

ALMOST SURE CONVERGENCE OF WEIGHTED SUMS FOR MARTINGALE DIFFERENCES

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Abstract: In this paper, we discuss a class of weighted sums for martingale difference sequence based on some elementary inequalities and the truncation technique. The almost sure convergence is obtained, which extends the result on the case of independent identically distributed random variables.

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1 Introduction and Main Results

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants. Weighted sums of the form $\sum_{i=1}^n a_{ni}X_i$ arise in a variety of applications, such as least squares estimation, nonparametric regression function estimate and jackknife estimate. Because of the wide applications of weighted sums in statistics, many researchers have paid much attention to its properties.

When $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed (i.i.d.) random variables, the almost sure convergence of weighted sums $\sum_{i=1}^n a_{ni}X_i$ was founded in Choi and Sung [1], Chow [2], Chow and Lai [3], Li et al. [7], Stout [8], Sung [9, 10], Teicher [11], Thrum [12] and so on.

This paper focus on the above almost sure convergence of the weighted sums for the case which $\{X_n, n \geq 1\}$ is a martingale difference sequence. To describe the results of this paper, suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and let $\{\mathcal{F}_n, n \geq 0\}$ be a family of σ -algebras, such that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}$.

Let $\{X_n, n \geq 1\}$ be a martingale difference sequence with respect to the filtration $\{\mathcal{F}_n, n \geq 0\}$ with $X_0 = 0$, i.e.,

- i) $\mathbb{E}|X_n| < \infty$, for all $n \geq 1$,

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- ii) X_n is \mathcal{F}_n -measurable,
- iii) $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = 0$ a.s. for every $n \geq 1$.

Throughout this paper, C denotes a positive constant, which may take different values whenever it appears in different expressions. 1_A denotes the indicator function of the event A . $\log x$ denotes $\ln \max\{x, e\}$, where \ln is the natural logarithm.

Now we state our main result as follows.

Theorem 1.1 Let $\{X_n, n \geq 1\}$ be a stationary martingale difference sequence with respect to the filtration $\{\mathcal{F}_n, n \geq 0\}$ with $X_0 = 0$. In addition, assume that

$$\mathbb{E}(X_n^2|\mathcal{F}_{n-1}) \leq 1 \quad \text{a.s.} \quad (1.1)$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying

$$\max_{1 \leq i \leq n} |a_{ni}| \leq \frac{1}{\sqrt{2n \log n}}. \quad (1.2)$$

Then we have

$$\limsup_{n \rightarrow \infty} \sum_{i=1}^n a_{ni} X_i \leq 1 \quad \text{a.s.} \quad (1.3)$$

We will obtain the following better consequence than Theorem 1.1 as condition (1.2) is replaced by the stronger condition (1.4).

Corollary 1.1 Let $\{X_n, n \geq 1\}$ be a stationary martingale difference sequence with respect to the filtration $\{\mathcal{F}_n, n \geq 0\}$ with $X_0 = 0$. In addition, assume that there exists a constant $C > 0$ such that $\mathbb{E}(X_n^2|\mathcal{F}_{n-1}) < C$ a.s.. If $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a triangular array of constants satisfying

$$\max_{1 \leq i \leq n} |a_{ni}| = o\left(\frac{1}{\sqrt{n \log n}}\right), \quad (1.4)$$

then we have $\sum_{i=1}^n a_{ni} X_i \rightarrow 0$ a.s..

2 The Proof of Main Results

In order to prove our main result, we need the following lemma.

Lemma 2.1 If $\mathbb{E}X^2 < \infty$, then for any $\epsilon > 0$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} \mathbb{E}|X| 1_{\{|X| > \epsilon \sqrt{\frac{n}{\log n}}\}} < \infty$.

Proof Noting that $\left\{\frac{n}{\log n}\right\}$ is an increasing sequence, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} \mathbb{E}|X| 1_{\{|X| > \epsilon \sqrt{\frac{n}{\log n}}\}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} \sum_{i=n}^{\infty} \mathbb{E}|X| 1_{\{\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}\}} \\ &= \sum_{i=1}^{\infty} \mathbb{E}|X| 1_{\{\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}\}} \sum_{n=1}^i \frac{1}{\sqrt{n \log n}} \leq C \sum_{i=1}^{\infty} \mathbb{E}|X| 1_{\{\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}\}} \sqrt{\frac{i}{\log i}} \\ &\leq C \sum_{i=1}^{\infty} \mathbb{P}\left(\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}\right) \frac{i}{\log i} \leq C \mathbb{E}X^2 < \infty, \end{aligned}$$

since the first inequality follows from the following fact:

$$\sum_{n=1}^i \frac{1}{\sqrt{n \log n}} \leq C \int_1^i \frac{1}{\sqrt{x \log x}} dx \leq C \sqrt{\frac{i}{\log i}}.$$

