

# MEROMORPHIC SOLUTIONS OF A TYPE OF SYSTEMS OF COMPLEX DIFFERENCE EQUATIONS

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**Abstract:** This paper investigates the growth order of a type of systems of complex difference equations. By using the Nevanlinna distribution theory of meromorphic functions, we obtain a main result on meromorphic solutions of systems of complex difference equations, which extends some results concerning complex difference equation to the systems of complex difference equations.

**Keywords:** growth order; meromorphic solutions; systems of complex difference equations

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## 1 Introduction

We use the standard notations of the Nevanlinna theory of meromorphic functions(see e.g.[1]). In addition we denote the order of  $f(z)$  by  $\rho(f)$ .

We set up some notations on difference. Let  $c$  be a fixed, non-zero complex number,  $\Delta_c f(z) = f(z+c) - f(z)$ , and  $\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z)$  for each integer  $n \geq 2$ . In order to simplify our notations, we shall use the same notation  $\Delta$  for both a general  $c$  and when  $c = 1$ . The context will make clear which quantity is under discussion. Equations written with the above difference operators  $\Delta_c^n f(z)$  are difference discussions. Let  $E$  be a subset on the positive real axis. We define the logarithmic measure of  $E$  to be

$$\log(E) = \int_{E \cap (1+\infty)} \frac{dr}{r}.$$

A set  $E \subseteq (1, +\infty)$  is said to have finite logarithmic measure if  $\log(E) < \infty$ .

Recently, Laine, Rieppo [5], Heittokangas, Korhonen [3], Halburd and Korhonen [4], Chiang and Feng [2] etc. investigated the existence or growth of solutions of complex difference equations, Gao [9–12] investigated the existence or growth of solutions of systems of complex difference equations, they obtain some main results.

In 2005, I. Laine, J. Rieppo[5] considered the following difference equation

$$\sum_{j=1}^n \alpha_j(z) f(z + c_j) = \frac{P(z, f(z))}{Q(z, f(z))},$$

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where the coefficients  $\alpha_j(z)$  are non-vanishing small functions relative to  $f$  and where  $P, Q$  are relatively prime polynomials in  $f$  over the field of small functions relative to  $f$ . He obtained

**Theorem A** [5] Suppose that  $c_1, c_2, \dots, c_n$  are distinct, non-zero complex numbers and that  $f$  is a transcendental meromorphic solution of the above equation,  $q = \deg_f^Q > 0, n = \max\{p, q\} = \max\{\deg_f^P, \deg_f^Q\}$  and that, without restricting generality,  $Q$  is a monic polynomial. If there exists  $\alpha \in [0, n)$  such that for all  $r$  sufficiently large,

$$\overline{N}(r, \sum_{j=1}^n \alpha_j(z)f(z + c_j)) \leq \alpha \overline{N}(r + C, f(z)) + S(r, f(z)),$$

where  $C = \max\{|c_1|, |c_2|, \dots, |c_n|\}$ , then either the order  $\rho(f) = \infty$ , or

$$Q(z, f(z)) \equiv (f(z) + h(z))^q,$$

where  $h(z)$  is a small meromorphic function.

The natural question arises whether or not the assertion of Theorem A remains valid, if we replace the difference equations with systems of complex difference equations. In this paper, our aim is to consider the problem of the growth of solution of systems of complex difference equations of the form

$$\begin{cases} \sum_{j=1}^{n_1} \alpha_j(z)f_1(z + c_j) = \frac{P_1(z, f_2)}{Q_1(z, f_2)}, \\ \sum_{j=1}^{n_2} \beta_j(z)f_2(z + c_j) = \frac{P_2(z, f_1)}{Q_2(z, f_1)}, \end{cases} \tag{1.1}$$

where the coefficients  $\alpha_j(z), \beta_j(z)$  are non-vanishing small functions relative to  $f_1, f_2$  and  $P_1, Q_1$  are relatively prime polynomials in  $f_2, P_2, Q_2$  are relatively prime polynomials in  $f_1$ , where  $q_k = \deg_f^{Q_k} > 0, p_k = \deg_f^{P_k} (k = 1, 2)$ . We assume that  $n_1 = \max\{p_1, q_1\}, n_2 = \max\{p_2, q_2\}, Q_1, Q_2$  are monic polynomials.

