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A CHARACTERIZATION OF BLOCH-TYPE SPACES

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Abstract: In this paper, we investigate the properties of functions in the Bloch-type spaces. By using the pseudo-hyperbolic metric and some inequalities, we obtain a new characterization of Bloch-type spaces $\mathcal{B}^\alpha(\mathbb{B}_n)$ with $0 < \alpha \le 1$, which generalizes the Holland-Walsh characterization in a higher order version for Bloch-type spaces $\mathcal{B}^\alpha(\mathbb{B}_n)$.

Keywords: Bloch-type spaces; Holland-Walsh characterization; the pseudo-hyperbolic metric

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1 Introduction

Let $\mathbb{B}_n$ be the unit ball in the n-dimensional complex Euclidean space $\mathbb{C}^n$. For $0 < \alpha < \infty$, the Bloch-type space $\mathcal{B}^\alpha(\mathbb{B}_n)$ consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\nabla f(z)| < \infty$$

or equivalently

$$\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\mathcal{R}f(z)| < \infty,$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z))$ and $\mathcal{R}f(z) = \sum\limits_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z)$. The Bloch-type space $\mathcal{B}^\alpha(\mathbb{B}_n)$ becomes a Banach space with the norm

$$\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} = |f(0)| + \sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\nabla f(z)|,\ f \in \mathcal{B}^\alpha(\mathbb{B}_n).$$

When $\alpha = 1$, $\mathcal{B}^\alpha(\mathbb{B}_n)$ is classical Bloch space $\mathcal{B}(\mathbb{B}_n)$. For the general theory of Bloch-type spaces we refer to [1] and [2]. The Zygmund class $\Lambda_1(\mathbb{B}_n)$ is the space of holomorphic functions in $\mathbb{B}_n$ whose first order partial derivatives are in the Bloch space $\mathcal{B}(\mathbb{B}_n)$. See page 246 of [2].

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The Holland-Walsh characterizations for Bloch-type spaces were extensively studied. See [3–7]. For example, in [6] Zhao obtained the following theorem.

Theorem 1.1 Let $0 < \alpha \leq 2$. Let λ be any real number satisfying the following properties: (1) $0 \leq \lambda \leq \alpha$ if $0 < \alpha < 1$; (2) $0 < \lambda < 1$ if $\alpha = 1$; (3) $\alpha - 1 \leq \lambda \leq 1$ if $1 < \alpha \leq 2$. Then a holomorphic function $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if

$$S_\lambda(f) = \sup_{\substack{z,w\in\mathbb{B}_n \\ z\neq w}} (1-|z|^2)^\lambda (1-|w|^2)^{\alpha-\lambda} \frac{|f(z)-f(w)|}{|z-w|} < \infty.$$

Moreover, for any α and λ satisfying above conditions two seminorms $\sup\limits_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\nabla f(z)|$ and $S_\lambda(f)$ are equivalent.

It was observed that a holomorphic function $f \in \Lambda_1(\mathbb{B}_n)$ if and only if

$$\sup_{\substack{z,w\in\mathbb{B}_n \\ z\neq w}} \frac{|f(z)+f(w)-2f(\frac{z+w}{2})|}{|z-w|} < \infty.$$

See exercise 7.15 of page 261 of [2]. In this paper, we generalize Theorem 1.1 and give the characterization of Bloch-type spaces $\mathcal{B}^\alpha(\mathbb{B}_n)$ with $0 < \alpha \leq 1$ in terms of $|f(z)+f(w)-2f(\frac{z+w}{2})|/|z-w|^2$, which has not appeared in the literature. It should be mentioned that in the proof of sufficiency of our main result we mainly use the pseudo-hyperbolic metric different from the real techniques used in [5, 6]. We will give our main result in Section 3.

As usual, the letter C will denote a positive constant, possibly different on each occurrence. The notation $A \approx B$ means that A/B is bounded above and below by some positive constants.

