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COMPARISON THEOREM AND ITS APPLICATIONS FOR SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

JIANG Guo, LI Bi-wen

(*School of Mathematics and statistics, Hubei Normal University, Huangshi 435002, China*)

Abstract: In this article, we study stochastic differential equations (SDEs) with different drift and diffusion coefficients which are driven by fractional Brownian motions. By using the generalized sample solutions of SDEs, two comparison theorems are obtained. moreover, we give their applications and propose an asymptotic optimal strategy.

Keywords: stochastic differential equation; generalized sample solution; comparison theorem; fractional Brownian motion; optimal strategy

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1 Introduction

As for the investigation of stochastic differential equations(SDEs) on solutions and system stabilities, the comparison theorems have been evolving not only as very powerful tools but also nice train of thought. Abundant contributions on comparison theorems have been already concerned with different random models or different stochastic resources, for more detail we refer to [1–6] and the reference therein.

In 1970s, Doss [7], Sussmann [8], Krasnoselski and Pokrovski [9] gave a new method that convert SDE to ODE with parameter. Huang [10] gave the definition of generalized sample solution of more general SDE. Jiang [11] also obtained the generalized sample solution of Volterra integral equations. Applying directly the theory of ODE, this method was very valid whether in the theory of SDE or in the numerical calculus of SDE. Inspired by these, we engage to study the comparison theorem of generalized sample solution for stochastic differential equation (SDE) with different drift and diffusion coefficients which are driven by fractional Brownian motions (fBm). That is

$$X_t = x_0 + \int_{t_0}^{t} f(s, B_s^H, X_s)ds + \int_{t_0}^{t} \sigma(s, B_s^H, X_s)dB_s^H, \quad t \geq t_0, \tag{1.1}$$

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Biography: Jiang Guo(1976–), male, born in Xishui, Hubei, associate professor, doctor, major in stochastic analysis and its applications.

where B_t^H is one dimensional fBm on a suitable complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $H(\geq \frac{1}{2})$ is the Hurst parameter, and the stochastic integral throughout the article is taken as pathwise integral in the sense of [12].

In terms of fractional Brownian motion B_t^H with Hurst parameter $H \in (0,1)$, it is a continuous-time Gaussian process with mean zero, starting at zero and having the following correlation function

$$E[B_s^H B_t^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Particularly, it is not semimartingale nor Markovian process, but when $H = \frac{1}{2}$ it reduces to the standard Brownian motion. Since its' self-similarity and long-range dependence, fBm has been a driving noise in models arising in physics, telecommunication network, finance and other fields and give rise to much more research interest recently. Numerous results on fBm have appeared, such as [13, 14] and so on.

The paper is organized as follows: in Section 2, we recall some basic results on the SDE. In Section 3, we characterize the generalized sample solution and show the comparison theorem for the solution of SDE driven by fractional Brownian motions. In Section 4, we give an examples of the comparison theorem and propose an asymptotic optimal strategy for the solutions of SDEs.

2 Preliminaries

In this section, we recall some important results of SDE which are used in the following contents. (For the detail, we refer to [14])

Lemma 2.1 Suppose the following SDE with one dimensional fBm B_t^H $(H \in (\frac{1}{2}, 1))$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$,

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s^H, \ \ t \in [0,T], \tag{2.1}$$

where the function $\sigma = \sigma(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the assumptions: σ is differentiable in x, there exist $M > 0, 0 < \gamma, \kappa \leq 1$ and for any $N > 0$ there exists M_N such that

(i) σ is Lipschitz continuous in x:

$$|\sigma(t,x) - \sigma(t,y)| \leq M|x-y|, \ \ \forall t \in [0,T] \ \ x, y \in \mathbb{R};$$

(ii) x-derivative of σ is local Hölder continuous in x:

$$|\sigma_x(t,x) - \sigma_x(t,y)| \leq M_N|x-y|^\kappa, \ \ \forall |x|, |y| \leq N \ \ t \in [0,T];$$

(iii) σ is Hölder continuous in time t:

$$|\sigma(t,x) - \sigma(s,x)| + |\sigma_x(t,x) - \sigma_x(s,x)| \leq M|t-s|^\gamma \ \ \forall x \in \mathbb{R}, \ \ t, s \in [0,T].$$