Let  $C = \max\{|c_1|, |c_2|, \dots, |c_n|\}$ . We obtain the following result

**Theorem 1** Let  $(f_1(z), f_2(z))$  be transcendental meromorphic solution of (1.1). If there exist  $\alpha, \beta \in [0, n), n = \min\{n_1, n_2\}$  such that for all  $r$  sufficiently large,

$$\begin{cases} \overline{N}(r, \sum_{j=1}^{n_1} \alpha_j(z)f_1(z + c_j)) \leq \alpha \overline{N}(r + C, f_1(z)) + S(r, f_1), \\ \overline{N}(r, \sum_{j=1}^{n_2} \beta_j(z)f_2(z + c_j)) \leq \beta \overline{N}(r + C, f_2(z)) + S(r, f_2), \end{cases} \tag{1.2}$$

then either the order  $\rho(f_1) = \infty, \rho(f_2) = \infty$  at least one of them will be true, or

$$Q_1(z, f_2) \equiv (f_2 + h_1(z))^{q_1}, Q_2(z, f_1) \equiv (f_1 + h_2(z))^{q_2}$$

at least one of them will be true where  $h_1(z), h_2(z)$  are small meromorphic functions.

## 2 Some Lemmas

**Lemma 1** [6] Let  $f(z)$  be a meromorphic function and let  $\phi$  be given by

$$\begin{aligned}\phi &= f^n + a_{n-1}f^{n-1} + \cdots + a_0, \\ T(r, a_j) &= S(r, f), j = 0, 1, \cdots, n-1.\end{aligned}$$

Then either

$$\phi \equiv \left(f + \frac{a_{n-1}}{a_n}\right)^n$$

or

$$T(r, f) \leq \bar{N}\left(r, \frac{1}{\phi}\right) + \bar{N}(r, f) + S(r, f).$$

**Lemma 2** [5] Let  $f(z)$  be a non-constant meromorphic function and let  $P(z, f), Q(z, f)$  be two polynomials in  $f$  with meromorphic coefficients small relative to  $f$ . If  $P$  and  $Q$  have no common factors of positive degree in  $f$  over the field of small functions relative to  $f$ , then

$$\bar{N}\left(r, \frac{1}{Q(z, f)}\right) \leq \bar{N}\left(r, \frac{P(z, f)}{Q(z, f)}\right) + S(r, f).$$

**Lemma 3** [7] Let  $T : [0, +\infty) \rightarrow [0, +\infty)$  be a non-decreasing continuous function,  $\delta \in (0, 1)$  and  $s \in (0, +\infty]$ . If  $T$  is of finite order, i.e.,

$$\lim_{r \rightarrow \infty} \frac{\log T(r)}{\log r} < \infty,$$

then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

where  $r$  runs to infinity outside of a set of finite logarithmic measure.

**Lemma 4** [8] Let  $f(z)$  be a non-constant meromorphic function. Then for all irreducible rational functions in  $f(z)$ ,

$$R(z, f(z)) = \frac{\sum_{j=0}^p a_j(z) f(z)^j}{\sum_{j=0}^q b_j(z) f(z)^j}$$

with meromorphic coefficients  $a_j(z), b_j(z)$ , the characteristic function of  $R(z, f(z))$  satisfies

$$T(r, R(z, f(z))) = \max\{p, q\}T(r, f) + O\left\{\sum T(r, a_i) + \sum T(r, b_j)\right\}.$$