2 Preliminaries

In this section we gather the necessary technical results and lemmas that will be need for the proof of main result. First, we give the Möbius transformation. For every point $a \in \mathbb{B}_n$, the Möbius transformation $\varphi_a : \mathbb{B}_n \longrightarrow \mathbb{B}_n$ is defined by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1-\langle z,a\rangle}, \quad z \in \mathbb{B}_n,$$

where $s_a = \sqrt{1-|a|^2}$, $P_a(z) = \frac{\langle z,a\rangle}{|a|^2}a$, $P_0(z) = 0$, $Q_a = I - P_a$. The map φ_a is also called involution of $\mathbb{B}_n$, or involutive automorphism.

Recall that the pseudo-hyperbolic metric on $\mathbb{B}_n$ is given by $\rho(z,w) = |\varphi_z(w)|$ for $z,w \in \mathbb{B}_n$. For $\delta \in (0,1)$, the pseudo-hyperbolic metric ball $E(z,\delta) = \{w \in \mathbb{B}_n : \rho(z,w) = |\varphi_z(w)| < \delta\}$. Let $|E(z,\delta)|$ denote the volume of $E(z,\delta)$. It is well known that for fixed $\delta \in (0,1)$,

$$(1-|z|^2)^{n+1} \approx (1-|w|^2)^{n+1} \approx |1-\langle z,w\rangle|^{n+1} \approx |E(z,\delta)|, \tag{2.1}$$

whenever $\rho(z,w) \leq \delta$. For example, see [2]. Let $B(z,r)$ be a Euclidean ball of radius r centered at z. For $\delta \in (0,1)$, it is easy to see that $B(z,\delta(1-|z|)) \subset E(z,\delta)$. In fact, for any

$w \in B(z, \delta(1-|z|))$ we have

$$\begin{aligned}\rho(z,w) &= |\varphi_z(w)| = \frac{|z - P_z(w) - s_z Q_z(w)|}{|1-\langle w,z\rangle|} = \frac{|P_z(z-w) + s_z Q_z(z-w)|}{|1-\langle w,z\rangle|} \\ &\le \frac{|z-w|}{|1-\langle w,z\rangle|} < \frac{\delta(1-|z|)}{(1-|z||w|)} < \delta. \end{aligned} \tag{2.2}$$

On the other hand, we will also need the following lemmas.

Lemma 2.1 Let $\delta \in (0,1)$, $z \in \mathbb{B}_n$ and let f be a holomorphic function in the ball $B(z, \delta(1-|z|))$. Then

$$(1-|z|^2)|\nabla f(z)| \le \frac{C}{(1-|z|^2)^{2n}} \int_{B(z,\delta(1-|z|))} |f(w)| d\nu(w),$$

where $z \in \mathbb{B}_n$ and $d\nu$ is the normalized volume measure on $\mathbb{B}_n$.

Proof This immediately follows from the Cauchy estimate and the subharmonicity. Indeed, we have

$$\begin{aligned}(1-|z|^2)|\nabla f(z)| &\le C \sup_{w\in B(z,\frac{\delta}{2}(1-|z|))} |f(w)| \\ &\le \frac{C}{(1-|z|^2)^{2n}} \int_{B(z,\delta(1-|z|))} |f(w)| d\nu(w).\end{aligned}$$

This completes the proof.

Lemma 2.2 (see [2, 8]) Let $\alpha > 0$ and k be a positive integer. Then a holomorphic function f in $\mathbb{B}_n$ belongs to $\mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if the function $(1-|z|^2)^{\alpha+m-1}\partial^{|m|} f/\partial z^m$ is bounded in $\mathbb{B}_n$ for each multi-index m of nonnegative integers with $|m| = k$.

Lemma 2.3 (see [6]) Let $0 < \alpha \le 2$. Let λ be any real number satisfying the properties:

(1) $0 \le \lambda \le \alpha$ if $0 < \alpha < 1$;

(2) $0 < \lambda < 1$ if $\alpha = 1$;

(3) $\alpha - 1 \le \lambda \le 1$ if $1 < \alpha \le 2$.

Let

$$H(x,y) = \frac{x^\lambda y^{\alpha-\lambda}}{y-x} \int_x^y \frac{d\tau}{\tau^\alpha}.$$

Then there exists a constant $C > 0$ such that $H(x,y) \le C$ for any x and y satisfying $0 < x, y < \infty$ and $x \ne y$.

3 Main Result

In this section, we give the characterization of Bloch-type spaces in terms of $|f(z) + f(w) - 2f(\frac{z+w}{2})|/|z-w|^2$, which generalizes the Holland-Walsh characterization. We state our main result as follows.