The function $f = f(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(iv) for any $N \geq 0$, there exists C_N such that

$$|f(t,x) - f(t,y)| \leq C_N |x-y|, \ \ \forall |x|, |y| \leq N \ \ t, s \in [0,T];$$

(v) there exists a function $f_0 \in L_p[0,T]$ and $C > 0$ such that

$$|f(t,x)| \leq C|x| + f_0(t), \ \ \forall \ \ (t,x) \in [0,T] \times \mathbb{R}.$$

And $p = (1-H+\epsilon)^{-1}$ with some $0 < \epsilon < H - \frac{1}{2}, \gamma > 1-H, \kappa > H^{-1} - 1$ (the constants M, M_N, N, C_N and the function f_0 can depend on ω). Then there exists an unique solution $\{X_t; t \in [0,T]\}$ of equation (2.1).

Lemma 2.2 For $\alpha > 1-H$, let $a(s) \in C^{\alpha}[0,t]$ and $v(s) \in L^1([0,T])$. Define

$$Y_t = Y_0 + \int_0^t v(s)ds + \int_0^t a(s)dB_s^H, \ \ H \in (\frac{1}{2}, 1),$$

moreover, assume $F = F(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $F \in C^1(\mathbb{R}_+) \times C^2(\mathbb{R})$ such that $F'_x(s, Y_s)a(s) \in C^{\beta}[0,t]$ for $\beta + H > 1$ a.s.. Then

$$\begin{aligned} F(t,Y_t) =& F(0,Y_0) + \int_0^t F'_t(s,Y_s)\mathrm{d}s + \int_0^t F'_x(s,Y_s)v(s)\mathrm{d}s \\ &+ \int_0^t F'_x(s,Y_s)a(s)\mathrm{d}B_s^H. \end{aligned}$$

In particular, for $F \in C^2(\mathbb{R})$, we have

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H)\mathrm{d}B_s^H.$$

Lemma 2.3 Assume $f(t,x)$, $\tilde{f}(t,x)$ satisfy Carathéodory conditions on $G^{\star} \subset \mathbb{R}^2$(i.e. they are measurable in t, continuous in x and dominated by a locally integrable function $m(t)$ in the domain $G^{\star}$). Let $(t_0, x_0) \in G^{\star}$, $(t_0, \tilde{x}_0) \in G^{\star}$ be two points such that $x_0 \leq \tilde{x}_0$, and $x(t)$ be any solution to the initial value problem $\dot{x}_t = f(t, x_t), x(t_0) = x_0$; $\tilde{x}(t)$ be the maximal solution [15] to the problem $\dot{x}_t = \tilde{f}(t, x_t), x(t_0) = \tilde{x}_0$. Moreover,

$$(t-t_0)f(t,x) \leq (t-t_0)\tilde{f}(t,x)$$

holds in $G^{\star}$. Then $x(t) \leq \tilde{x}(t)$ in the interval of both them exist.

3 Generalized Sample Solution and Comparison Theorem

In what follows, we consider the SDE (1.1) driven by one dimensional fBm B_t^H with $H \in (\frac{1}{2}, 1)$ on a suitable complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $\{\mathcal{F}_t, t \geq t_0\}$ an increasing family of subalgebras of $\mathcal{F}$ satisfying the usual conditions. The coefficients satisfy the following conditions $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$:

($\mathbb{H}_1$) $f = f(t,b,x)$, $\sigma = \sigma(t,b,x)$ are two real valued functions defined on $G \subset [t_0,\infty) \times \mathbb{R}^2$;

($\mathbb{H}_2$) $\sigma \in C^2(G)$ and $f \in C^0(G)$ and locally Lipschitz continuous with respect to x in G.

For detailed study on the well-defined SDE (1.1), we refer to [14, 16].