### 3 Proof of Theorem 1

When the second alternative of assertion is not hand. Then by Lemma 1 and Lemma 2, we obtain

$$T(r, f_2) \leq \bar{N}\left(r, \frac{1}{Q_1}\right) + \bar{N}(r, f_2) + S(r, f_2) \leq \bar{N}\left(r, \frac{P_1}{Q_1}\right) + \bar{N}(r, f_2) + S(r, f_2).$$

Then by (1.1) and (1.2), we have

$$\begin{aligned} T(r, f_2) - \bar{N}(r, f_2) &\leq \bar{N}\left(r, \frac{P_1}{Q_1}\right) + S(r, f_2) \\ &= \bar{N}\left(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)\right) + S(r, f_2) \\ &\leq \alpha \bar{N}(r + C, f_1(z)) + S(r, f_1) + S(r, f_2). \end{aligned}$$

Namely,

$$T(r, f_2) - \bar{N}(r, f_2) \leq \alpha \bar{N}(r + C, f_1(z)) + S(r, f_1) + S(r, f_2),$$

where  $\alpha \in [0, n), n = \min\{n_1, n_2\}$ . Similarly, we obtain

$$T(r, f_1) - \bar{N}(r, f_1) \leq \beta \bar{N}(r + C, f_2(z)) + S(r, f_1) + S(r, f_2),$$

where  $\beta \in [0, n), n = \min\{n_1, n_2\}$ .

Assuming, contrary to the assertion, that  $\rho(f_i) < \infty, i = 1, 2$ . Then it implies that

$$S(r, f_1(z + c_j)) = S(r, f_1), \quad S(r, f_1(z + c_j)) = S(r, f_2).$$

Hence

$$T(r, f_2(z + c_j)) - \bar{N}(r, f_2(z + c_j)) \leq \alpha \bar{N}(r + C, f_1(z + c_j)) + S(r, f_1(z)) + S(r, f_2(z)), \quad (3.1)$$

$$T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j)) \leq \beta \bar{N}(r + C, f_2(z + c_j)) + S(r, f_1(z)) + S(r, f_2(z)). \quad (3.2)$$

By (1.1), (1.2), (3.2) and Lemma 4, we have

$$\begin{aligned} n_1 T(r, f_2) &\leq T\left(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)\right) + S(r, f_1) \\ &\leq \sum_{j=1}^{n_1} [T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j))] + \bar{N}\left(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)\right) + S(r, f_1) \\ &\leq \sum_{j=1}^{n_1} \beta \bar{N}(r + C, f_2(z + c_j)) + \alpha \bar{N}(r + C, f_1) + S(r, f_1) \\ &\leq n_1 \beta \bar{N}(r + 2C, f_2(z)) + \alpha \bar{N}(r + C, f_1) + S(r, f_1) + S(r, f_2). \end{aligned}$$

Namely,

$$T(r, f_2) \leq \beta \bar{N}(r + 2C, f_2(z)) + \frac{\alpha}{n_1} \bar{N}(r + C, f_1) + S(r, f_1) + S(r, f_2). \quad (3.3)$$

Similarly, we have

$$T(r, f_1) \leq \alpha \bar{N}(r + 2C, f_1(z)) + \frac{\beta}{n_2} \bar{N}(r + C, f_2) + S(r, f_1) + S(r, f_2). \quad (3.4)$$

Hence

$$\begin{aligned} & T(r, f_2) - \bar{N}(r, f_2) \\ & \leq \beta \bar{N}(r + 2C, f_2(z)) + \frac{\alpha}{n_1} \bar{N}(r + C, f_1) - \bar{N}(r, f_2) + S(r, f_1) + S(r, f_2), \end{aligned} \quad (3.5)$$

$$\begin{aligned} & T(r, f_1) - \bar{N}(r, f_1) \\ & \leq \alpha \bar{N}(r + 2C, f_1(z)) + \frac{\beta}{n_2} \bar{N}(r + C, f_2) - \bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \end{aligned} \quad (3.6)$$