Theorem 3.1 Let $0 < \alpha \le 1$ and $\alpha \le \lambda \le 1$. Then a holomorphic function $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if

$$\sup_{\substack{z,w\in\mathbb{B}_n \\ z\ne w}} (1-|z|^2)^\lambda (1-|w|^2)^{\alpha+1-\lambda} \frac{|f(z)+f(w)-2f(\frac{z+w}{2})|}{|z-w|^2} < \infty. \tag{3.1}$$

Proof Assume that $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$. For any $z, w \in \mathbb{B}_n$, denote $\zeta = \frac{z+w}{2}$. Then we have

$$\begin{aligned}
& f(z) + f(w) - 2f(\frac{z+w}{2}) = (f(z) - f(\frac{z+w}{2})) + (f(w) - f(\frac{z+w}{2})) \\
= & (f(z) - f(\zeta)) + (f(w) - f(\zeta)) \\
= & \int_0^1 \left[\frac{df}{dt}(tz + (1-t)\zeta) + \frac{df}{dt}(tw + (1-t)\zeta) \right] dt \\
= & \frac{1}{2} \sum_{k=1}^n (z_k - w_k) \int_0^1 \left[\frac{\partial f}{\partial z_k}(tz + (1-t)\zeta) - \frac{\partial f}{\partial z_k}(tw + (1-t)\zeta) \right] dt \\
= & \frac{1}{2} \sum_{k=1}^n \sum_{j=1}^n (z_k - w_k)(z_j - w_j) \int_0^1 t \int_0^1 \frac{\partial^2 f}{\partial z_k \partial z_j} \left(stz + s(1-t)\zeta + (1-s)(tw + (1-t)\zeta) \right) dsdt.
\end{aligned}$$

By Lemma 2.2 and the facts that $|z_k - w_k| \le |z - w|$ $(k = 1, 2, \cdots, n)$, we obtain

$$\begin{aligned}
& \left| f(z) + f(w) - 2f(\frac{z+w}{2}) \right| \\
= & \frac{1}{2} \left| \sum_{k=1}^n \sum_{j=1}^n (z_k - w_k)(z_j - w_j) \int_0^1 t \int_0^1 \frac{\partial^2 f}{\partial z_k \partial z_j} \left(stz + (1-s)tw + (1-t)\zeta \right) dsdt \right| \\
\le & C \sum_{k=1}^n \sum_{j=1}^n |z_k - w_k||z_j - w_j| \|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \int_0^1 t \int_0^1 \frac{1}{(1 - |stz + (1-s)tw + (1-t)\zeta|^2)^{\alpha+1}} dsdt \\
\le & C|z - w|^2 \|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \int_0^1 t \int_0^1 \frac{1}{(1 - |stz + (1-s)tw + (1-t)\zeta|^2)^{\alpha+1}} dsdt. \qquad (3.2)
\end{aligned}$$

Now suppose that $|z| \le |w|$. Then we get

$$\begin{aligned}
& 1 - |stz + (1-s)tw + (1-t)\zeta|^2 \ge 1 - |stz + (1-s)tw + (1-t)\zeta| \\
= & 1 - |(st + \frac{1-t}{2})z + (1 - st - \frac{1-t}{2})w| \\
\ge & 1 - (st + \frac{1-t}{2})|z| - (1 - st - \frac{1-t}{2})|w| \\
= & 1 - st|z| - (1-st)|w| + \frac{1-t}{2}(|w| - |z|) \ge 1 - st|z| - (1-st)|w|. \qquad (3.3)
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
& \int_0^1 t \int_0^1 \frac{1}{(1 - |stz + (1-s)tw + (1-t)\zeta|^2)^{\alpha+1}} dsdt \\
\le & \int_0^1 t \int_0^1 \frac{1}{(1 - st|z| - (1-st)|w|)^{\alpha+1}} dsdt \\
= & \int_0^1 \int_0^t \frac{1}{(1 - s|z| - (1-s)|w|)^{\alpha+1}} dsdt \le \int_0^1 \frac{1}{(1 - s|z| - (1-s)|w|)^{\alpha+1}} ds. \qquad (3.4)
\end{aligned}$$