Now, we adopt three steps to characterize the generalized sample solution of SDE (1.1). First, we consider the following deterministic initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}b} = \sigma(t,b,y), \quad y(b_0) = \xi, \tag{3.1}$$

where t is a parameter. Since $\sigma \in C^2(G)$, by common theories of ordinary differential equation, we imply that equation (3.1) have an unique C^2-solution $y = \Phi(t,b,\xi,b_0)$ in some domain $\tilde{G} \subset [t_0,\infty) \times \mathbb{R}^2$. Denote

$$h(t,\xi,\omega) = \frac{f\left(t, B_t^H(\omega), \Phi\left(t, B_t^H(\omega), \xi, b_0\right)\right) - \Phi_t\left(t, B_t^H(\omega), \xi, b_0\right)}{\Phi_\xi\left(t, B_t^H(\omega), \xi, b_0)\right)},$$

where Φ_t, Φ_ξ indicate partial derivatives with the variables t and ξ, respectively.

Second, we consider the following initial value problem

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = h(t,\xi,\omega), \quad \xi(t_0) = x_0(\omega). \tag{3.2}$$

Due to the function $f(t,b,x)$ is locally Lipschitz continuous with respect to x in G, the equation (3.2) exists an unique solution $\xi(t,\omega)$, $t_0 \le t < \zeta(\omega)$, where ζ is the "explosion time".

Finally, we intend to show that $X(t,\omega) = \Phi(t, B_t^H(\omega), \xi(t,\omega), b_0)$ is just the unique solution of equation (1.1).

Theorem 3.1 Assuming that the drift and diffusion coefficient f, σ satisfy the foregoing conditions ($\mathbb{H}_1$) and ($\mathbb{H}_2$), there exists an unique strong solution for SDE (1.1),

$$X(t,\omega) = \Phi(t, B_t^H(\omega), \xi(t,\omega), b_0), \quad t_0 \le t < \zeta(\omega), \tag{3.3}$$

where Φ and ξ are the solutions to the problems (3.1) and (3.2), respectively, and ζ is the "explosion time" for X_t.

Proof Since $\xi(t,\omega)$ is an adapted continuous process, we could find a sequence of compact sets K_n in G such that $K_n \uparrow G$ and the well-defined stopping time

$$\zeta_n(\omega) = \inf\left\{t > 0 : \left(t, B_t^H(\omega), \Phi\left(t, B_t^H(\omega), \xi(t,\omega), b_0\right)\right) \notin K_n\right\},$$

which converges to ζ almost surely. Therefore, by applying Lemma 2.2 to the process

$$X(t \wedge \zeta) = \Phi(t \wedge \zeta, B_t^H(\omega), \xi(t \wedge \zeta), b_0),$$

we have

$$\begin{aligned}X(t\wedge\zeta)-x_0&=\int_{t_0}^{t\wedge\zeta}\Phi_b(s,B_s^H,\xi_s,b_0)\mathrm{d}B_s^H+\int_{t_0}^{t\wedge\zeta}\Phi_t\mathrm{d}s+\int_{t_0}^{t\wedge\zeta}\Phi_\xi\mathrm{d}\xi_s\\&=\int_{t_0}^{t\wedge\zeta}\sigma(s,B_s^H,X_s)\mathrm{d}B_s^H+\int_{t_0}^{t\wedge\zeta}\Phi_t\mathrm{d}s+\int_{t_0}^{t\wedge\zeta}\Phi_\xi h(s,\xi_s,\omega)\mathrm{d}s\\&=\int_{t_0}^{t\wedge\zeta}\sigma(s,B_s^H,X_s)\mathrm{d}B_s^H+\int_{t_0}^{t\wedge\zeta}f(s,B_s^H,X_s)\mathrm{d}s,\qquad\text{a.s.,}\end{aligned}$$

which shows that $\{X_t, 0\le t<\zeta\}$ is a solution to SDE (1.1).

Conversely, if $\{\tilde X_t, 0\le t<\tilde\zeta\}$ is an another solution to (1.1), we can also use Lemma 2.2 to the process

$$\tilde\xi(t\wedge\tilde\zeta)\equiv\Phi(t\wedge\tilde\zeta,\tilde X(t\wedge\tilde\zeta),B^H_{t\wedge\tilde\zeta},b_0)$$

to verify similarly that $\{\tilde\xi(t), 0\le t<\tilde\zeta\}$ is a solution to problem (3.2)(cf [2]). It follows from the uniqueness of the solution to problem (3.2) that $\tilde\zeta\le\zeta$ and $\tilde\xi(t)$ coincides with $\xi(t)$ in the interval $[0,\tilde\zeta)$. Therefore,

$$\begin{aligned}\tilde X(t\wedge\tilde\zeta)&=\Phi(t\wedge\tilde\zeta,B^H_{t\wedge\tilde\zeta},\tilde\xi(t\wedge\tilde\zeta),b_0)\\&=\Phi(t\wedge\tilde\zeta,B^H_{t\wedge\tilde\zeta},\xi(t\wedge\tilde\zeta),b_0)=X(t\wedge\tilde\zeta),\qquad\text{a.s.,}\end{aligned}$$

which completes the proof.