By (3.5) and Lemma 3, we have

$$\begin{aligned} & T(r, f_2(z + c_j)) - \bar{N}(r, f_2(z + c_j)) \\ & \leq \beta \bar{N}(r + 2C, f_2(z + c_j)) + \frac{\alpha}{n_1} \bar{N}(r + C, f_1(z + c_j)) - \bar{N}(r, f_2(z + c_j)) \\ & \quad + S(r, f_1(z + c_j)) + S(r, f_2(z + c_j)) \\ & \leq \beta \bar{N}(r + 3C, f_2) + \frac{\alpha}{n_1} \bar{N}(r + 2C, f_1) - \bar{N}(r - C, f_2) + S(r, f_1) + S(r, f_2) \\ & \leq \beta \bar{N}(r + 3C, f_2) + \frac{\alpha}{n_1} \bar{N}(r, f_1) - \bar{N}(r, f_2) + S(r, f_1) + S(r, f_2). \end{aligned}$$

Namely,

$$T(r, f_2(z + c_j)) - \bar{N}(r, f_2(z + c_j)) \leq \beta \bar{N}(r + 3C, f_2) + \frac{\alpha}{n_1} \bar{N}(r, f_1) - \bar{N}(r, f_2) + S(r, f_1) + S(r, f_2). \quad (3.7)$$

Similarly, we have

$$T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j)) \leq \alpha \bar{N}(r + 3C, f_1) + \frac{\beta}{n_2} \bar{N}(r, f_2) - \bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \quad (3.8)$$

By (1.1), (1.2), (3.8) and Lemma 4, we have

$$\begin{aligned} n_1 T(r, f_2) & \leq T(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)) + S(r, f_1) \\ & \leq \sum_{j=1}^{n_1} [T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j))] + \bar{N}(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)) + S(r, f_1) \\ & \leq \sum_{j=1}^{n_1} [\alpha \bar{N}(r + 3C, f_1(z)) + \frac{\beta}{n_2} \bar{N}(r, f_2) - \bar{N}(r, f_1)] + \alpha \bar{N}(r + C, f_1) \\ & \quad + S(r, f_1) + S(r, f_2) \\ & \leq n_1 \alpha \bar{N}(r + 3C, f_1(z)) + \frac{n_1 \beta}{n_2} \bar{N}(r, f_2) - n_1 \bar{N}(r, f_1) \\ & \quad + \alpha \bar{N}(r + C, f_1) + S(r, f_1) + S(r, f_2). \end{aligned} \quad (3.9)$$

Then by Lemma 3 and (3.9), we have

$$\begin{aligned} T(r, f_2) - \bar{N}(r, f_2) & \leq \alpha \bar{N}(r + 3C, f_1) + \frac{\beta}{n_2} \bar{N}(r, f_2) + \frac{\alpha}{n_1} \bar{N}(r, f_1) - \bar{N}(r, f_1) \\ & \quad - \bar{N}(r, f_2) + S(r, f_1) + S(r, f_2). \end{aligned}$$

Similarly, we have

$$T(r, f_1) - \bar{N}(r, f_1) \leq \beta \bar{N}(r + 3C, f_2) + \frac{\alpha}{n_1} \bar{N}(r, f_1) + \frac{\beta}{n_2} \bar{N}(r, f_2) - \bar{N}(r, f_2) - \bar{N}(r, f_1) + S(r, f_1) + S(r, f_2).$$

Namely,

$$\begin{cases} T(r, f_2) - \bar{N}(r, f_2) \leq \alpha \bar{N}(r + 3C, f_1) + \frac{\beta}{n_2} \bar{N}(r, f_2) + \frac{\alpha}{n_1} \bar{N}(r, f_1) - \bar{N}(r, f_1) - \bar{N}(r, f_2) + S(r, f_1) + S(r, f_2), \\ T(r, f_1) - \bar{N}(r, f_1) \leq \beta \bar{N}(r + 3C, f_2) + \frac{\alpha}{n_1} \bar{N}(r, f_1) + \frac{\beta}{n_2} \bar{N}(r, f_2) - \bar{N}(r, f_2) - \bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \end{cases} \tag{3.10}$$

