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# AFFINE EMBEDDINGS OF SELF-SIMILAR SETS WITH MULTI-BRANCHES

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**Abstract:** In this paper, we consider the affine embeddings of a class of self-similar sets with multi-branches. By applying the fixed point of the contractive mapping, we show that under certain circumstances, if a self-similar set can be affinely embedded into another one, then their contractive ratios of the corresponding IFS must satisfy some arithmetic conditions.

**Keywords:** affine embeddings; self-similar set; attractor

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## 1 Introduction

Let $A$ and $B$ be two subsets of $\mathbb{R}^d$. We say that $A$ can be affinely embedded into $B$ if there is an affine map $f$ on $\mathbb{R}^d$ of the form $f(x) = Mx + t$ such that $f(E) \subset F$, where $M$ is an invertible $d \times d$ matrix and $t \in \mathbb{R}^d$.

Let $\Phi = \{\varphi_i\}_{1\le i\le k}$ be a family of contractive mappings on $\mathbb{R}^d$, we say $\Phi$ is an iterated function system (IFS in short) on $\mathbb{R}^d$. There is a unique non-empty compact set $E \subset \mathbb{R}^d$, such that

$$E = \bigcup_{i=1}^{k} \varphi_i(E).$$

We call $E$ the attractor of the IFS $\Phi$. See [2, 6].

It is seen that the attractor $E$ is a singleton if and only if the contractive mappings $\varphi_i (1 \le i \le k)$ have the same fixed point. In this paper, we always suppose the attractor $E$ is not a singleton.

We say that $\Phi$ satisfies the strong separation condition (SSC in short) if $\varphi_i(E) (1 \le i \le k)$ are pairwise disjoint subsets of $E$.

A contractive mapping $\varphi$ on $\mathbb{R}^d$ is called a similitude if $\varphi$ is of the form $\varphi(x) = \rho A(x)+a$, where $0 < \rho < 1$ is the contractive ratio of $\varphi$, $A$ is an $d \times d$ orthogonal matrix and $a \in \mathbb{R}^d$. If all maps of an IFS $\Phi$ are similitudes, the attractor $E$ of $\Phi$ is called a self-similar set.

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Suppose that $\Phi=\{\varphi_i\}_{1\le i\le k}$ and $\Psi=\{\psi_j\}_{1\le j\le l}$ are two families of contractive similitudes of $\mathbb{R}^d$ with the form

$$\varphi_i(x)=\rho_iA_i(x)+a_i,\qquad \psi_j(x)=\lambda_jB_j(x)+b_j,\quad 1\le i\le k,\ 1\le j\le l, \tag{1.1}$$

where $0<\rho_i,\lambda_j<1, a_i,b_j\in\mathbb{R}^d$ and $A_i,B_j$ are orthogonal $d\times d$ matrixes. Let $E$ and $F$ be the attractors of $\Phi$ and $\Psi$, respectively.

In this paper, we discuss when the self-similar set $E$ can be affinely embedded into $F$. It was conjectured in Feng, Huang and Rao [4] that if $E$ and $F$ are totally disconnected and $E$ can be affinely embedded into $F$, then the contractive ratios $\rho_i,\lambda_j$ should satisfy the following arithmetic conditions.

**Conjecture 1.1** Suppose $E$ and $F$ are totally disconnected and $E$ can be affinely embedded into $F$. Then for each $1\le i\le k$, there exists non-negative $t_{i,j}\in\mathbb{Q}$ such that $\rho_i=\prod_{j=1}^l\lambda_j^{t_{i,j}}$. In particular, if $\lambda_j=\lambda(1\le j\le l)$, then $\log\rho_i/\log\lambda\in\mathbb{Q}$ for $1\le i\le k$.

Falconer and Marsh [3] proved the arithmetic conditions when $E,F$ satisfy the SSC and are Lipschitz equivalent. Deng, Wen, Xiong and Xi [1] showed that if $E,F$ satisfy the SSC and $\dim_H E<\dim_H F$, then $E$ can be Lipschitz embedded into $F$.

Let $E$ and $F$ be central Cantor sets in $\mathbb{R}$. That is, $E=C_\rho$ and $F=C_\lambda$ are the attractors of the IFS $\{\rho x,\rho x+1-\rho\}$ and $\{\lambda x,\lambda x+1-\lambda\}$, respectively. Feng, Huang and Rao [4] proved the following result.

**Theorem 1.2** If $0<\rho<\lambda<\sqrt{2}-1$, and $C_\rho$ can be affinely embedded into $C_\lambda$, then $\log\rho/\log\lambda$ is a rational number.