Remark 3.1 When the solutions of problems (3.1) and (3.2) exist, we call (3.3) is the generalized sample solution of SDE (1.1). For more detail on generalized sample solution, see [10, 11].

With the definition of generalized sample solution and the above lemmas, we now illustrate the main results on comparison theorems.

Theorem 3.2 Assume that the drift and diffusion coefficients f and σ of SDE (1.1) satisfy the conditions of Theorem 3.1. Moreover,

(i) there exists a function $\tilde\sigma=\tilde\sigma(t,b,x)$ defined on G satisfying the Carathéodory conditions in (b,x) (i.e., they are measurable in t, continuous in (b,x) and dominated by a locally integrable function $\tilde m(t)$ in the domain G) and the inequality

$$(b-b_0)\sigma(t,b,x)\le(b-b_0)\tilde\sigma(t,b,x)\ \text{ in }\ G;\tag{3.4}$$

(ii) there exists a function $\tilde h$ defined on D which is the domain of h, satisfying the Carathéorory conditions in (t,ξ) and the inequality

$$h(t,\xi,\omega)\le\tilde h(t,\xi,\omega)\ \text{ a.e. }\ \omega\ \text{ and }\ (t,\xi)\in D;\tag{3.5}$$

(iii) denoted by $\tilde\Phi$ and $\tilde\xi$ are the maximal solutions to the following two problems

$$\frac{\mathrm{d}y}{\mathrm{d}b}=\tilde\sigma(t,b,y),y(t_0)=\xi^\star;\tag{3.6}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \tilde{h}(t,\xi,\omega), \xi(t_0) = \tilde{x}_0(\omega), \tag{3.7}$$

respectively, and $x_0 \leq \tilde{x}_0$. Then for the unique solution X_t of SDE (1.1)

$$X(t) \leq \tilde{\Phi}(t, B_t^H(\omega), \tilde{\xi}(t,\omega), b_0)$$

holds for a.e. ω and all t on the common interval where both sides are well-defined.

Proof Using Lemma 2.3, we have

$$\Phi(t,b,\xi,b_0) \leq \tilde{\Phi}(t,b,\xi^\star,b_0),$$

provided $\xi \leq \xi^\star$. By the same reason and $x_0 \leq \tilde{x}_0$, we could also show that $\xi(t,\omega) \leq \tilde{\xi}(t,\omega)$ on the common interval where both sides are well-defined.

According to these two inequalities and Theorem 3.1, we obtain

$$X(t,\omega) = \Phi(t, B_t^H(\omega), \xi(t,\omega), b_0) \leq \tilde{\Phi}(t, B_t^H(\omega), \tilde{\xi}(t,\omega), b_0).$$

Remark 3.2 The conditions imposed on σ in Theorem 3.2 would be too restrictive, we can weaken them by a method of smooth approximation and obtain a similar result.

For any $\epsilon > 0$, set φ_ϵ be a modified on $\mathbb{R}^3$, that is, $\varphi_\epsilon \in C_0^\infty(\mathbb{R}^3)$, $\varphi_\epsilon \geq 0$, $\int \varphi_\epsilon = 1$ and the support of φ_ϵ is contained in the ball $U(0,\epsilon)$. If necessary, we can extend its domain and assume that σ is well-defined and continuous in $\mathbb{R}^3$. Let $\sigma^{(\epsilon)} = \sigma * \varphi_\epsilon$, where $*$ denotes the convolution. Hence, $\sigma^{(\epsilon)} \in C^\infty(\mathbb{R}^3)$ and $\sigma^{(\epsilon)}$ converges to σ uniformly on every compact subset in G as $\epsilon \to 0$.

