \tau) = (\int_{\Omega} g(x, y)v^{\frac{1}{m}}(y, \tau)dy)^m, & x \in \partial\Omega, \tau > 0, \\ v(x, 0) = v_0(x) = u_0^m(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $0 < r = (m - 1)/m < 1, p_1 = p/m, q_1 = q/m$. Under this transformation, assumptions (H1) and (H3)–(H4) become

(G1) $v_0(x) \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega}), \alpha \in (0, 1), v_0(x) > 0$ in $\Omega, v_0(x) = \int_{\Omega} g(x, y)v_0^l(y)dy$ on $\partial\Omega$.

(G2) $\Delta v_0 + av_0^{p_1} \int_{\Omega} v_0^{q_1} dx > 0, x \in \Omega$.

(G3) There exists a constant $\rho \geq \rho_1$ such that

$$\Delta v_0 + av_0^{p_1} \int_{\Omega} v_0^{q_1} dx - \rho v_0^{p_1+q_1} \geq 0,$$

where

$$\rho_1 = (a|\Omega|(r + p_1 + q_1 - 1)/r)((r + q_1)/(r + p_1 + 2q_1 - 1))^{(r+p_1+2q_1-1)/(r+p_1+q_1-1)}.$$

Suppose that the solution $v(x, \tau)$ of (4.1) blows up at finite time T^* and set

$$V(\tau) = \max_{x \in \bar{\Omega}} v(x, \tau),$$

then we have the following:

Lemma 3 Suppose that assumptions (H2) and (G1)–(G3) hold, then there exists a positive constant C_1 such that

$$V(\tau) \geq C_1(T^* - \tau)^{-\frac{1}{r+p_1+q_1-1}}. \tag{4.2}$$

Proof From (4.1), we have (see [14])

$$V'(\tau) \leq a|\Omega|V^{r+p_1+q_1}(\tau), \quad \text{a.e., } \tau \in (0, T^*).$$

Integrating above inequality over (τ, T^*) , then we draw the conclusion.

Lemma 4 Under the conditions of Lemma 3, then there exists a constant $\rho \geq \rho_1$, which is defined in (G3), such that

$$v_{\tau} - \rho v^{r+p_1+q_1} \geq 0, \quad (x, \tau) \in \Omega \times (0, T^*). \tag{4.3}$$

Proof Let $J(x, \tau) = v_\tau - \rho v^{r+p_1+q_1}$ for $(x, \tau) \in \Omega \times (0, T^*)$, a series of computations yields

$$\begin{aligned}
 & J_\tau - v^r \Delta J - (2r\rho v^{r+p_1+q_1-1} + ap_1 v^{r+p_1-1} \int_\Omega v^{q_1} dx) J - aq_1 v^{r+p_1} \int_\Omega v^{q_1-1} J dx \\
 \geq & a\rho(r + q_1) v^{r+p_1} \left(\frac{r\rho v^{r+p_1+2q_1-1}}{a(r + q_1)} + \frac{q_1}{r + q_1} \int_\Omega v^{r+p_1+2q_1-1} dx - v^{r+p_1+q_1-1} \int_\Omega v^{q_1} dx \right).
 \end{aligned} \tag{4.4}$$

By the Hölder inequality, we have

$$\int_\Omega v^{q_1} dx \leq |\Omega|^{\frac{r+p_1+q_1-1}{r+p_1+2q_1-1}} \left(\int_\Omega v^{r+p_1+2q_1-1} dx \right)^{\frac{q_1}{r+p_1+2q_1-1}}. \tag{4.5}$$

Noting that $\frac{r+p_1+q_1-1}{r+p_1+2q_1-1} + \frac{q_1}{r+p_1+2q_1-1} = 1$, for any $\varepsilon > 0$, by virtue of Young’s inequality, we obtain

$$\begin{aligned}
 & v^{r+p_1+q_1-1} \left(\int_\Omega v^{r+p_1+2q_1-1} dx \right)^{\frac{q_1}{r+p_1+2q_1-1}} \\
 \leq & \frac{(r + p_1 + q_1 - 1)\varepsilon}{r + p_1 + 2q_1 - 1} v^{r+p_1+2q_1-1} + \frac{q_1 \varepsilon^{-\frac{r+p_1+q_1-1}{q_1}}}{r + p_1 + 2q_1 - 1} \int_\Omega v^{r+p_1+2q_1-1} dx.
 \end{aligned} \tag{4.6}$$

Now, we take

$$\varepsilon = \left(\frac{r + q_1}{r + p_1 + 2q_1 - 1} |\Omega|^{\frac{r+p_1+q_1-1}{r+p_1+2q_1-1}} \right)^{\frac{q_1}{r+p_1+q_1-1}}.$$

Therefore, it follows from (4.4)–(4.6) that

$$\begin{aligned}
 & J_\tau - v^r \Delta J - (2r\rho v^{r+p_1+q_1-1} + ap_1 v^{r+p_1-1} \int_\Omega v^{q_1} dx) J - aq_1 v^{r+p_1} \int_\Omega v^{q_1-1} J dx \\
 \geq & r\rho(\rho - \rho_1) v^{2r+2p_1+2q_1-1} \geq 0.
 \end{aligned} \tag{4.7}$$