In this paper, we partially generalize the above Theorem 1.2 to the case of multi-branches when $d=1$. Let $E,F$ be attractors of the IFS defined by (1.1). It seen that $A_i,B_j$ equal to 1 or $-1$ for each $1\le i\le k$ and $1\le j\le l$. Without loss of generality, we always assume that $B_j=1$ for $1\le j\le l$ and

$$0=b_1<b_2<\cdots<b_l\le 1-\lambda. \tag{1.2}$$

**Theorem 1.3** Let $E,F$ be attractors of the IFS defined by (1.1) with $\lambda_j=\lambda$ for all $1\le j\le l$. If

$$b_j-b_{j-1}\ge 2\lambda,\quad 2\le j\le l, \tag{1.3}$$

and $E$ can be affinely embedded into $F$, then $\log\rho_i/\log\lambda$ is a rational number for $1\le i\le k$.

## 2 Proof of Theorem 1.3

Let $\varphi_i=\varphi$ be a mapping in the IFS $\Phi$, denote its contraction ratio $\rho_i$ by $\rho$. We shall show $\log\rho/\log\lambda\in\mathbb{Q}$.

Without loss of generality, we assume that $A_i=1$ in (1.1). In the case of $A_i=-1$, we just replace $-\rho$ by $\rho$.

Let $a$ be the fixed point of $\varphi$, i.e., $\varphi(a)=a$. Then it is easy to see that $\varphi$ has the form $\varphi(x)=\rho(x-a)+a$. Since $E$ is not a singleton, there exists $b\in E$ with $b\ne a$. By induction, it is easy to show that

$$\varphi^n(b) = \rho^n(b-a) + a, \quad n \geq 1,$$

where $\varphi^n$ stands for the $n$-th iteration of $\varphi$. Since $\varphi^n(b) \in \varphi^n(E) \subset E \subset F$, we infer that $\rho^n(b-a) + a \in F$ for all $n \geq 1$. Especially

$$\rho^{2n}(b-a) + a \in F, \quad \forall n \geq 1. \tag{2.1}$$

Suppose on the contrary that $\log\rho/\log\lambda$ is irrational. Then by the classical Kronecker Theorem the set

$$\left\{ m - n\frac{2\log\rho}{\log\lambda} : \ m, n \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$ [5]. We choose $\varepsilon$ such that $0 < \varepsilon < \rho^2$. Assume $b - a > 0$ in what follows. The case $b - a < 0$ can be treated exactly in the same way.

By the above dense property, there exist $m, n \in \mathbb{N}$ such that

$$\frac{\log(b-a)}{\log\lambda} < m - 2n\frac{\log\rho}{\log\lambda} < \frac{\log(b-a)}{\log\lambda} - \frac{\log(1+\varepsilon)}{\log\lambda},$$

in other words, $1 < \rho^{2n}(b-a)\lambda^{-m} < 1 + \varepsilon$, or

$$\lambda^m < \rho^{2n}(b-a) < \lambda^m(1+\varepsilon). \tag{2.2}$$

It is seen that $F \subset [0,1]$ by (1.2). We call $\psi_{j_1} \cdots \psi_{j_m}([0,1])$ a basic interval of order $m$ w.r.t. $\Psi$. Clearly the length of a basic interval of order $m$ is $\lambda^m$. By our assumption (1.3), the distance between two basic intervals of order $m$ is $\geq \lambda^m$.

Let $c = \rho^{2n}(b-a) + a$. Then $a, c \in F$. $c - a = \rho^{2n}(b-a) > \lambda^m$ implies that $a$ and $c$ belong to different basic intervals of order $m$. Let us denote the basic intervals of order $m$ containing $a$ and $c$ by $I$ and $J$, respectively.

Let us denote by $x$ the distance between $a$ and the right end point of $I$. Then

$$c - a \geq x + \lambda^m,$$

and by (2.2), we have

$$x \leq c - a - \lambda^m \leq \rho^{2n}(b-a) - \lambda^m < \varepsilon\lambda^m.$$

Let $y$ be the distance between $a$ and the left end point of $J'$, where $J'$ is the basic interval of order $m$ on the right hand side of $I$ and next to $I$. (It is possible that $J = J'$.) Then $y \geq x + \lambda^m$ and so that

$$\frac{x}{y} \leq \frac{x}{x+\lambda^m} < \frac{\varepsilon\lambda^m}{\varepsilon\lambda^m + \lambda^m} < \rho^2.$$

Hence there exists $s \geq 1$ such that

$$x < \rho^{2n+2s}(b-a) < \lambda^m.$$

It follows that $z = \rho^{2n+2s}(b-a) + a$ is between $a + x$ and $a + \lambda^m$, which is fallen into the gap between $I$ and $J'$. This contradicts (2.1) which asserts $z \in F$. This proves the theorem.

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# 多分支自相似集的自仿嵌入

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**摘要**: 本文研究了一类多分支自相似集的自仿嵌入. 利用压缩映射的不动点, 在一定条件下, 证明了若一个自相似集能自仿嵌入到另一个自相似集中, 则它们对应的迭代函数系的压缩比满足某种代数性质.

**关键词**: 自仿嵌入; 自相似集; 吸引子

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