Thus, by (3.2), (3.3) and (3.4) we obtain for $z \ne w$ and $|z| \le |w|$

$$\frac{|f(z) + f(w) - 2f(\frac{z+w}{2})|}{|z - w|^2} \le C \int_0^1 \frac{1}{(1 - s|z| - (1-s)|w|)^{\alpha+1}} ds. \qquad (3.5)$$

If $|z| = |w|$, then

$$\frac{|f(z)+f(w)-2f(\frac{z+w}{2})|}{|z-w|^2} \le C\int_0^1 \frac{1}{(1-s|z|-(1-s)|z|)^{\alpha+1}}ds$$
$$= C\int_0^1 \frac{1}{(1-|z|)^{\alpha+1}}ds = \frac{C}{(1-|z|)^{\lambda}(1-|w|)^{\alpha+1-\lambda}}. \tag{3.6}$$

If $|z| < |w|$, let $\tau = 1 - s|z| - (1-s)|w|$. By Lemma 2.3, the integral in (3.5) becomes

$$\begin{aligned}\int_0^1 \frac{1}{(1-s|z|-(1-s)|w|)^{\alpha+1}}ds &= \frac{1}{(1-|z|)-(1-|w|)}\int_{1-|w|}^{1-|z|}\frac{1}{\tau^{\alpha+1}}d\tau \\ &\le \frac{C_1}{(1-|z|)^{\lambda}(1-|w|)^{\alpha+1-\lambda}}.\end{aligned}$$

By Lemma 2.3, we also have

$$\begin{aligned}\int_0^1 \frac{1}{(1-s|z|-(1-s)|w|)^{\alpha+1}}ds &= \frac{1}{(1-|z|)-(1-|w|)}\int_{1-|w|}^{1-|z|}\frac{1}{\tau^{\alpha+1}}d\tau \\ &\le \frac{C_2}{(1-|z|)^{\alpha+1-\lambda}(1-|w|)^{\lambda}}.\end{aligned}$$

By symmetry of the roles of z and w, and combining this with (3.6), finally we get that for $z \ne w$,

$$\frac{|f(z)+f(w)-2f(\frac{z+w}{2})|}{|z-w|^2} \le \frac{C}{(1-|z|)^{\lambda}(1-|w|)^{\alpha+1-\lambda}}.$$

This proves the necessity.

Conversely, suppose that f is holomorphic on $\mathbb{B}_n$ and satisfies (3.1). Take $\delta \in (0,1)$ such that $\delta(2+\delta) < 1$. By Lemma 2.1, for $k = 1,2,\cdots,n$, $j = 1,2,\cdots,n$, then we have

$$\begin{aligned}&(1-|z|^2)^{\alpha+1}\left|\frac{\partial^2 f}{\partial z_k \partial z_j}(z)\right| \\ \le\ & \frac{C(1-|z|^2)^{\alpha}}{(1-|z|^2)^{2n}}\int_{B(z,\delta(1-|z|))}\left|\frac{\partial f}{\partial z_k}(w) - \frac{\partial f}{\partial z_k}(\frac{z+w}{2})\right|d\nu(w) \\ \le\ & \frac{C(1-|z|^2)^{\alpha}}{(1-|z|^2)^{2n}}\int_{B(z,\delta(1-|z|))}\frac{1}{(1-|w|^2)^{2n+1}}\int_{B(w,\delta(1-|w|))}|f(z)+f(\xi)-2f(\frac{z+\xi}{2})|d\nu(\xi)d\nu(w).\end{aligned}$$

Notice that for $\xi \in B(w,\delta(1-|w|))$ and $w \in B(z,\delta(1-|z|))$, we have

$$\begin{aligned}\frac{|z-\xi|}{|1-<z,\xi>|} &\le \frac{|z-w|+|w-\xi|}{(1-|z|)} < \frac{\delta(1-|z|)+\delta(1-|w|)}{(1-|z|)} \\ &< \frac{\delta(1-|z|)+\delta(1+\delta)(1-|z|)}{(1-|z|)} = \delta(2+\delta). \end{aligned} \tag{3.7}$$