Let $\Phi^{(\epsilon)}$ be the unique solution to the problem

$$\frac{\mathrm{d}\Phi}{\mathrm{d}b} = \sigma^{(\epsilon)}(t,b,\Phi), \ \ \Phi(b_0) = \xi;$$

and

$$h^{(\epsilon)}(t,\xi,\omega) \equiv \frac{f\left(t, B_t^H(\omega), \Phi^{(\epsilon)}(t, B_t^H(\omega), \xi, b_0)\right) - \Phi_t^{(\epsilon)}\left(t, B_t^H(\omega), \xi, b_0\right)}{\Phi_\xi^{(\epsilon)}\left(t, B_t^H(\omega), \xi, b_0\right)}.$$

Hence, we can obtain the following result.

Theorem 3.3 Assume that the drift and diffusion coefficients $f(t,b,x)$ and $\sigma(t,b,x)$ of SDE (1.1) are continuous in G and locally Lipschitz continuous w.r.t x. Let $X(t)$ and $\Phi(t,b,\xi,b_0)$ be the unique solutions to SDE (1.1) and problem (3.1) respectively. Moreover, there exist two functions $\tilde{\sigma}$ satisfying inequality (3.4) and $\tilde{h}$ satisfying

$$h^{(\epsilon)}(t,\xi,\omega) \leq \tilde{h}(t,\xi,\omega) \ \ (t,\xi,\omega) \in D$$

for any sufficiently small positive number ϵ. Then

$$X(t) \leq \tilde{\Phi}(t, B_t^H(\omega), \tilde{\xi}(t,\omega), b_0)$$

holds for a.e. ω and all t on the common interval where both sides are well-defined, where X_t is the unique solution of SDE (1.1) and $\tilde{\Phi}$, $\tilde{\xi}$ are the maximal solutions to (3.6),(3.7).

Proof The existence and uniqueness of solution $X(t)$ to SDE (1.1) are easily obtained. Denoted by ς its "explosion time" and choose a sequence of compact sets $\{K_n\}$ such that $K_n \uparrow G$. Define

$$\varsigma_n(\omega) = \inf\left\{t > t_0 : \left(t, B_t^H(\omega), X(t,\omega)\right) \notin K_n\right\},$$

then each ς_n is a stopping time and $\varsigma_n \uparrow \varsigma$. Since f and $\sigma^{(\epsilon)}$ satisfy the conditions stated in Theorem 3.2, and for $\epsilon > 0$ sufficiently small, the solution $X^{(\epsilon)}(t)$ to the SDE

$$X_t = x_0 + \int_{t_0}^t f(s, B_s^H, X_s)\mathrm{d}s + \int_{t_0}^t \sigma^{(\epsilon)}(s, B_s^H, X_s)\mathrm{d}B_s^H, \qquad t \geq t_0,$$

satisfies the inequality

$$X^{(\epsilon)}(t) \leq \tilde{\Phi}(t, B_t^H(\omega), \tilde{\xi}(t,\omega), b_0). \tag{3.8}$$

For every sequence $\{\epsilon_m\}$ such that $\epsilon_m \downarrow 0$ and every n, we have

$$\lim_{m\to\infty} E\left\{\sup_{t\leq\varsigma_n} \mid X^{(\epsilon_m)}(t) - X(t) \mid^2\right\} = 0.$$

Hence, there exists a subsequence, again denoted by $\{\epsilon_m\}$, such that

$$P\left\{\lim_{m\to\infty} \sup_{t\leq\varsigma_n} \mid X^{(\epsilon_m)}(t) - X(t) \mid = 0\right\} = 1.$$

Upon passage to limit in (3.8), the proof is complete.

4 Examples and Optimal Strategy

In the section, we present an examples of the comparison theorem and propose an asymptotic optimal strategy for the solutions of SDEs.

Example 1 Consider the following two SDEs

$$\begin{cases} \mathrm{d}X_i(t) = f_i(t, B_t^H, X_i(t))\mathrm{d}t + \sigma(t, B_t^H, X_i(t))\mathrm{d}B_s^H, \\ X_i(t_0) = x_0^i \ \ (i = 1, 2), \end{cases}$$

where σ, f_1, and f_2 satisfy the conditions in Theorem 2.3 and $x_0^1 \leq x_0^2$, $f_1 \leq f_2$ in G.