Proof of Theorem 1.1 By Lemma 2.1 there exists a sequence of positive real numbers $\{\epsilon_n\}$ satisfying $\epsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} \mathbb{E}|X_n| 1_{\{|X_n| > \epsilon_n \sqrt{\frac{n}{\log n}}\}} < \infty. \tag{2.1}$$

For $1 \leq i \leq n$, define $Y_i = X_i 1_{\{|X_i| \leq \epsilon_i \sqrt{\frac{i}{\log i}}\}}$, $Z_i = X_i 1_{\{|X_i| > \epsilon_i \sqrt{\frac{i}{\log i}}\}}$, and

$$X'_i = Y_i - \mathbb{E}(Y_i | \mathcal{F}_{i-1}), \quad X''_i = Z_i - \mathbb{E}(Z_i | \mathcal{F}_{i-1}),$$

then $X_i = X'_i + X''_i$ and $\{X'_i, i \geq 1\}$, $\{X''_i, i \geq 1\}$ are two martingale difference sequences with respect to the filtration $\{\mathcal{F}_i, i \geq 0\}$.

For any $1 \leq i \leq n$, let us define $X_{ni} = a_{ni} b_n X'_i$, where $b_n = \sqrt{2n \log n}$. By (1.1) and (1.2), it is easy to check

$$\begin{aligned} \mathbb{E}(X_{ni}^2 | \mathcal{F}_{i-1}) &\leq \mathbb{E}(X_i'^2 | \mathcal{F}_{i-1}) = \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}) - (\mathbb{E}(Y_i | \mathcal{F}_{i-1}))^2 \\ &\leq \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}) \leq \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}) \leq 1 \quad \text{a.s.} \end{aligned} \tag{2.2}$$

and

$$|X_{ni}| \leq 2 \max_{1 \leq i \leq n} \epsilon_i \sqrt{\frac{i}{\log i}} = o\left(\sqrt{\frac{n}{\log n}}\right).$$

So there exists a positive decreasing sequence $k_n \rightarrow 0$ such that

$$|X_{ni}| \leq 2 \max_{1 \leq i \leq n} \epsilon_i \sqrt{\frac{i}{\log i}} = k_n \sqrt{\frac{n}{\log n}}.$$

For any $\lambda > 0$, from the following elementary inequalities

$$e^x \leq 1 + x + \frac{x^2}{2} e^{|x|}, \quad \forall x \in \mathbb{R} \quad \text{and} \quad e^x \geq 1 + x, \quad \forall x > 0,$$

and the definition of martingale difference, we have

$$\begin{aligned} &\mathbb{E} \left(\exp \left\{ \frac{\lambda X_{ni}}{b_n} \right\} \middle| \mathcal{F}_{i-1} \right) \\ &\leq \mathbb{E} \left(1 + \frac{\lambda X_{ni}}{b_n} + \frac{\lambda^2 X_{ni}^2}{2b_n^2} \exp \left\{ \frac{\lambda |X_{ni}|}{b_n} \right\} \middle| \mathcal{F}_{i-1} \right) \\ &= 1 + \frac{\lambda^2}{2b_n^2} \mathbb{E} \left(X_{ni}^2 \exp \left\{ \frac{\lambda |X_{ni}|}{b_n} \right\} \middle| \mathcal{F}_{i-1} \right) \\ &\leq 1 + \frac{\lambda^2}{2b_n^2} \exp \left\{ \frac{\lambda k_n}{b_n} \sqrt{\frac{n}{\log n}} \right\} \\ &\leq \exp \left\{ \frac{\lambda^2}{2b_n^2} \exp \left\{ \frac{\lambda k_n}{b_n} \sqrt{\frac{n}{\log n}} \right\} \right\}. \end{aligned}$$

By iterating the same procedure and using the properties of conditional expectation, we can give

$$\begin{aligned} \mathbb{E} \exp \left\{ \frac{\lambda}{b_n} \sum_{i=1}^n X_{ni} \right\} &= \mathbb{E} \left\{ \mathbb{E} \left(\exp \left\{ \frac{\lambda}{b_n} \sum_{i=1}^n X_{ni} \right\} \middle| \mathcal{F}_{n-1} \right) \right\} \\ &= \mathbb{E} \left\{ \exp \left\{ \frac{\lambda}{b_n} \sum_{i=1}^{n-1} X_{ni} \right\} \mathbb{E} \left(\exp \left\{ \lambda \frac{X_{nn}}{b_n} \right\} \middle| \mathcal{F}_{n-1} \right) \right\} \\ &\leq \exp \left\{ n \frac{\lambda^2}{2b_n^2} \exp \left\{ \frac{\lambda k_n}{b_n} \sqrt{\frac{n}{\log n}} \right\} \right\}. \end{aligned}$$