From above inequality, (2.1) and (2.2), when $w \in B(z,\delta(1-|z|))$ and $\xi \in B(w,\delta(1-|w|))$, we have $\xi \in B(z,\delta(2+\delta)(1-|z|))$ and

$$1-|z|^2 \approx 1-|w|^2 \approx 1-|\xi|^2 \approx |1-\langle z,\xi\rangle|. \tag{3.8}$$

Combined with (3.1), (3.7) and (3.8), we consequently obtain

$$
\begin{aligned}
& (1-|z|^2)^{\alpha+1}|\frac{\partial^2 f}{\partial z_k \partial z_j}(z)| \\
\leq\ & \frac{C(1-|z|^2)^{\alpha}}{(1-|z|^2)^{2n}} \int_{B(z,\delta(1-|z|))} \frac{1}{(1-|w|^2)^{2n+1}} \int_{B(w,\delta(1-|w|))} \frac{\delta^2(2+\delta)^2|1-\langle z,\xi\rangle|^2}{(1-|z|^2)^{\lambda}(1-|\xi|^2)^{\alpha+1-\lambda}} \\
& \frac{(1-|z|^2)^{\lambda}(1-|\xi|^2)^{\alpha+1-\lambda}}{|z-\xi|^2}|f(z)+f(\xi)-2f(\frac{z+\xi}{2})|d\nu(\xi)d\nu(w) \\
\leq\ & \frac{C(1-|z|^2)^{\alpha}}{(1-|z|^2)^{2n}} \int_{B(z,\delta(1-|z|))} \frac{1}{(1-|w|^2)^{2n+1}} \int_{B(w,\delta(1-|w|))} \frac{|1-\langle z,\xi\rangle|^2}{(1-|z|^2)^{\lambda}(1-|\xi|^2)^{\alpha+1-\lambda}} d\nu(\xi)d\nu(w) \\
\leq\ & \frac{C(1-|z|^2)^{\alpha}}{(1-|z|^2)^{2n}} \int_{B(z,\delta(1-|z|))} \frac{1}{(1-|w|^2)^{2n+1}} \int_{B(w,\delta(1-|w|))} \frac{(1-|w|^2)^2}{(1-|z|^2)^{\alpha}(1-|w|^2)} d\nu(\xi)d\nu(w) \\
\leq\ & C.
\end{aligned}
$$

From Lemma 2.2, this implies that $f \in \mathcal{B}^{\alpha}(\mathbb{B}_n)$. The proof is completed.

References

[1] Timoney R M. Bloch functions in several complex variables I[J]. Bull. London Math. Soc., 1980, 12: 241–267.

[2] Zhu Kehe. Spaces of holomorphic functions in the unit ball[M]. New York: Springer-Verlag, 2004.

[3] Holland F, Walsh D. Criteria for membership of Bloch space and its subspace, BMOA[J]. Math. Ann., 1986, 273: 317–335.

[4] Nowak M. Bloch space and Möbius invariant Besov spaces on the unit ball of $\mathbb{C}^n$[J]. Complex Variables, 2001, 44: 1–12.

[5] Ren Guangbin, Tu Caifeng. Bloch spaces in the unit ball of $\mathbb{C}^n$[J]. Proc. Amer. Math. Soc., 2005, 133: 719–726.

[6] Zhao Ruhan. A characterization of Bloch-type spaces on the unit ball of $\mathbb{C}^n$[J]. J. Math. Anal. Appl., 2007, 330: 291–297.

[7] Li Songxiao, Wulan H. Characterizations of α-Bloch spaces on the unit ball[J]. J. Math. Anal. Appl., 2008, 343: 58–63.

[8] Li Bo, Ouyang Caiheng. Higher radial derivative of Bloch type functions[J]. Acta Math. Scientia, 2002, 22B: 433–445.

Bloch型空间的一个刻画

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摘 要: 本文研究了Bloch型空间中函数性质问题. 利用拟双曲度量及一些不等式得到了Bloch型空间$\mathcal{B}^{\alpha}(\mathbb{B}_n)$ $(0<\alpha\leq 1)$的一个新的刻画, 该刻画将Bloch型空间$\mathcal{B}^{\alpha}(\mathbb{B}_n)$的Holland-Walsh刻画推广到一个高阶形式.

关键词: Bloch型空间; Holland-Walsh刻画; 拟双曲度量

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