We now proceed, inductively, to prove

$$\begin{cases} T(r, f_2) - \bar{N}(r, f_2) \leq \alpha \bar{N}(r + (2m + 1)C, f_1) + \frac{m\beta}{n_2} \bar{N}(r, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) - m\bar{N}(r, f_1) - m\bar{N}(r, f_2) + S(r, f_1) + S(r, f_2), \\ T(r, f_1) - \bar{N}(r, f_1) \leq \beta \bar{N}(r + (2m + 1)C, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) + \frac{m\beta}{n_2} \bar{N}(r, f_2) - m\bar{N}(r, f_2) - m\bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \end{cases}$$

Having already proved the case  $m = 1$  in (3.10), we continue to the inductive step. To this end, by the above inequalities, we obtain

$$\begin{aligned} & T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j)) \\ & \leq \beta \bar{N}(r + (2m + 1)C, f_2(z + c_j)) + \frac{m\alpha}{n_1} \bar{N}(r, f_1(z + c_j)) + \frac{m\beta}{n_2} \bar{N}(r, f_2(z + c_j)) \\ & \quad - m\bar{N}(r, f_2(z + c_j)) - m\bar{N}(r, f_1(z + c_j)) + S(r, f_1(z + c_j)) + S(r, f_2(z + c_j)) \\ & \leq \beta \bar{N}(r + (2m + 2)C, f_2) + \frac{m\alpha}{n_1} \bar{N}(r + C, f_1) + \frac{m\beta}{n_2} \bar{N}(r + C, f_2) - m\bar{N}(r - C, f_2) \\ & \quad - m\bar{N}(r - C, f_1) + S(r, f_1) + S(r, f_2). \end{aligned} \tag{3.11}$$

Applying (3.11) and Lemma 3, and using (1.1) and (1.2) we conclude that

$$\begin{aligned} n_1 T(r, f_2) & \leq T(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)) + S(r, f_1) \\ & \leq \sum_{j=1}^{n_1} [T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j))] + \bar{N}(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)) + S(r, f_1) \\ & \leq \sum_{j=1}^{n_1} [\beta \bar{N}(r + (2m + 2)C, f_2) + \frac{m\alpha}{n_1} \bar{N}(r + C, f_1) + \frac{m\beta}{n_2} \bar{N}(r + C, f_2) \\ & \quad - m\bar{N}(r - C, f_2) - m\bar{N}(r - C, f_1)] + \alpha \bar{N}(r + C, f_1) + S(r, f_1) \\ & \leq n_1 \beta \bar{N}(r + (2m + 2)C, f_2) + \frac{n_1 m \alpha}{n_1} \bar{N}(r, f_1) + \frac{n_1 m \beta}{n_2} \bar{N}(r, f_2) \\ & \quad - mn_1 \bar{N}(r, f_2) - mn_1 \bar{N}(r, f_1) + \alpha \bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \end{aligned}$$

Therefore, we obtain

$$T(r, f_2) - \bar{N}(r, f_2) \leq \beta \bar{N}(r + (2m + 2)C, f_2) + \frac{(m + 1)\alpha}{n_1} \bar{N}(r, f_1) + \frac{m\beta}{n_2} \bar{N}(r, f_2) \\ - (m + 1)\bar{N}(r, f_2) - m\bar{N}(r, f_1) + S(r, f_1) + S(r, f_2).$$

Similarly, we have

$$T(r, f_1) - \bar{N}(r, f_1) \leq \alpha \bar{N}(r + (2m + 2)C, f_1) + \frac{(m + 1)\beta}{n_2} \bar{N}(r, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) \\ - (m + 1)\bar{N}(r, f_1) - m\bar{N}(r, f_2) + S(r, f_2) + S(r, f_1).$$