Hence, by Markov's inequality, it follows that

$$\begin{aligned} &\mathbb{P} \left(\frac{1}{b_n} \sum_{i=1}^n X_{ni} > 1 + r \right) \\ &\leq \inf_{\lambda > 0} e^{-\lambda(1+r)} \mathbb{E} \exp \left\{ \frac{\lambda}{b_n} \sum_{i=1}^n X_{ni} \right\} \\ &\leq \inf_{\lambda > 0} \exp \left\{ -\lambda(1+r) + n \frac{\lambda^2}{2b_n^2} \exp \left\{ \frac{\lambda k_n}{b_n} \sqrt{\frac{n}{\log n}} \right\} \right\} \\ &= n^{-(1+r)^2(2 - \exp\{\sqrt{2}(1+r)k_n\})}, \end{aligned}$$

where we choose $\lambda = 2(1+r) \log n$. Since $k_n \rightarrow 0$ implies that

$$(1+r)^2(2 - \exp\{\sqrt{2}(1+r)k_n\}) > 1,$$

for n sufficiently large. Then we have

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\sum_{i=1}^n a_{ni} X'_i > 1 + r \right) = \sum_{n=1}^{\infty} \mathbb{P} \left(\frac{1}{b_n} \sum_{i=1}^n X_{ni} > 1 + r \right) < \infty.$$

Therefore by the Borel-Cantelli lemma, we have

$$\limsup_{n \rightarrow \infty} \sum_{i=1}^n a_{ni} X'_i \leq 1 \quad \text{a.s.}$$

To finish the proof, it is enough to show that

$$\sum_{i=1}^n a_{ni} X''_i \rightarrow 0 \quad \text{a.s.} \quad (2.3)$$

From Markov's inequality, for any $r > 0$, we can obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbb{P} \left(\frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} (|Z_i| + \mathbb{E}(|Z_i| | \mathcal{F}_{i-1})) > r \right) \\ &\leq \frac{2}{r} \sum_{k=1}^{\infty} \frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} \mathbb{E}|Z_i| \\ &= \frac{2}{r} \sum_{i=1}^{\infty} \mathbb{E}|Z_i| \sum_{\{k: 2^k \geq i\}} \frac{1}{\sqrt{2^k \log 2^k}}. \end{aligned}$$

Note the fact

$$\sum_{\{k:2^k \geq i\}} \frac{1}{\sqrt{2^k \log 2^k}} \leq \frac{1}{\sqrt{\log i}} \sum_{\{k:2^k \geq i\}} \frac{1}{\sqrt{2^k}} \leq \frac{\sqrt{2}}{\sqrt{2}-1} \frac{1}{\sqrt{i \log i}}.$$

This implies

$$\sum_{k=1}^{\infty} \mathbb{P} \left(\frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} (|Z_i| + \mathbb{E}(|Z_i| | \mathcal{F}_{i-1})) > r \right) \leq C \sum_{i=1}^{\infty} \frac{\mathbb{E}|Z_i|}{\sqrt{i \log i}} < \infty,$$

where the last inequality follows from (2.1). Thus, by the Borel-Cantelli lemma, we have

$$\frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} (|Z_i| + \mathbb{E}(|Z_i| | \mathcal{F}_{i-1})) \rightarrow 0 \text{ a.s.} \tag{2.4}$$

From the condition (1.2), we get the following inequality

$$\begin{aligned} & \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=1}^n a_{ni} X_i'' \right| \\ & \leq \frac{1}{\sqrt{2^{k+1} \log 2^k}} \sum_{i=1}^{2^{k+1}-1} (|Z_i| + \mathbb{E}(|Z_i| | \mathcal{F}_{i-1})) \\ & \leq \frac{C}{\sqrt{2^{k+1} \log 2^{k+1}}} \sum_{i=1}^{2^{k+1}} (|Z_i| + \mathbb{E}(|Z_i| | \mathcal{F}_{i-1})) \text{ a.s.} \end{aligned} \tag{2.5}$$

From the inequalities (2.4) and (2.5), we get

$$\max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=1}^n a_{ni} X_i'' \right| \rightarrow 0 \text{ a.s.,}$$

which implies (2.3). So the proof is completed.

Proof of Corollary 1.1 From the condition (1.4), there exists a sequence of real numbers denoted by $\{t_n\}$ such that $t_n \downarrow 0$ and

$$\max_{1 \leq i \leq n} |a_{ni}| \leq \frac{t_n}{\sqrt{2n \log n}}.$$

By Theorem 1.1, we get

$$\limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_{ni} X_i}{t_n} \leq 1 \text{ a.s.,}$$

which implies

$$\limsup_{n \rightarrow \infty} \sum_{i=1}^n a_{ni} X_i \leq 0 \text{ a.s..}$$

Replacing X_i by $-X_i$, we can get

$$\liminf_{n \rightarrow \infty} \sum_{i=1}^n a_{ni} X_i \geq 0 \text{ a.s..}$$

Thus the desired result can be obtained.

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鞅差序列加权和的几乎处处收敛

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摘要: 本文研究了一类鞅差序列加权和的收敛性的问题. 利用一些基本不等式和截尾技术, 获得了加权和的几乎处处收敛性, 推广了关于独立同分布的随机变量序列的相关结果.

关键词: 几乎处处收敛; 加权和; 鞅差

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