Set $\tilde{\sigma} = \sigma$ and $h_i^{(\epsilon)}(t,\xi,\omega) \equiv (f_i - \Phi_t^{(\epsilon)})/\Phi_\xi^{(\epsilon)}(i = 1, 2)$, we can easily show that

$$h_1^{(\epsilon)}(t,\xi,\omega) \leq h_2^{(\epsilon)}(t,\xi,\omega) \ \ \forall\epsilon > 0.$$

Hence, applying Theorem 3.1 and 3.2 to the equations

$$\begin{cases} \mathrm{d}X_i^{(\epsilon)}(t) = f_i(t, B_t^H, X_i^{(\epsilon)}(t))\mathrm{d}t + \sigma^{(\epsilon)}(t, B_t^H, X_i^{(\epsilon)}(t))\mathrm{d}B_s^H, \\ X_i^{(\epsilon)}(t_0) = x_0^i \ \ (i = 1, 2), \end{cases}$$

we see that $X_1^{(\epsilon)}(t) \leq X_2^{(\epsilon)}(t)$ holds for $\forall\epsilon > 0$ on the interval where they are both well-defined. Let $\epsilon \to 0$, by virtue of Theorem 3.3, we obtain that $X_1(t) \leq X_2(t)$.

Theorem 4.1 Let $G \subset [t_0, \infty) \times \mathbb{R}^2$, assume f, σ are finite measurable function on G and satisfy the conditions in Theorem 3.1, then there exists two finite measurable function sequences f_n and σ_n such that

$$f_n \leq f,\ \sigma_n \leq \sigma,\ \lim_{n\to\infty} f_n = f,\quad \lim_{n\to\infty} \sigma_n = \sigma.$$

Moreover, $\lim\limits_{n\to\infty} X_t^n = X_t$ a.s., where X_t^n and X_t are solutions of equations

$$X_t^n = x_0 + \int_{t_0}^t f_n(s, B_s^H, X_s^n)ds + \int_{t_0}^t \sigma_n(s, B_s^H, X_s^n)dB_s^H,\ \ t \geq t_0 \tag{4.1}$$

and

$$X_t = x_0 + \int_{t_0}^t f(s, B_s^H, X_s)ds + \int_{t_0}^t \sigma(s, B_s^H, X_s)dB_s^H,\ \ t \geq t_0, \tag{4.2}$$

respectively.

Proof Since f, σ are finite measurable functions on G, there exists two sequences of simple functions $\{\psi_n\}$ with $|\psi_1| \leq |\psi_2| \leq \cdots \leq |\psi_n| \leq \cdots$, and $\{\varphi_n\}$ with $|\varphi_1| \leq |\varphi_2| \leq \cdots \leq |\varphi_n| \leq \cdots$, such that

$$\lim_{n\to\infty} \psi_n = f;\quad \lim_{n\to\infty} \varphi_n = \sigma.$$

Let $f_n = \min\{\psi_n, f\}$, $\sigma_n = \min\{\varphi_n, \sigma\}$, then

$$f_n \leq f, \lim_{n\to\infty} f_n = f;\quad \sigma_n \leq \sigma, \lim_{n\to\infty} \sigma_n = \sigma.$$

Notice that equations (4.1) and (4.2) are well-defined with regard to the foregoing drift and diffusion coefficients, applying the above elaborated comparison theorems, we have $X_t^n \leq X_t$, and $\lim\limits_{n\to\infty} X_t^n = X_t$.

Remark 4.1 By suitable designing to approximate the drift and diffusion coefficients numerically and using comparison theorems, we could obtain the optimal state of the stochastic dynamical systems.

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由分数布朗运动驱动的随机微分方程的比较定理及其应用

姜 国 , 李必文

(湖北师范学院数学与统计学院, 湖北 黄石 435002)

摘 要: 本文研究了由分数布朗运动驱动的不同扩散和漂移系数随机微分方程. 利用随机微分方程广义样本解的方法, 得到了两个比较定理. 进一步, 给出了他们的应用和一个最优逼近策略.

关 键 词: 随机微分方程; 广义样本解; 比较定理; 分数布朗运动; 最优策略

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