In addition, for $(x, \tau) \in \partial\Omega \times (0, T^*)$, from (4.1), we have

$$\begin{aligned}
 v_\tau &= \left(\int_\Omega g(x, y) v^{\frac{1}{m}}(y, \tau) dy \right)^{m-1} \int_\Omega g(x, y) v^{\frac{1-m}{m}}(y, \tau) v_\tau(y, \tau) dy \\
 &= \left(\int_\Omega g(x, y) v^{\frac{1}{m}}(y, \tau) dy \right)^{m-1} \int_\Omega g(x, y) v^{\frac{1-m}{m}}(y, \tau) (J(y, \tau) + \rho v^{r+p_1+q_1}(y, \tau)) dy \\
 &= \left(\int_\Omega g(x, y) v^{\frac{1}{m}}(y, \tau) dy \right)^{m-1} \left(\int_\Omega g(x, y) v^{\frac{1-m}{m}}(y, \tau) J(y, \tau) dy + \rho \int_\Omega g(x, y) v^{\frac{p+q}{m}}(y, \tau) dy \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 J(x, \tau) &= v_\tau - \rho v^{r+p_1+q_1} \\
 &= \left(\int_\Omega g v^{\frac{1}{m}} dy \right)^{m-1} \left(\int_\Omega g v^{\frac{1-m}{m}} J dy + \rho \left(\int_\Omega g v^{\frac{p+q}{m}} dy - \left(\int_\Omega g v^{\frac{1}{m}} dy \right)^{p+q} \right) \right).
 \end{aligned} \tag{4.8}$$

On the other hand, according to the Hölder inequality, we obtain

$$\int_\Omega g v^{\frac{p+q}{m}} dy - \left(\int_\Omega g v^{\frac{1}{m}} dy \right)^{p+q} \geq \int_\Omega g dy \left(\frac{\int_\Omega g v^{\frac{1}{m}} dy}{\int_\Omega g dy} \right)^{p+q} - \left(\int_\Omega g v^{\frac{1}{m}} dy \right)^{p+q} \geq 0.$$

Again by the Hölder inequality,

$$\left(\int_{\Omega} g J^{\frac{1}{m}} dy\right)^m \leq \int_{\Omega} g v^{\frac{1-m}{m}} J dy \left(\int_{\Omega} g v^{\frac{1}{m}} dy\right)^{m-1}.$$

Hence, (4.8) becomes

$$J(x, \tau) \geq \left(\int_{\Omega} g v^{\frac{1}{m}} dy\right)^{m-1} \int_{\Omega} g v^{\frac{1-m}{m}} J dy \geq \left(\int_{\Omega} g(x, y) J^{\frac{1}{m}}(y, \tau) dy\right)^m. \quad (4.9)$$

In addition, (G3) implies that $J(x, 0) \geq 0$. By Lemma 1 we obtain

$$J(x, \tau) \geq 0, (x, \tau) \in \Omega \times (0, T^*).$$

Integrating (4.3) over (τ, T^*) , we have

$$v(x, \tau) \leq \left(\rho(r + p_1 + q_1 - 1)\right)^{-\frac{1}{r+p_1+q_1-1}} (T^* - \tau)^{-\frac{1}{r+p_1+q_1-1}}. \quad (4.10)$$

Setting $C_2 = \left(\rho(r + p_1 + q_1 - 1)\right)^{-\frac{1}{r+p_1+q_1-1}}$ and combining (4.10) with (4.2), we obtain the following:

Theorem 5 Suppose that conditions (H1), (G1)–(G3) hold and the solution of (4.1) blows up in finite time T^* . Then there exist two positive constants C_1, C_2 such that

$$C_1(T^* - \tau)^{-\frac{1}{r+p_1+q_1-1}} \leq \max_{x \in \Omega} v(x, \tau) \leq C_2(T^* - \tau)^{-\frac{1}{r+p_1+q_1-1}}. \quad (4.11)$$

Substituting $r = (m - 1)/m, p_1 = p/m, q_1 = q/m, v = u^m, \tau = mt$ into (4.10), we get the following:

Corollary 1 Suppose that conditions (H1)–(H4) hold and the solution of (1.4) blows up in finite time. Then there exist two positive constants C_3, C_4 such that

$$C_3(T - t)^{-\frac{1}{p+q-1}} \leq \max_{x \in \Omega} u(x, t) \leq C_4(T - t)^{-\frac{1}{p+q-1}},$$

where $T = T^*/m$ is the blow-up time of $u(x, t)$ and

$$C_3 = C_1^{1/m} m^{-1/(p+q-1)}, \quad C_4 = C_2^{1/m} m^{-1/(p+q-1)}.$$

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具非线性非局部边界条件的一类多孔介质方程解的整体存在与爆破

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摘要: 本文研究了具有非线性非局部边界条件的一类退化型多孔介质方程. 利用比较原理和上下解的方法, 获得了方程的解是否在有限时刻爆破或整体存在的准则, 这些结果表明, 权重函数 $g(x, y)$ 及指数 l 的大小对于问题解的爆破与否起着关键的作用. 最后研究了爆破解的爆破率.

关键词: 多孔介质方程; 非线性非局部边界条件; 整体存在; 爆破; 爆破率

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