The above inequality applies to the functions  $f(z + c_j)$  as well instead of  $f(z)$ . Therefore, by Lemma 3, we obtain

$$T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j)) \\ \leq \alpha \bar{N}(r + (2m + 2)C, f_1(z + c_j)) + \frac{(m + 1)\beta}{n_2} \bar{N}(r, f_2(z + c_j)) + \frac{m\alpha}{n_1} \bar{N}(r, f_1(z + c_j)) \\ - (m + 1)\bar{N}(r, f_1(z + c_j)) - m\bar{N}(r, f_2(z + c_j)) + S(r, f_2) + S(r, f_1) \\ \leq \alpha \bar{N}(r + (2m + 3)C, f_1) + \frac{(m + 1)\beta}{n_2} \bar{N}(r + C, f_2) + \frac{m\alpha}{n_1} \bar{N}(r + C, f_1) \\ - (m + 1)\bar{N}(r - C, f_1) - m\bar{N}(r - C, f_2) + S(r, f_2) + S(r, f_1) \\ \leq \alpha \bar{N}(r + (2m + 3)C, f_1) + \frac{(m + 1)\beta}{n_2} \bar{N}(r, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) \\ - (m + 1)\bar{N}(r, f_1) - m\bar{N}(r, f_2) + S(r, f_2) + S(r, f_1). \quad (3.12)$$

Applying (3.12) and Lemma 3, and using (1.1) and (1.2) we conclude that

$$n_1 T(r, f_2) \leq T(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)) + S(r, f_1) \\ \leq \sum_{j=1}^{n_1} [T(r, f_1(z + c_j)) - \bar{N}(r, f_1(z + c_j))] + \bar{N}(r, \sum_{j=1}^{n_1} \alpha_j(z) f_1(z + c_j)) + S(r, f_1) \\ \leq \sum_{j=1}^{n_1} [\alpha \bar{N}(r + (2m + 3)C, f_1) + \frac{(m + 1)\beta}{n_2} \bar{N}(r, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) \\ - (m + 1)\bar{N}(r, f_1) - m\bar{N}(r, f_2)] + \alpha \bar{N}(r + C, f_1) + S(r, f_1) + S(r, f_2) \\ \leq n_1 \alpha \bar{N}(r + (2m + 3)C, f_1) + \frac{n_1(m + 1)\beta}{n_2} \bar{N}(r, f_2) + \frac{n_1 m \alpha}{n_1} \bar{N}(r, f_1) \\ - n_1(m + 1)\bar{N}(r, f_1) - n_1 m \bar{N}(r, f_2) + \alpha \bar{N}(r, f_1) + S(r, f_1) + S(r, f_2).$$

This implies that

$$T(r, f_2) - \bar{N}(r, f_2) \leq \alpha \bar{N}(r + [2(m + 1) + 1]C, f_2) + \frac{(m + 1)\alpha}{n_1} \bar{N}(r, f_1) + \frac{(m + 1)\beta}{n_2} \bar{N}(r, f_2) \\ - (m + 1)\bar{N}(r, f_2) - (m + 1)\bar{N}(r, f_1) + S(r, f_1) + S(r, f_2).$$

Similarly, we obtain

$$T(r, f_1) - \bar{N}(r, f_1) \leq \beta \bar{N}(r + [2(m + 1) + 1]C, f_1) + \frac{(m + 1)\beta}{n_2} \bar{N}(r, f_2) + \frac{(m + 1)\alpha}{n_1} \bar{N}(r, f_1) - (m + 1)\bar{N}(r, f_1) - (m + 1)\bar{N}(r, f_2) + S(r, f_1) + S(r, f_2).$$

We complete the induction. Moreover, we immediately see from (3.10) that

$$\begin{cases} T(r, f_2) - \bar{N}(r, f_2) \leq \alpha \bar{N}(r + (2m + 1)C, f_1) + \frac{m\beta}{n_2} \bar{N}(r, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) \\ \quad - m\bar{N}(r, f_1) - m\bar{N}(r, f_2) + S(r, f_1) + S(r, f_2), \\ T(r, f_1) - \bar{N}(r, f_1) \leq \beta \bar{N}(r + (2m + 1)C, f_2) + \frac{m\alpha}{n_1} \bar{N}(r, f_1) + \frac{m\beta}{n_2} \bar{N}(r, f_2) \\ \quad - m\bar{N}(r, f_2) - m\bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \end{cases} \tag{3.13}$$

By (3.13) and Lemma 3, we have

$$\begin{cases} \bar{N}(r, f_1) + \bar{N}(r, f_2) \leq (\frac{\alpha}{m} + \frac{\alpha}{n_1})\bar{N}(r, f_1) + \frac{\beta}{n_2}\bar{N}(r, f_2) + S(r, f_1) + S(r, f_2), \\ \bar{N}(r, f_2) + \bar{N}(r, f_1) \leq (\frac{\beta}{m} + \frac{\beta}{n_2})\bar{N}(r, f_2) + \frac{\alpha}{n_1}\bar{N}(r, f_1) + S(r, f_1) + S(r, f_2). \end{cases} \tag{3.14}$$

For sufficiently large  $m$  we see that

$$\begin{aligned} \frac{\alpha}{m} + \frac{\alpha}{n_1} &= \alpha(\frac{1}{m} + \frac{1}{n_1}) < 1, \frac{\beta}{n_2} < 1, \\ \frac{\beta}{m} + \frac{\beta}{n_2} &= \beta(\frac{1}{m} + \frac{1}{n_2}) < 1, \frac{\alpha}{n_1} < 1. \end{aligned}$$

From (3.14), we have

$$\begin{cases} [1 - (\frac{\alpha}{m} + \frac{\alpha}{n_1})] \bar{N}(r, f_1) + [1 - \frac{\beta}{n_2}] \bar{N}(r, f_2) \leq S(r, f_1) + S(r, f_2), \\ [1 - (\frac{\beta}{m} + \frac{\beta}{n_2})] \bar{N}(r, f_2) + [1 - \frac{\alpha}{n_1}] \bar{N}(r, f_1) \leq S(r, f_1) + S(r, f_2). \end{cases}$$

Thus, for sufficiently large  $m$ , we have

$$\bar{N}(r, f_1) = S(r, f_1) + S(r, f_2), \bar{N}(r, f_2) = S(r, f_1) + S(r, f_2). \tag{3.15}$$

By (3.3), (3.4), (3.15) and Lemma 3, we have

$$T(r, f_1) = S(r, f_1) + S(r, f_2), T(r, f_2) = S(r, f_1) + S(r, f_2).$$

Namely,

$$[1 + o(1)]T(r, f_1) = S(r, f_2), [1 + o(1)]T(r, f_2) = S(r, f_1).$$

Hence, we have

$$[1 + o(1)][1 + o(1)]T(r, f_1)T(r, f_2) = S(r, f_1)S(r, f_2) = o(T(r, f_1))o(T(r, f_2)).$$

Namely,

$$[1 + o(1)]T(r, f_1)T(r, f_2) = o(T(r, f_1))o(T(r, f_2)).$$

Then, we have  $1 = 0$ , which is a contradiction. Thus, the order  $\rho(f_1) = \infty, \rho(f_2) = \infty$  at least one of them will be true.

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## 一类复差分方程组的亚纯解

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**摘要:** 本文研究了一类复差分方程组的增长性的问题. 利用亚纯函数的Nevanlinna 值分布理论, 得到了有关复差分方程组的亚纯解的一个重要结果, 将复差分方程的某些结果推广到复差分方程组中.

**关键词:** 增长性; 亚纯解; 复差分方程组

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